Midterm Test (Fall 1997)

This test lasts 50 minutes.

1. (5 marks) Explain why the following wff is not a tautology.

$$[(A \wedge B)' \wedge (B \vee C \vee D')] \to [(A \wedge D) \to C']$$

2. (5 marks) Write the following English sentence in predicate logic. Specify your domain of interpretation and give meanings for all the predicates you use.

There is a ghost who is not afraid of any witch, but who is afraid of all spiders.

3. (5 marks) Is the wff $\exists y \forall x (P(x) \to P(y)')$ true or false under the interpretation whose domain is the set of integers and where $P(x)$ means that "x is even"? Justify your answer.

4. (5 marks) Prove the following result, explicitly mentioning the rules of logical reasoning you are using.

Theorem. For all integers p and q, if $\sqrt{pq} \neq (p+q)/2$, then $p \neq q$.

5. (10 marks) Prove by induction that, for any positive integer n divisible by 3, a $2 \times n$ chessboard can be tiled using L-shaped pieces, each of which covers 3 squares of the chessboard. Your proof must have the correct form (as specified in class).

Term Test #1 (Spring 1998)

This closed book *test lasts* 50 minutes.

1. [15 marks] Consider the following predicate symbols and their intended meaning over the domain consisting of all people.

predicate symbol	intended meaning
$m(X)$	"X is male"
$f(X)$	"X is female"
$p(X, Y)$	"X is a biological parent of Y"

[3] **(a)** What is the English meaning of the following wff?

$$\forall X\,\big[[m(X) \vee f(X)] \wedge [\neg m(X) \vee \neg f(X)]\big]$$

[3] **(b)** Is there an interpretation under which the wff in part (a) is false? If so, describe such an interpretation and briefly explain why it works; if not, explain why.

[3] **(c)** Write a wff that expresses the statement "Everyone has a father and a mother".

[3] **(d)** Is it possible to write a wff with free variables X and Y that expresses the statement "X is the sister of Y"? If so, give such a wff; if not, explain why.

[3] **(e)** Is it possible to write a wff with free variables X and Y that expresses the statement "X is the uncle of Y"? If so, give such a wff; if not, explain why.

2. [10 marks] For each part below, you may refer to the following predicate and propositional equivalences (where p and q stand for any propositional variables, E_1 and E_2 stand for any predicate wffs such that X does not occur free in E_2).

1. idempotence, commutativity, and associativity of $\wedge$ and $\vee$
2. $\neg\neg p \equiv p$
3. $p \rightarrow q \equiv \neg p \vee q$
4. $\neg(p \vee q) \equiv \neg p \wedge \neg q$
5. $\neg(p \wedge q) \equiv \neg p \vee \neg q$
6. $\neg\exists X E_1 \equiv \forall X \neg E_1$
7. $\neg\forall X E_1 \equiv \exists X \neg E_1$
8. $E_2 \wedge \exists X E_1 \equiv \exists X[E_2 \wedge E_1]$
9. $E_2 \vee \exists X E_1 \equiv \exists X[E_2 \vee E_1]$
10. $E_2 \wedge \forall X E_1 \equiv \forall X[E_2 \wedge E_1]$
11. $E_2 \vee \forall X E_1 \equiv \forall X[E_2 \vee E_1]$

[5] **(a)** Is it the case that $\forall Y q(Y) \rightarrow \exists X p(X) \equiv \exists X[\forall Y q(Y) \rightarrow p(X)]$? Justify your answer.

[5] **(b)** Is it the case that $\exists X p(X) \rightarrow \forall Y q(Y) \equiv \exists X[p(X) \rightarrow \forall Y q(Y)]$? Justify your answer.

3. [10 marks] Prove that for every positive integer n,

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n \cdot (n+1)} \;=\; \frac{n}{n+1}.$$

(You will be marked on form as well as content.)

Term Test #2 (Spring 1998)

This closed book *test lasts* 50 minutes.

1. [15 marks]

Consider the algorithm on the right (where "$\div$" is used for integer division, *i.e.*, $c \div d = \lfloor c/d \rfloor$), under the precondition that x and y are positive integers.

(NOTE: For this question, you may use without proof the facts that when b is even, $b \div 2 = b/2$, and when b is odd, $b \div 2 = (b-1)/2$.)

```
z := 1;
a := x;
b := y;
while b > 0 do
    if b mod 2 = 1 then
        z := z × a;
    end if
    a := a × a;
    b := b ÷ 2;
end while
```

[10] **(a)** Prove that the statement "$z_k \cdot a_k^{b_k} = x^y$" is a loop invariant for this algorithm.

[2] **(b)** What can you conclude about the value of z at the end of the algorithm? Briefly justify (no need for a formal proof).

[3] **(c)** Estimate the running time of the algorithm, as a function of x and y, using big-Oh notation. (The better your bound, the better your mark.)

2. [10 marks] Following is a list of functions and a list of big-Oh expressions. Match each function $f(n)$ with a big-Oh expression $\mathcal{O}(g(n))$ such that $f(n) \in \mathcal{O}(g(n))$, using each big-Oh expression only once (write your answer on the line following each function).

(a) $2/n + 5 \in$ ____________	1. $\mathcal{O}(n!)$
(b) $(n+1)! \in$ ____________	2. $\mathcal{O}(n^2)$
(c) $(5n \log_3 n)^2 \in$ ____________	3. $\mathcal{O}(1/n)$
(d) $3^{n+1} \in$ ____________	4. $\mathcal{O}(\log n)$
(e) $(\log_2 n)^3 \in$ ____________	5. $\mathcal{O}(5^n)$
(f) $5n^2 + 1/\log_2 n \in$ ____________	6. $\mathcal{O}(n)$
(g) $3/(5n^2) \in$ ____________	7. $\mathcal{O}(n^n)$
(h) $n^{\log_2 n} \in$ ____________	8. $\mathcal{O}(n^5)$
(i) $\log_5(5n^3 + 2n) \in$ ____________	9. $\mathcal{O}(2^n)$
(j) $3^{2n} \in$ ____________	10. $\mathcal{O}(1)$

3. [5 marks]

Write a recurrence relation for the running time of the algorithm to the right, as a function of $n = last - first + 1$ (you may assume that $first \leq last + 1$).

Briefly justify your answer (you do not need to solve the recurrence).

```
NORM(A, first, last)
    if first > last then
        return 0;
    else
        mid := (first + last) ÷ 2;
        a := NORM(A, first, mid − 1);
        b := NORM(A, mid + 1, last);
        return sqrt(a² + (A[mid])² + b²);
    end if
END NORM
```