CSC 238F 1996

Notes on Proofs in Predicate Logic

We will allow as axioms any tautology from propositional logic with predicate wffs substituted for the propositions. (All occurrences of each particular proposition within the tautology must be replaced by the same wff, although different propositions can be replaced by different wffs.) There are also some new axioms:

a. $(\forall x[P(x) \rightarrow Q(x)]) \rightarrow [\forall x P(x) \rightarrow \forall x Q(x)]$

b. $(\forall x P(x)) \rightarrow P(x)$

c. $(\forall x P(x)) \rightarrow P(a)$ where a is a constant

d. $(\exists x P(x)) \rightarrow P(t)$ where t is a variable name not previously used in the proof

e. $P(x) \rightarrow (\exists x P(x))$

f. $P(a) \rightarrow (\exists x P(x))$, where a is a constant and x does not occur in $P(a)$

g. $\sim [\exists x P(x)] \leftrightarrow [\forall x (\sim P(x))]$

Substitution and natural deduction may be performed to obtain other lines in a proof, as in propositional logic. All rules of inference from propositional logic, such as modus ponens, may also be performed.

Generalization, the rule of inference added for predicate logic, has a number of restrictions on when it may be applied. To make these conditions easier to understand, we will *mark* certain free variables in certain wffs. A marked variable will correspond to a free variable that is not completely arbitrary. Then generalization can be stated as follows:

> $\forall x Q$ can be inferrred from the wff Q provided that x is not marked in Q.

There are a number of ways in which a variable is marked in a wff:

- If Q is a hypothesis and x is free in Q, then x is marked in Q.
- In any instance $\exists x P(x) \rightarrow P(t)$ of axiom d, all free variables of $P(t)$ (including t) are marked.
- If x is marked in P, x is free in Q, and Q is derived from P, then x is marked in Q.

Finally, the last line in a proof cannot contain any marked variables. Otherwise, for example, the axiom $(\exists x P(x)) \rightarrow P(t)$ would be a theorem. This is problematic, since it is not valid. For example, consider the interpretation in which the domain of interpretation is the set of integers and $P(x)$ denotes the predicate "x is positive". Then $\exists x P(x)$ is true, but $P(t)$ may be either true or false, so this wff has no truth value in this interpretation.

An Example of A Correct Proof

Consider the following proof of the valid wff

$$\exists y \forall x Q(x, y) \rightarrow \forall x \exists y Q(x, y)$$

.

1.	$\exists y \forall x Q(x,y)$	hypothesis
2.	$\exists y \forall x Q(x,y) \rightarrow \forall x Q(x,t)$	axiom d [t marked]
3.	$\forall x Q(x,t)$	1,2 modus ponens [t marked]
4.	$\forall x Q(x,t) \rightarrow Q(x,t)$	axiom b
5.	$Q(x,t)$	3,4 modus ponens [t marked]
6.	$Q(x,t) \rightarrow \exists y Q(x,y)$	axiom f
7.	$\exists y Q(x,y)$	5,6 modus ponens
8.	$\forall x \exists y Q(x,y)$	7, generalization
9.	$\exists y \forall x Q(x,y) \rightarrow \forall x \exists y Q(x,y)$	1-8, natural deduction

Note that when generalization is used to deduce the wff on line 8 from the wff on line 7, the wff on line 7 did not have x as a marked variable.

Examples of Incorrect Proofs

1. Recall that, in class, we showed that

$$\forall x \exists y Q(x,y) \rightarrow \exists y \forall x Q(x,y)$$

is not valid (by exhibiting an interpretation that makes it false). Consider the following incorrect proof that purports to show this wff is valid.

1.	$\forall x \exists y Q(x,y)$	hypothesis
2.	$\forall x \exists y Q(x,y) \rightarrow \exists y Q(x,y)$	axiom b
3.	$\exists y Q(x,y)$	1,2 modus ponens
4.	$\exists y Q(x,y) \rightarrow Q(x,t)$	axiom d [x,t marked]
5.	$Q(x,t)$	3,4 modus ponens [x,t marked]
6.	$\forall x Q(x,t)$	5, generalization [t marked]
7.	$\forall x Q(x,t) \rightarrow \exists y \forall x Q(x,y)$	axiom f
8.	$\exists y \forall x Q(x,y)$	6,7 modus ponens
9.	$\forall x \exists y Q(x,y) \rightarrow \exists y \forall x Q(x,y)$	1-8, natural deduction

The mistake is that x is marked on line 5, so generalization cannot be applied to obtain line 6.

2. Consider the following incorrect proof of validity of the invalid wff $[(\exists x P(x)) \wedge (\exists y Q(y))] \rightarrow \exists z (P(z) \wedge Q(z))$. The mistake is that z is used again on line 5 instead of a new variable. Thus, without the restriction in axiom d, invalid wffs can be deduced.

1.	$\exists x P(x)$	hypothesis
2.	$\exists y Q(y)$	hypothesis
3.	$(\exists x P(x)) \rightarrow P(z)$	axiom d [z marked]
4.	$P(z)$	1,3 modus ponens
5.	$(\exists y Q(y)) \rightarrow Q(z)$	axiom d [z marked]
6.	$Q(z)$	2,5 modus ponens
7.	$P(z) \wedge Q(z)$	conjunctive amplification
8.	$(P(z) \wedge Q(z)) \rightarrow \forall z (P(z) \wedge Q(z))$	axiom e
9.	$\forall z (P(z) \wedge Q(z))$	7,8 modus ponens
10.	$[(\exists x P(x)) \wedge (\exists y Q(y))] \rightarrow \exists z (P(z) \wedge Q(z))$	1-9 natural deduction

3. Now consider the wff

$$[\forall x (P(x) \wedge \exists y G(y,x))] \rightarrow [\exists y (P(y) \wedge \exists y G(y,y))].$$

This is invalid, as can be seen by considering the interpretation where the domain of interpretation is the set of integers, $P(x)$ is the predicate that is always true, and $G(y,x)$ is the predicate "$y > x$". Since $x+1 > x$ for all integers x, the antecedent $\forall x(P(x) \wedge \exists y G(y,x))$ is true. However, the consequence $\exists y(P(y) \wedge \exists y G(y,y))$ is false, because no integer is greater than itself. Thus, under this interpretation, the wff is false. An incorrect proof of the validity of this wff appears below. The mistake is that the variable y already appears in the wff $\forall y(P(y) \wedge G(y,x))$, so the use of axiom f on line 4 is incorrect. Thus, without the restriction in axiom f, invalid wffs can be deduced.

1.	$\forall x(P(x) \wedge \exists y G(y,x))$	hypothesis
2.	$[\forall x(P(x) \wedge \exists y G(y,x))] \rightarrow (P(a) \wedge \exists y G(y,a))$	axiom c
3.	$P(a) \wedge \exists y G(y,a)$	1,2 modus ponens
4.	$[P(a) \wedge \exists y G(y,a)] \rightarrow \exists y(P(y) \wedge \exists y G(y,y))$	axiom f
5.	$\exists y(P(y) \wedge \exists y G(y,y))$	3,4 modus ponens
6.	$[\forall x(P(x) \wedge \exists y G(y,x))] \rightarrow [\exists y(P(y) \wedge \exists y G(y,y))]$	1-5 natural deduction