Computer Science CSC 238H — Wednesday June 24 1998
St. George Campus — University of Toronto

Midterm Test Solutions

1. [13 marks] Consider the following symbols with their intended meanings. (The domain is the set of all people.)

Predicate	Meaning
$A(x)$	x is an artist
$G(x)$	x goes mad
$P(x)$	x is a poet
$M(x)$	x is a mathematician
$F(x,y)$	x is a friend of y

Constant	Meaning
d	Leonardo da Vinci

[8] **(a)** Write predicate formulae that represent the following statements.

(i) Poets and artists do not go mad but some mathematicians do.[1]

$$\forall x[(P(x) \vee A(x)) \rightarrow \sim G(x)] \wedge \exists x[M(x) \wedge G(x)]$$

(ii) Leonardo da Vinci is an artist and mathematician, not a poet, although he is the friend of a poet.

$$A(d) \wedge M(d) \wedge \sim P(d) \wedge \exists x(P(x) \wedge F(d,x))$$

[5] **(b)** What is the meaning of the following predicate formula? You should give your answer in *plain* English.

$$\forall x[M(x) \rightarrow (P(x) \vee A(x))] \wedge \forall x \sim [G(x) \wedge M(x)]$$

All mathematicians are poets or artists (or both) but there is no mathematician who goes mad.

[1] "Poets do not go mad; but chess-players do. Mathematicians go mad, and cashiers; but creative artists very seldom. I am not, as will be seen, in any sense attacking logic: I only say that this danger does lie in logic, not in imagination." —G. K. Chesterton (1874–1936) in *Orthodoxy*.

2. [10 marks] Is the following formula valid, unsatisfiable or neither? Justify your answer.

$$\forall x[A(x) \leftrightarrow \exists y B(x,y)] \wedge \forall x[B(x, f(x))]$$

It is neither. Consider the interpretation with domain $\mathbb{N}$ where $f(x)$ represents $x+1$, $A(x)$ represents $x \geq 0$ and $B(x,y)$ represents $y > x$. Under this interpretation, the formula is true, so the formula is not unsatisfiable.

Now consider another interpretation which is the same as the first, except that $A(x)$ now represents $x > 0$. Under this second interpretation, the formula is false since the first conjunct is false for $x = 0$. Thus, the formula is not valid.

3. [14 marks] Beside each function $f(n)$ in the left column, write down the number of the **first** function $g(n)$ in the right column such that $f(n) \in O(g(n))$.

(a) $n^{1.9}\log_2 n$ __8__	1. 1
(b) $1/n$ __1__	2. $\log_2 n$
(c) $(n+7)^3$ __9__	3. $(\log_2 n)^2$
(d) 2^{n+5} __10__	4. $\sqrt{n}$
(e) $\log_4(n^2+42n+3)$ __2__	5. $\sqrt{n}\log_2 n$
(f) $10 \cdot 2^{\log_4 n}$ __4__	6. n
(g) $\sum_{i=1}^{n} \sqrt{i}\log_2 i$ __8__	7. $n\log_2 n$
	8. n^2
	9. n^3
	10. 2^n
	11. n^n

4. [13 marks] Let α be a *propositional* formula. We shall say that α is a *special* formula if each symbol appearing in α is either a propositional variable, (,), $\top$, $\vee$ or $\wedge$.

[11] **(a)** Prove that every special formula is satisfiable. Hint: use mathematical induction on the length of α and try to think of one particular tva that will satisfy any special formula.

Let σ be the truth value assignment that makes all propositional variables true.

Let $P(n)$ be the statement "The tva σ satisfies every special formula of length n," where n is a positive integer. It will be shown that $P(n)$ is true for all positive integers n by strong induction on n.

Base Case ($n = 1$): Let α be a special formula of length 1. Then α is either a single propositional variable or the symbol $\top$. In either case, α is clearly true under the tva σ.

Inductive Step: Let n be a positive integer. For the inductive hypothesis, assume $\bigwedge_{i=1}^{n} P(i)$ is true.

Let α be any special formula of length $n+1$. Since $n+1 > 1$, α has the form $\beta \wedge \gamma$ or $\beta \vee \gamma$ or (β).

Case I (α has the form (β)): Then β is a special formula whose length is less than n. By the inductive hypothesis, σ satisfies β, and therefore σ satisfies α.

Case II (α has the form $\beta \wedge \gamma$ or $\beta \vee \gamma$): In these cases, β and γ have length less than n. By the inductive hypothesis, both β and γ are satisfied by the tva σ. Thus, the formula α is also satisfied by σ (since $\top \wedge \top \equiv \top$ and $\top \vee \top \equiv \top$).

$P(n+1)$ is true (by universal generalization).

By natural deduction, $\bigwedge_{i=1}^{n} P(i) \to P(n+1)$ is true (for any positive n, by universal generalization again).

By the strong principle of mathematical induction, the statement $P(n)$ is true for all positive integers n.

[2] **(b)** Give an example of a propositional formula that is not equivalent to any special formula.

$\bot$ (This formula is unsatisfiable, so it cannot be equivalent to a special formula since all special formulae are satisfiable.)