Convergence Remedies for Option Pricing on Sparse Grids

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Overview

1. Introduction
2. Sparse Grids
3. Smoothing Methods & Quantization Error
4. Coordinate Transformation
5. Numerical Results
6. Conclusions

Introduction

We study the problem of Option Pricing on Sparse Grids.

Option Pricing - gives rise to Black-Scholes equations and variants, continuous ($\mathcal{C}^0$) and discontinuous ($\mathcal{C}^{-1}$) initial conditions, many dimensions.

Sparse Grid framework - mitigates curse of dimensionality for multidimensional problems - stringent requirements on smoothness of initial conditions.

Typically use smoothing operators [Hendricks, 2016] to resolve insufficiently smooth initial conditions.

Our contributions [Wu and Christara, 2023]:

- compare different remedies of convergence for option pricing on sparse grids.
- relate smoothness requirements of some problems to the one-dimensional theory of quantization error [Christara and Leung, 2018].
- apply sparse grid methods to nonlinear PDE from pricing American options.
- provide examples where discontinuities in the payoff cannot be aligned with a coordinate axis.

Black-Scholes PDE

European Options lead to a Black-Scholes PDE in d dimensions:

$$V_\tau = \mathcal{L}V \equiv \frac{1}{2}\sum_{i,j=1}^{d} \rho_{i,j}\sigma_i\sigma_j S_i S_j V_{S_i,S_j} + \sum_{i=1}^{d} rSV_{S_i} - rV, \tag{1}$$

where

- V denotes the option price, S_i the price of the i-th underlying, τ the reverse time counted from expiry T,
- σ_i the volatility of the i-th underlying, $\rho_{i,j}$ the correlation between S_i and S_j, r the risk free interest rate, and
- $V_{S_i,S_j} \equiv \partial^2 V/\partial S_i \partial S_j$.

American Options lead to a penalized (nonlinear) PDE

$$V_\tau = \mathcal{L}V + \rho \max(V^* - V, 0), \tag{2}$$

where V^* denotes the initial condition and ρ the reciprocal of a desired accuracy.

Spatial domain is semi-infinite ($S_i \in [0, \infty)$), but truncated to finite domain for computation.

Initial and Boundary conditions

Payoffs correspond to initial conditions:

- Basket Put:

$$V^* = V(\tau = 0, S_i) = \max(K - \sum w_i S_i, 0) \tag{3}$$

- Min Put:

$$V^* = V(\tau = 0, S_i) = \max(K - \min(S_i), 0) \tag{4}$$

- Binary Basket Call:

$$V^* = V(\tau = 0, S_i) = \begin{cases} 1 & \text{if } w_i S_i - K > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

where w_i are weights, and K is strike price.

Dirichlet boundary conditions, time-discounted payoffs:

$$V(\tau, \partial\Omega) = V(0, \partial\Omega)\exp(-r\tau) \tag{6}$$

Discretizations

We use finite differences in space, and Crank-Nicolson-Rannacher timestepping.

$$(I - \theta \Delta\tau L) u^j = (I + (1-\theta)\Delta\tau L) u^{j-1} \tag{7}$$

Here, L is the matrix that is assembled from the finite differences arising from $\mathcal{L}$, and superscripts denote timesteps.

- Crank-Nicolson: $\theta = 1/2$
- Fully Implicit: $\theta = 1$

American PDE: $\tilde{\tau} = \tau^2/T$ to restore quadratic convergence [Reisinger and Whitley, 2014].

Solve nonlinear equations with penalty (generalized Newton) iteration [Forsyth and Vetzal, 2002]

$$(I - \theta \Delta\tau L + P^j) u^j = (I + (1-\theta)\Delta\tau L) u^{j-1} + P^j V^* \tag{8}$$

where P is a diagonal matrix arising from the discretization of the nonlinear penalty term.

Multi-dimensional grids are constructed by tensor products of one-dimensional grids.

Sparse Grid Combination Method

In d dimensions with n unknowns in each dimension, tensor product grids lead to $\mathcal{O}(n^d)$ unknowns.

Sparse Grid combination method solution [Griebel et al., 1992] is defined by

$$u_q^c = \sum_{p=0}^{d-1} (-1)^p \binom{d-1}{p} \sum_{\sum l_i = q+(d-1)-p} u_{l_1,l_2,\ldots,l_d}. \tag{9}$$

Requires solution of $\mathcal{O}((\log n)^{d-1})$ subproblems and gives the following tradeoffs:

- Order of accuracy reduced from $\mathcal{O}(h^\beta)$ to $\mathcal{O}(h^\beta (\log h)^{d-1})$.
- Number of unknowns reduced from $\mathcal{O}(n^d)$ to $\mathcal{O}(n(\log n)^{d-1})$.

Notation:

- q is the grid refinement level
- $(l_1, l_2, \ldots, l_d)$ denotes the multi-index, defining the grid size of the subproblem.
- Examples:

$$u_q^c = \sum_{l_1+l_2=q+1} u_{l_1,l_2} - \sum_{l_1+l_2=q} u_{l_1,l_2} \text{ (2D)}, \quad u_q^c = \sum_{l_1+l_2+l_3=q+2} u_{l_1,l_2,l_3} - 2 \sum_{l_1+l_2+l_3=q+1} u_{l_1,l_2,l_3} + \sum_{l_1+l_2+l_3=q} u_{l_1,l_2,l_3} \text{ (3D)}$$

Advantages of the sparse grid method

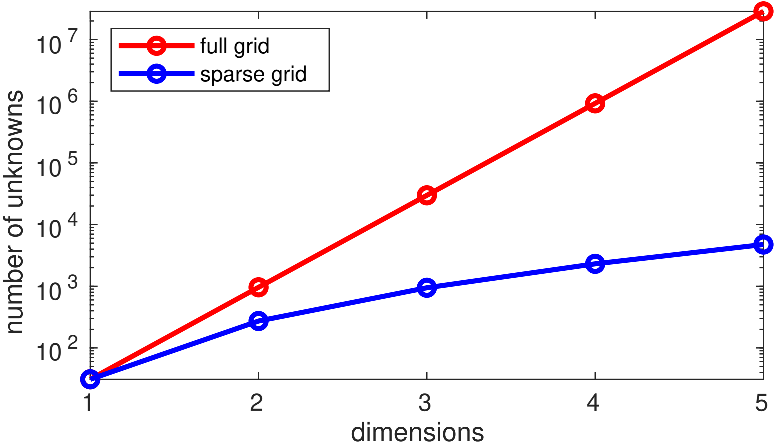


Figure 1: Number of unknowns for grid level $q = 5$, for sparse and full grids of dimensions one through five. Note that the error, governed by the discretization parameter $h = 2^{-q}$, would be kept relatively consistent for some hypothetical problem on this domain.

Proofs of order of accuracy assume that the initial condition is sufficiently smooth.

The sparse grid combination method is particularly sensitive to smoothness.

In finance, we typically have nonsmooth (including discontinuous) initial conditions - although they give rise to smooth solutions. These initial conditions can cause unstable or reduced order of convergence.

Additionally, these initial conditions can introduce "quantization error" - error arising from the placement of the point of non-smoothness on the discrete grid.

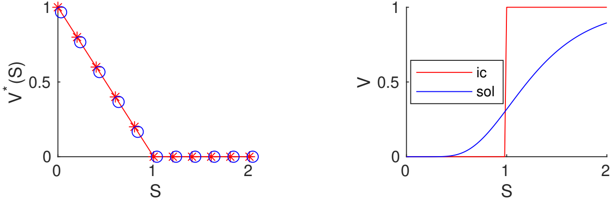


Figure 2: Examples of common payoffs, with discretizations (left) and the PDE solution at a time $t > 0$ (right).

Smoothing in Fourier Space

Smoothing is a popular approach to obtain stable convergence and restore optimal order of convergence in the presence of insufficiently smooth initial conditions.

We look at smoothing operators introduced in [Kreiss et al., 1970]. In particular, we are interested in the first two smoothing operators. In the spatial domain, the first-order operator is given by

$$(\Phi_1 * v)(x) = \frac{1}{h}\int_{-h/2}^{h/2} v(x-y)\, dy \tag{10}$$

and the second-order operator is given by

$$(\Phi_2 * v)(x) = \frac{1}{h}\int_{-h}^{h} (1-\frac{|y|}{h}) v(x-y)\, dy \tag{11}$$

In multiple dimensions, the smoothings operators result in nested integrals:

$$(\Phi_2 * (\Phi_2 * v))(x,y) = \frac{1}{h_x h_y}\int_{-h_y}^{h_y}\int_{-h_x}^{h_x} (1-\frac{|w|}{h_x})(1-\frac{|z|}{h_y}) v(x-w, y-z)\, dw\ dz \tag{12}$$

More on Quantization Error

In one dimension, with one point of nonsmoothness, let $\alpha \in (0, 1]$ denote the relative offset of the point of nonsmoothness with the grid. From [Christara and Leung, 2018], the quantization error arising from a continuous ($\mathcal{C}^0$) but nonsmooth initial condition,

- with no smoothing is $\mathcal{O}(h^2)$, with coefficient depending on α, and
- with Φ_1 smoothing is $\mathcal{O}(h^2)$, with coefficient independent of α.

The quantization error arising from a discontinuous ($\mathcal{C}^{-1}$) initial condition,

- with no smoothing is $\mathcal{O}(h)$, with coefficient $\alpha - 1/2$,
- with Φ_1 smoothing is $\mathcal{O}(h^2)$, with coefficient depending on α, and
- with Φ_2 smoothing is $\mathcal{O}(h^2)$, with coefficient independent of α.

Note that α can change with grid refinement, except when point of nonsmoothness is on the grid.

In multiple dimensions, with lines or hypersurfaces of nonsmoothness, defining α is difficult, and maintaining α across refinements is difficult if not impossible.

Coordinate Transformation for Basket Options removes dependence of the payoff function on all variables except one. In original coordinates, the Basket Put payoff function is

$$V(0, S_i) = \max(K - \sum w_i S_i, 0). \tag{13}$$

In new coordinates, the Basket Put payoff function becomes

$$V(0, x_i) = \max(K - x_1, 0). \tag{14}$$

Coefficients of the transformed PDE given in [Leentvaar and Oosterlee, 2008].

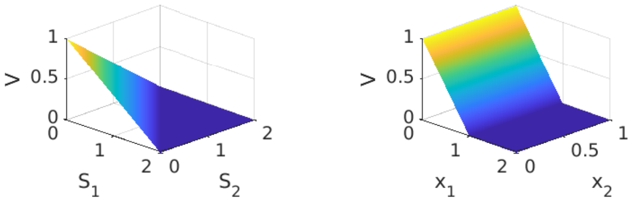


Figure 3: The Basket Put payoff, in original and transformed coordinate systems.

Combining it all together

The transformed payoff function $V(0, x_i) = \max(K - x_1, 0)$ allows us to apply the one-dimensional theory of quantization error to certain problems in multiple dimensions.

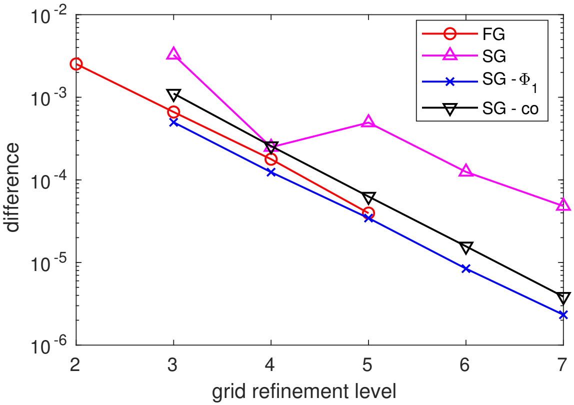
Since the initial condition only depends on x_1, there would only be quantization error in that dimension.

This allows us to apply one less order of smoothing compared to not using a grid transformation and still achieve stable convergence with the sparse grid combination method.

Type of IC	Remedy for stable convergence
Discontinuous $\mathcal{C}^{-1}$	Φ_2 smoothing, OR Φ_1 smoothing with grid transformation and maintaining α.
Continuous $\mathcal{C}^{0}$	Φ_1 smoothing, OR no smoothing with grid transformation and maintaining α.

Table 1: Summary of remedies for stable convergence for common payoffs

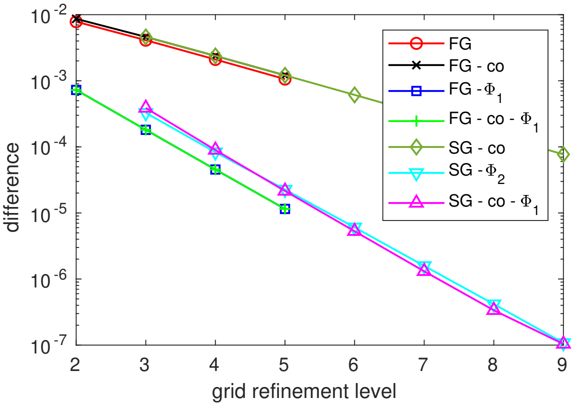
American Basket Put Option



q	FG	SG*	SG-Φ_1	SG-co
3	1.92	—	—	—
4	1.90	3.71	2.01	2.12
5	2.17	-0.99	1.85	2.03
6	—	1.98	2.03	2.01
7	—	1.38	1.86	2.02

Figure 4: 2D American Basket Put Option. Parameters: $\sigma_i = 0.4$, $\rho = 0.2$, $r = 0.1$, $T = 1$, $K = 10$.

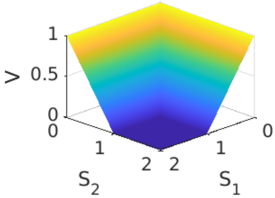
Digital Basket Call Option

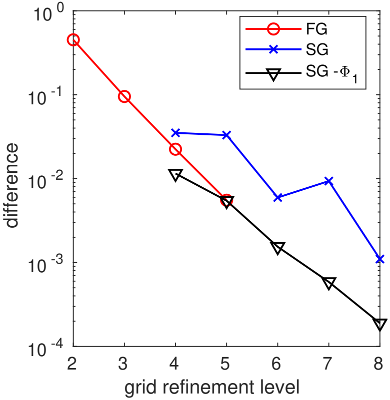


q	FG none	FG Φ_1	SG co	SG co-Φ_1	SG Φ_2
3	0.93	2.00	—	—	—
4	0.97	2.00	0.95	2.11	1.97
5	0.98	1.98	0.98	2.06	1.88
6	—	—	0.99	2.03	1.90
7	—	—	0.99	2.01	1.93
8	—	—	1.00	1.96	1.92
9	—	—	1.00	1.70	1.94

Figure 5: 2D European Binary Basket Call Option. Parameters: $\sigma_i = 0.4$, $\rho = 0.2$, $r = 0.1$, $T = 1$, $K = 10$.

Min-Put Option





q	FG	SG*	SG-Φ_1
3	2.24	—	—
4	2.09	—	—
5	2.02	0.09	1.08
6	—	2.47	1.82
7	—	-0.65	1.40
8	—	3.09	1.64

Figure 6: 3D Min-Put Option. Parameters: $\sigma_i = 0.4$, $\rho = 0.2$, $r = 0.1$, $T = 1$, $K = 10$.

Conclusions

Nonsmooth/discontinuous initial conditions cause trouble in the sparse grid combination method.

Smoothing techniques and coordinate transformation are known remedies to restore the order of convergence.

We connect these techniques with the one-dimensional grid-alignment theory in [Christara and Leung, 2018], which allows us to obtain the necessary order of smoothings for typical initial conditions.

We show that the coordinate transformation allows us to use one less order of smoothing, compared to solving the PDE in the original coordinates.

Our numerical experiments demonstrate stable $\mathcal{O}(h^2)$ convergence for nonlinear problems (American options), discontinuous initial conditions (binary options), and initial conditions where the coordinate transformation cannot be applied (min-put options).

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Elliptic PDE

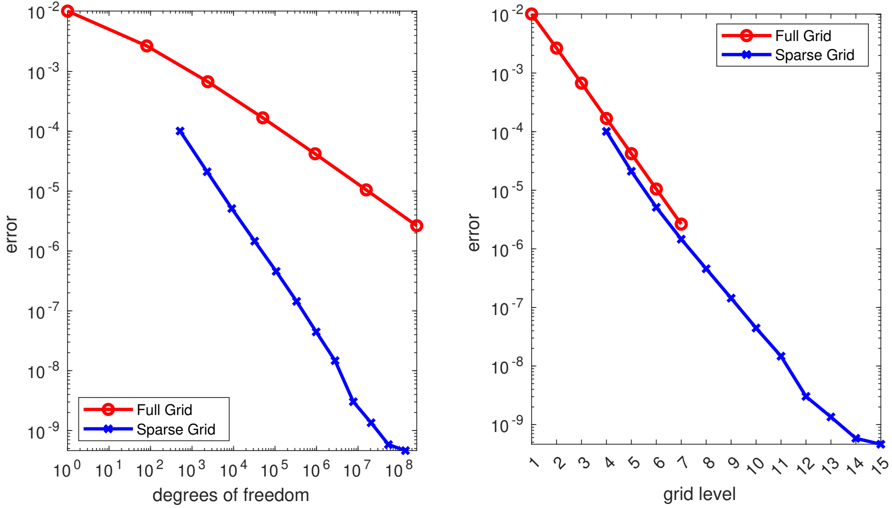

Figure 7: Convergence of the numerical solution of an elliptic PDE using the full grid method (red) and the sparse grid method (blue). As can be seen, the sparse grid method can use fewer unknown parameters to