Computer Science CSC 238H — Wednesday July 22 1998
St. George Campus — University of Toronto

Quiz #7 Solution

First Name: ______________________

Last Name: ______________________

Student Number: ______________________

1. [1 mark] In lecture last week, we saw that the function T defined by the recurrence $T(n) = a_1T(\lceil n/b\rceil) + a_2T(\lfloor n/b\rfloor) + cn^d$ (where a_1, a_2, b, c and d are constants satisfying $b^d > a_1 + a_2 \geq 1, b \geq 2$ and $c > 0$) has the property $T(n) \in \Theta(n^d)$.
Let S be the function defined by the recurrence

$S(0) = 1$
$S(n) = 2S(\lfloor n/3\rfloor) + n^3 + 3n^2 + 7n$ (for $n > 0$).

Let U be the function defined by the recurrence

$U(0) = 1$
$U(n) = 2U(\lfloor n/3\rfloor) + 11n^3$ (for $n > 0$).

First prove that $S(n) \leq U(n)$ for all $n \in \mathbb{N}$. Then use the fact mentioned above to show that $S(n) \in O(n^3)$

Let $P(n)$ be the statement "$S(n) \leq U(n)$". We shall prove $P(n)$ for all $n \in \mathbb{N}$ by mathematical induction.

Base case: When $n = 0$, $S(0) = 1 \leq U(0)$.

Inductive Step: Let n be a positive integer. Suppose $P(0), P(1), \ldots P(n-1)$ are true. We want to prove $P(n)$.

$$\begin{aligned}
S(n) &= 2S(\lfloor n/3\rfloor) + n^3 + 3n^2 + 7n \\
&\leq 2S(\lfloor n/3\rfloor) + n^3 + 3n^3 + 7n^3 \\
&= 2S(\lfloor n/3\rfloor) + 11n^3 \\
&\leq 2U(\lfloor n/3\rfloor) + 11n^3 \qquad \text{(by induction hypothesis, since } \lfloor n/3\rfloor < n) \\
&= U(n).
\end{aligned}$$

This completes the proof of $P(n)$ for all $n \in \mathbb{N}$. By the general divide-and-conquer recurrence, we know that $U(n) \in \Theta(n^3)$. Hence, there is some constant c such that, for large n, $S(n) \leq U(n) \leq cn^3$. So, $S(n) \in O(n^3)$.

Extra exercise: Prove that $S(n) \in \Omega(n^3)$.