STA 414/2104, Spring 2012, Practice Problem Set #2

Note: these problems are not for credit, and not to be handed in

Question 1: Recall that a multilayer perceptron network with m hidden units using the tanh activation function computes a function defined as follows:

$$f(x,w) \;=\; w_0^{(2)} \;+\; \sum_{j=1}^{m} w_j^{(2)} \phi_j(x,w), \qquad \phi_j(x,w) \;=\; \tanh\Big(w_{0j}^{(1)} \;+\; \sum_{k=1}^{p} w_{kj}^{(1)} x_k\Big)$$

where w is the set of parameters (weights) for the network, and x is the vector of p inputs to the network.

Suppose we train such a network with $m = 1$ hidden units on the following set of $n = 4$ training cases, with $p = 1$ input, x_1, and one real-valued response, y:

x_1	y
-1	1
0	1
1	5
2	5

We use a Gaussian model for the response, in which y given x has a Gaussian distribution with mean $y(x,w)$ and variance one.

a) Suppose that we initialize the weights to $w_{01}^{(1)} = 0$, $w_{11}^{(1)} = 0$, $w_0^{(2)} = 0$, and $w_1^{(2)} = 0.1$. Define $E(w)$ to be the minus the log likelihood, dropping terms that don't depend on w, so that $E(w)$ is 1/2 times the sum of the squares of the residuals in the four training cases.

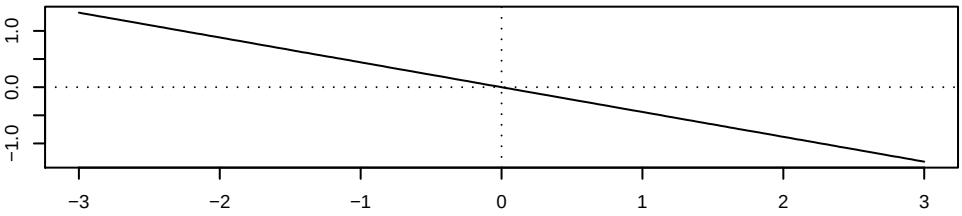
Find the gradient of $E(w)$, as would be needed to do gradient descent learning, evaluated at the initial value of w specified above. In other words, find the partial derivatives of E with respect to all the components of w, at the initial value of w.

b) If gradient descent learning to minimize minus the log likelihood is done from the initial weights specified in part (a) above, what weights will the learning converge to (assuming that the learning rate used is small enough to ensure stability)? You may not be able to say exactly what the values of all the weights will be, but say as much as you can.

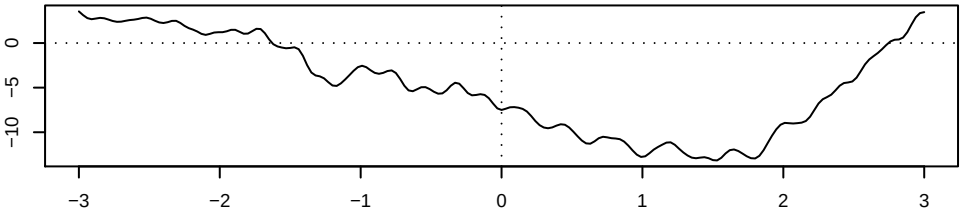
c) Suppose that gradient descent learning is done from the initial weights in part (a), but with a penalty of $\lambda[w_{11}^{(1)}]^2$ added to minus the log likelihood. If λ is a small positive number, what will the learning converge to (assuming that the learning rate used is small enough to ensure stability)? You may not be able to say exactly what the values of all the weights will be, but say as much as you can.

Question 2: Below are five functions randomly drawn from five different Gaussian processes. For all five Gaussian processes, the mean function is zero. The covariance functions are one of those listed below.

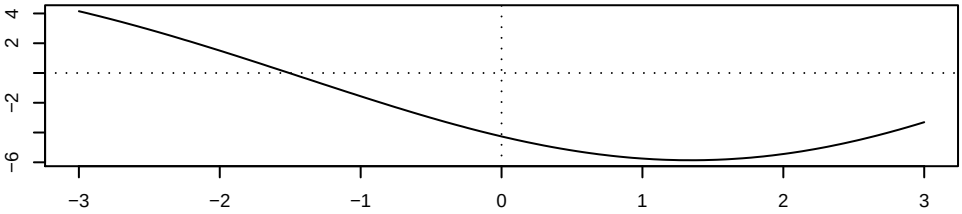
a)



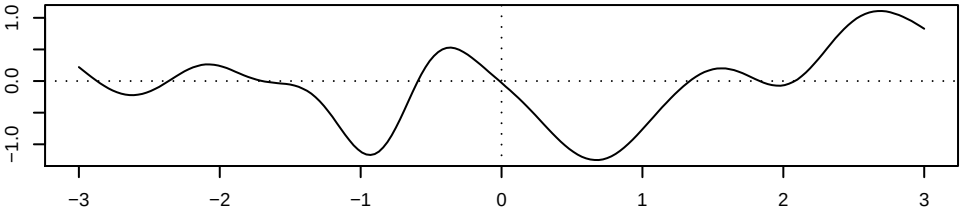
b)



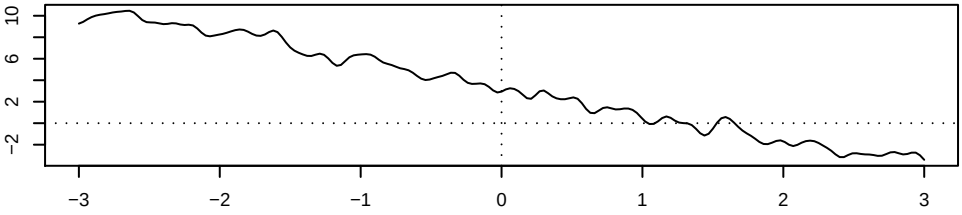
c)



d)



e)



For each of the five covariance functions below, indicate which of the five functions above is most likely to have been drawn from the Gaussian process with that covariance function.

1) $\text{Cov}(y_{i_1}, y_{i_2}) \;=\; 0.5^2 \,\exp(-((x_{i_1} - x_{i_2})\,/\,0.5)^2\,)$

2) $\text{Cov}(y_{i_1}, y_{i_2}) \;=\; x_{i_1}\, x_{i_2}$

3) $\text{Cov}(y_{i_1}, y_{i_2}) \;=\; 5^2 \;+\; 5^2\, x_{i_1}\, x_{i_2} \;+\; 0.5^2 \,\exp(-((x_{i_1} - x_{i_2})\,/\,0.1)^2\,)$

4) $\text{Cov}(y_{i_1}, y_{i_2}) \;=\; 0.7^2 \,\exp(-((x_{i_1} - x_{i_2})\,/\,0.1)^2\,) \;+\; 8^2 \,\exp(-((x_{i_1} - x_{i_2})\,/\,2)^2\,)$

5) $\text{Cov}(y_{i_1}, y_{i_2}) \;=\; 8^2 \,\exp(-((x_{i_1} - x_{i_2})\,/\,5)^2\,)$

Question 3: Suppose we model the relationship of a real-valued response variable, y, to a single real input, x, using a Gaussian process model in which the mean is zero and the covariances of the observed responses are given by

$$\text{Cov}(y_i, y_{i'}) \;=\; 0.5^2 \delta_{i,i'} \;+\; K(x_i, x_{i'})$$

with the noise-free covariance function, K, defined by

$$K(x, x') \;=\; \begin{cases} 1 - |x - x'| & \text{if } |x - x'| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Suppose we have four training cases, as follows:

x	y
0.5	2.0
2.8	3.3
1.6	3.0
3.9	2.7

Recall that the conditional mean of the response in a test case with input x_*, given the responses in the training cases, is $k^T C^{-1} y$, where y is the vector of training responses, C is the covariance matrix of training responses, and k is the vector of convariances of training responses with the response in the test case.

Find the predictive mean for the response in a test case in which the input is $x_* = 1.2$.

Question 4: Recall that for a Gaussian process model the predictive distribution for the response y^* in a test case with inputs x^* has mean and variance given by

$$\begin{aligned} E[y^* \,|\, x^*, \text{ training data}] &= k^T C^{-1} y \\ \text{Var}[y^* \,|\, x^*, \text{ training data}] &= v \;-\; k^T C^{-1} k \end{aligned}$$

where y is the vector of observed responses in training cases, C is the matrix of covariances for the responses in training cases, k is the vector of covariances of the response in the test case with the responses in training cases, and v is the prior variance of the response in the test case.

a) Suppose we have just one training case, with $x_1 = 3$ and $y_1 = 4$. Suppose also that the noise-free covariance function is $K(x, x') = 2^{-|x - x'|}$, and the variance of the noise is $1/2$. Find the mean and variance of the predictive distribution for the response in a test case for which the value of the input is 5.

b) Repeat the calculations for (a), but using $K(x, x') = 2^{+|x - x'|}$. What can you conclude from the result of this calculation?

Question 5: Consider a binary classification problem in which two inputs are available for predicting the class — input x_1, which is binary, and input x_2, which is real-valued. Suppose we use a naive Bayes model in which x_1 and x_2 are assumed to be independent within each class. Let $P(x_1 = 1 \,|\, C_0) = \theta_0$ and $P(x_1 = 1 \,|\, C_1) = \theta_1$, and assume that $x_2|C_0 \sim N(\mu_0, \sigma^2)$ and $x_2|C_1 \sim N(\mu_1, \sigma^2)$, where θ_0, θ_1, μ_0, μ_1, and σ are parameters to be estimated from the training data.

Supposing that these parameters have been estimated, as $\hat{\theta}_0$, $\hat{\theta}_1$, $\hat{\mu}_0$, $\hat{\mu}_1$, and $\hat{\sigma}$, and that some estimate for the "prior" probability of class 1, $P(C_1)$ is available, work out an expression for the probability of class 1 for a test case with inputs (x_1^*, x_2^*).

Question 6: Recall that the maximum margin separating hyperplane, defined by $w^T x + b = 0$, can be found by solving the following optimiziation problem:

$$\text{minimize } ||w||^2, \quad \text{subject to } y_i(w^T x_i + b) \geq 1 \text{ for } i = 1, \ldots, n$$

Use this to find the maximum margin hyperplane for the following $n = 3$ data points, (x, y), in which x is one-dimensional:

$$(-1, -1), \ (2, +1), \ (3, +1)$$

(Note that a separating "hyperplane" when x is one-dimensional is a single point.)

You should produce a two-dimensional plot of the linear inequality constraints on w and b, and from this find the minimum of $||w||^2$. You should then plot the function $wx + b$, verify that the point where $wx + b = 0$ separates the classes, and find its margin.