STA 247 — Solutions to the mid-term test

Note that the makeup test differed slightly from this version.

Question 1. [30 marks] Suppose you roll a fair, six-sided die ten times, with each roll producing a number from one to six. Define the following events:

$$\begin{aligned} A &= \text{The first roll is a six} \\ B &= \text{The last roll is a six} \\ C &= \text{The first and last rolls are the same} \\ D &= \text{The sum of the ten rolls is eleven or less} \end{aligned}$$

a) Are events A and B independent? Explain why or why not.

Yes, they are independent. Intuitively, we expect that an event involving only one of the rolls and an event involving only another roll will be independent. We can confirm this from the definition of independence, since $P(A) = 1/6$, $P(B) = 1/6$, *and* $P(A \cap B) = 1/36 = P(A)P(B)$.

b) Are events A and C independent? Explain why or why not.

Yes, they are independent. Although event C *is related to event* A, *since they both involve the first roll, finding out that* A *occurred would tell you nothing new about how likely* C *is to occur, since the last roll has a 1/6 chance of matching the first roll regardless of what the first roll is. There are 36 possible outcomes if we ignore rolls two through nine, in six of which the first and last roll are the same — outcomes of* $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, $(6,6)$. *We therefore have* $P(C) = 6/36 = 1/6$, *as well as* $P(A) = 1/6$. *The event* $A \cap C$ *corresponds to only one outcome —* $(6,6)$ *— and therefore* $P(A \cap C) = 1/36 = P(A)P(B)$.

c) Are events A and D independent? Explain why or why not.

No, they are not independent. There is no possible outcome in which A *and* D *are both true, since the sum of the ten rolls must be at least 15 if the first roll is six. So* $A \cap D = \emptyset$, *and hence* $P(A \cap D) = 0$. *But it is possible for* D *to occur, since if all rolls are one, the sum will be ten. So* $P(D) > 0$, *since this outcome (like all the others) has non-zero probability. Also,* $P(A) = 1/6$. *So* $P(A)P(D) \neq 0$, *but* $P(A \cap D) = 0$, *so* A *and* D *can't be independent.*

d) Find $P(D)$.

There are 6^{10} *possible outcomes from rolling ten dice, all of which are equally likely. The outcomes for which the sum of the rolls is eleven or less are* $(1,1,1,1,1,1,1,1,1,1)$, *for which the sum is ten, and the ten outcomes* $(2,1,1,1,1,1,1,1,1,1)$, $(1,2,1,1,1,1,1,1,1,1)$, $\ldots$, $(1,1,1,1,1,1,1,1,1,2)$, *for which the sum is eleven. The number of outcomes in* D *is therefore 11, from which* $P(D) = 11/6^{10}$.

3) Find $P(C|D)$.

By the definition of conditional probability, $P(C|D) = P(C \cap D) \,/\, P(D)$. The event $C \cap D$ consists of all the outcomes in D except those for which the first and last rolls are different. The only outcomes in D for which the first and last rolls differ are $(2, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ and $(1, 1, 1, 1, 1, 1, 1, 1, 1, 2)$. This leaves 9 outcomes, so $P(C \cap D) = 9/6^{10}$. Combining this with the value for $P(D)$ found above, we get that $P(C|D) = (9/6^{10}) \,/\, (11/6^{10}) = 9/11$.

Question 2. [20 marks] Joe puts either one white marble or one black marble in a paper bag, and then hands the bag to Susan. Susan adds a white marble to the bag, shakes it to mix up the two marbles, and then draws one marble from the bag. If the marble Susan draws is white, how likely is it that the other marble (still in the bag) is also white?

Assume that Joe is equally likely to put a white marble or a black marble in the bag, and that Susan is equally likely to draw the marble she put in the bag or the marble Joe put in the bag.

First solution: *We can use a sample space with four outcomes, $S = \{BJ, BS, WJ, WS\}$, where B or W indicates whether Joe put a black or a white marble in the bag, and J or S indicates whether Susan drew the marble Joe put in the bag or the white marble she put in the bag herself. These outcomes are equally likely. The event that Susan draws a white marble corresponds to the subset $D = \{BS, WJ, WS\}$. The event that there is a white marble left in the bag corresponds to the subset $L = \{BJ, WJ, WS\}$. The probability we are interested in is $P(L|D) = P(L \cap D)/P(D) = P(\{WJ, WS\})/P(\{BS, WJ, WS\} = (2/4)/(3/4) = 2/3$.*

Second solution: *Let the random variable W be 0 if Joe puts a black marble in the bag and 1 if he puts a white marble in the bag. Let the random variable D be 0 if Susan draws a black marble and 1 if Susan draws a white marble. We know that $P(W = 0) = P(W = 1) = 1/2$. We also know that $P(D = 1|W = 0) = 1/2$, since if $W = 0$, Susan is drawing from a bag with one white and one black marble, and that $P(D = 1|W = 1) = 1$, since if $W = 1$, both the marbles in the bag are white. The probability we are trying to find can be written as $P(W = 1|D = 1)$, since if Susan draws a white marble, the other marble can be white only if Joe put in a white marble. We can now find the answer using Bayes' Rule:*

$$
\begin{aligned}
P(W=1 \,|\, D=1) &= \frac{P(D=1 \,|\, W=1)P(W=1)}{P(D=1 \,|\, W=1)P(W=1) \;+\; P(D=1 \,|\, W=0)P(W=0)} \\
&= \frac{1 \times (1/2)}{1 \times (1/2) \;+\; (1/2) \times (1/2)} \;=\; \frac{2}{3}
\end{aligned}
$$

Question 3. [25 marks] Suppose you roll two fair, six-sided dice, one of which is red and the other of which is green. Define the following random variables:

$$
X \;=\; \text{The number shown on the red die}
$$

$$
Y \;=\; \begin{cases} 0 & \text{if the two dice show the same number} \\ 1 & \text{if the number on the green die is bigger than the number on the red die} \\ 2 & \text{if the number on the red die is bigger than the number on the green die} \end{cases}
$$

a) Write down a table showing the joint probability mass function for X and Y.

We can find the joint probability mass function by counting how many of the 36 equally likely outcomes produce each combination of values for X and Y. The result is as follows:

		$X =$					
		1	*2*	*3*	*4*	*5*	*6*
	0	*1/36*	*1/36*	*1/36*	*1/36*	*1/36*	*1/36*
$Y =$	*1*	*5/36*	*4/36*	*3/36*	*2/36*	*1/36*	*0*
	2	*0*	*1/36*	*2/36*	*3/36*	*4/36*	*5/36*

b) Find the marginal probability mass function for Y, and compute its expected value.

We find the marginal probability mass function for Y by summing the rows in the table of the joint probability mass function above. The result is:

y	*0*	*1*	*2*
$p(y)$	*6/36*	*15/36*	*15/36*

The expected value of Y is

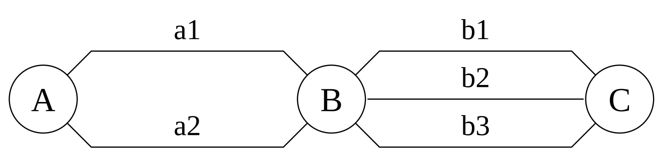
$$\begin{aligned} E(Y) &= 0 \times P(Y=0) \ + \ 1 \times P(Y=1) \ + \ 2 \times P(Y=2) \\ &= 0 \times 6/36 \ + \ 1 \times 15/36 \ + \ 2 \times 15/36 \ = \ 45/36 \ = \ 5/4 \end{aligned}$$

c) Find the conditional probability mass function for X given $Y = 1$.

We can find this by taking the row for $Y = 1$ from the table of the joint probability mass function in (a) above, and dividing all the numbers in this row by $P(Y = 1)$ from the marginal probability mass function in (b) above. The result is

x	*1*	*2*	*3*	*4*	*5*	*6*
$P(X = x \mid Y = 1)$	*5/15*	*4/15*	*3/15*	*2/15*	*1/15*	*0*

Question 4. [25 marks] Three computers, A, B, and C, are linked by network connections as shown below:



Two network connections, a1 and a2, link computer A and computer B. Three network connections, b1, b2, and b3, link computer B and computer C. Since there is no direct connection from computer A to computer C, messages sent from computer A to computer C must pass through computer B.

When computer A sends a message to computer B, it randomly chooses whether to use connection a1 or connection a2, with probabilities of 2/3 for a1 and 1/3 for a2. When computer B sends a message to computer C, it randomly chooses whether to use connection b1, connection b2, or connection b3, with probabilities of 1/2 for b1, 1/4 for b2, and 1/4 for b3.

The time to send a message through each connection is 1ms for a1, 2ms for a2, 1ms for b1, 2ms for b2, and 3ms for b3. When a message is sent from computer A to computer C through computer B, it takes no time for computer B to take the message received from connection a1 or a2 and send it on connection b1, b2, or b3.

Let X be the random variable whose value is the total time (in milliseconds) taken for a message sent from computer A to computer C to arrive.

a) Find $P(X = 4)$.

The total time taken will be 4ms if either connections a1 and b3 are used or if connections a2 and b2 are used. We add the probabilities for these two possibilities (which are mutually exclusive) to get the answer:

$$P(X = 4) \;=\; (2/3)(1/4) \;+\; (1/3)(1/4) \;=\; 1/4$$

b) Find $E(X)$.

One way to answer this is to find the probability mass function for X and then directly use the definition of expectation. Another way is to note that $X = Y + Z$, where Y is the time taken to go from A to B and Z is the time taken to go from B to C. It follows that $E(X) = E(Y) + E(Z)$. We can find $E(X)$ and $E(Y)$ as follows:

$$E(Y) \;=\; 1\times(2/3)+2\times(1/3) \;=\; 4/3, \quad E(Z) \;=\; 1\times(1/2)+2\times(1/4)+3\times(1/4) \;=\; 7/4$$

The answer is therefore $E(X) = 4/3 + 7/4 = 37/12$.

c) Suppose that a message sent from computer A to computer C takes 4ms or more to arrive (ie, $X \geq 4$). How likely is it that this message was sent from computer B to computer C using connection b3?

There are three combinations of connections used for which the total time taken is 4ms or more: a1 & b3, a2 & b3, and a2 & b2. The first two of these use connection b3; the last doesn't. The answer is therefore

$$\begin{aligned} P(\text{b3 used} \,|\, X \geq 4) &= \frac{P(X \geq 4 \text{ and b3 used})}{P(X \geq 4)} \\ &= \frac{(2/3)(1/4) \;+\; (1/3)(1/4)}{(2/3)(1/4) \;+\; (1/3)(1/4) \;+\; (1/3)(1/4)} \\ &= 3/4 \end{aligned}$$