Computer Science CSC 238H
St. George Campus

Wednesday May 13 1998
University of Toronto

Homework Assignment #1
Due: Wednesday June 10 1998 at 6:10pm

Don't forget to attach the completed COVER PAGE *to the* front *of your assignment!*

1. [9 marks] Translate each of the following statements into *propositional* formulae. Clearly define any propositional variables you use.

[3] **(a)** Wilbur is sleeping and he dreams whenever he sleeps.

[3] **(b)** Dr. Magnusson's first name is either Willemina or Helga.

[3] **(c)** Ludwig loves his dog even though it smells bad.

2. [14 marks] For each of the following pairs of formulae, determine whether two formulae are logically equivalent. If so, prove it using the tautological equivalences on page 10 of the text, justifying each step. Otherwise, give a truth value assignment that proves they are not equivalent.

[7] **(a)** $L \wedge B$
$(L \rightarrow B) \wedge \sim [(L \rightarrow \sim Q) \wedge (L \rightarrow Q)]$

[7] **(b)** $\sim A \wedge B \wedge \sim (C \rightarrow D) \wedge (E \rightarrow E)$
$[\sim (C \rightarrow E) \vee (E \wedge C)] \wedge \sim D \wedge (A \rightarrow \sim B) \wedge (\sim A \rightarrow B)$

3. [25+5 marks] A directed graph on n vertices is a set $\{1, \ldots, n\}$ whose elements are called vertices, together with a set of ordered pairs of vertices. Each such ordered pair is called an edge of the graph. The first vertex of an edge is called the edge's tail and the second vertex is called the edge's head. Consider a graph G with n vertices. Let E_{ij} be a propositional variable which represents the statement "the ordered pair (i, j) is one of the edges of G" for $1 \leq i, j \leq n$. For this question, assume $n \geq 5$.

Now consider a sequence σ of n ordered pairs of vertices. Let S_{ij}^k represent the statement "the kth edge in the sequence σ is (i, j)" for $1 \leq i, j, k \leq n$. The fact that the variables S_{ij}^k represent a sequence σ is expressed by the propositional formula

$$\alpha = \left(\bigwedge_{k=1}^{n} \bigvee_{i=1}^{n} \bigvee_{j=1}^{n} S_{ij}^k \right) \wedge \left(\bigwedge_{k=1}^{n} \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} \bigwedge_{(l,m) \neq (i,j)} \sim (S_{ij}^k \wedge S_{lm}^k) \right),$$

which says that there is *exactly* one S variable with superscript k that is true for each k.

Show that the following statements can be expressed by *propositional* formulae using the propositional variables S_{ij}^k and E_{ij}. The propositional formulae should have the indicated lengths. The length of a formula is just the total number of atoms, connectives and parentheses that appear in the formula. For example, the length of α is $O(n^5)$. You may assume that the formula α is satisfied; you do not have to include it as part of your formulae. Make your formulae as simple as possible (but no simpler).

[5] **(a)** The elements of σ form a subgraph of G (*i.e.* every ordered pair that appears in the sequence σ is also an edge of G); $O(n^3)$.

[5] **(b)** The second vertex in the third ordered pair of σ is 4; $O(n)$.

[5] **(c)** The head of the third ordered pair of σ is the same vertex as the tail of the fourth ordered pair of σ; $O(n^2)$.

[5] **(d)** The sequence σ is a path in graph G; $O(n^3)$. The sequence σ is called a path in G if all of its elements are edges of G and the head of each edge (except the last) is the same as the tail of the next edge.

[5] **(e)** The sequence σ is a path in graph G that begins at vertex 1 and ends at vertex n; $O(n^3)$. (*I.e.* σ is a path in G where the tail of the first edge is 1 and the head of the last edge is n.)

[Bonus 5] **(f)** The sequence σ forms a Hamiltonian cycle of G; $O(n^4)$. The sequence σ is called a Hamiltonian cycle if it is a path in G that begins and ends at the same vertex and each vertex appears in exactly two edges.

4. [10 marks] Use the formal proof system given on page 53 of the text to prove the statement $[A \wedge (B \rightarrow \sim A)] \rightarrow (\sim B \vee C)$. (The shorter your proof, the better your mark.)

5. [24 marks] Now you will be required to give much more concise representations of the statements in question 3 using *predicate* calculus. Consider a graph G on n vertices and a sequence σ of n ordered pairs of vertices. Let the universe of discourse be $\{1, \ldots, n\}$. Let us first define the symbols to be used, along with their intended interpretations. You may use two predicates:

$E(x, y)$, which means (x, y) is an edge of G, and
$x = y$, which has its usual meaning

and three function symbols:

$t(x)$, which represents the tail of the xth edge of σ,
$h(x)$, which represents the head of the xth edge of σ, and
$p(x)$ which represents the predecessor of x (*i.e.* $p(x)$ represents $x - 1$) if $x > 1$; by convention, the predecessor of 1 will be n

and the constant symbol 1.

Notice that we no longer need a statement like α, since the symbols t and h always represent functions. Thus, for each k, there is a unique pair $(t(x), h(x))$, which is the kth element of σ.

[6] **(a)** Express the statement in part (a) of question 3 as a predicate calculus formula. Your formula should be of length $O(1)$: the length should be independent of the number of vertices in the graph.

[6] **(b)** Do the same for part (d) of question 3.

[6] **(c)** Do the same for part (e) of question 3.

[6] **(d)** What does the following formula express about the graph G? Give your answer in a *simple* English sentence.
$\forall x \exists y [E(x,y) \wedge \forall z (E(x,z) \rightarrow y = z)]$

6. [18 marks] For each of the following formulae, determine whether the formula is
(i) unsatisfiable,
(ii) valid, or
(iii) neither of the above.
Briefly justify your answer.

[9] **(a)** $\forall x \forall y (R(x,y) \rightarrow R(y,x)) \wedge \exists x (\sim R(x,x))$

[9] **(b)** $\forall x \forall y (R(x,y) \rightarrow R(y,x)) \rightarrow \forall x \forall y (R(x,y) \leftrightarrow R(y,x))$