STA 410/2102 — Practice questions for the second test

1. Find the Cholesky decomposition of the following matrix:

$$\begin{bmatrix} 9 & -3 & 6 \\ -3 & 2 & -3 \\ 6 & -3 & 6 \end{bmatrix}$$

2. Recall that the Householder transformation based on a vector u is the orthogonal transformation that is defined by the following matrix:

$$H(u) \;=\; I \;-\; 2\frac{uu'}{|u|^2}$$

Find a value for u that will cause the vector below to be transformed as indicated:

$$H(u)\begin{bmatrix} 7 \\ 4 \\ 1 \\ 2 \\ -2 \end{bmatrix} \;=\; \begin{bmatrix} 7 \\ x \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

where x can be anything you wish that will make this possible. You should say what x is for your choice of u.

3. Consider the regression model $Y = X\beta + \epsilon$, with ϵ having mean 0 and variance $\sigma^2 I$. The variance-covariance matrix for the sampling distribution of $\hat\beta$, the least-squares estimator for β, is $\sigma^2(X'X)^{-1}$. Typically, this variance-covariance matrix for the estimator is estimated by $\hat\sigma^2(X'X)^{-1}$, where $\hat\sigma$ is the estimate for the standard deviation of the residuals based on the actual residuals found using $\hat\beta$ — ie, $\hat\sigma$ is found from $Y - X\hat\beta$.

Suppose we orthogonally transform X and Y by multiplying them by an orthogonal matrix, Q, giving the model $Y^* = X^*\beta + \epsilon^*$, where $Y^* = QY$, $X^* = QX$, and $\epsilon^* = Q\epsilon$. The least squares estimate for β in this model will be the same as for the original model. Suppose that we estimate the variance-covariance matrix for its sampling distribution by $\hat\sigma_*^2(X^{*\prime}X^*)^{-1}$, where $\hat\sigma_*$ is the estimate for σ based on the actual residuals found for this model using $\hat\beta$ — ie, it is based on $Y^* - X^*\hat\beta$. Will this estimate for the variance-covariance matrix for $\hat\beta$ be the same as that found using the original model? Show why or why not.

4. Suppose we try to solve the equation $x^4 - 81 = 0$ using Newton-Raphson iteration, starting from an initial value of $x_0 = 5$. Here are the values found in the first four iterations:

$$\begin{aligned} x_1 &= 3.912 \\ x_2 &= 3.2722427442656 \\ x_3 &= 3.03212968848923 \\ x_4 &= 3.00050709092522 \end{aligned}$$

The exact answer is of course 3 (for the solution in this neighborhood).

Estimate what value of x_5 will be the result of doing one more iteration, without actually doing this iteration. Try to get as accurate an answer as you can (without doing lots of arithmetic), by considering in detail the rate at which the error ought to be going down.

5. Suppose we have i.i.d. data points $a_1, \ldots, a_n$ that are measurements of angles in radians, in the range of 0 to 2π. We decide to model this data with a form of the "von Mises" distribution that assigns probability density $K \exp(\cos(a_i - \theta))$ to data point a_i, where $K = 0.1257\ldots$, and θ is an unknown model parameter.

Derive the formulas needed to use Newton-Raphson iteration to find the maximum likelihood estimate for θ, and write an S/R program that takes as arguments a data vector a, an initial guess for θ, and the number of Newton-Raphson iterations to do, and which returns a list consisting of the maximum likelihood estimate for θ along with its standard error, found using the observed likelihood.

Note: It's actually possible to find the MLE analytically, but you're not supposed to do it that way!