Question #1: Network flow with maximum and *minimum flow* constraints on edges

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Given a flow network (i.e. directed graph) $G = (V, E)$, and two functions: $lower : E \to \mathbb{R}^+$ and $upper : E \to R^+$ find a flow f such that the flow of each edge e is between the lower(e) and upper(e). That is find a function $f : E \to R$ such that:

1. The flow into and out of each node are the same,

2. for any $e \in E(G)$ we have $lower(e) <= f(e) <= upper(e)$.

Notice that this is just the max flow problem except that we don't have any source or sinks and we have a lower bound as well as the upper bound (capacity) on each edge.

Solution: Add a source s and sink t to the graph and for any edge $e : u \to v$ add the following three edges to the graph:

- $e_1 : u \to v$ with capacity $upper(e) - lower(e)$,
- $e_2 : s \to v$ with capacity $lower(e)$,
- $e_3 : u \to t$ with capacity $lower(e)$.

Find the max flow in this flow network. If the max flow is $\sum_e lower(e)$ then the original network had a feasible flow.

Theorem 1. *If f is a flow of the new network of value $\sum_e lower(e)$ then f' defined as (for every edge $e \in E$) $f'(e) = f(e_1) + lower(e)$ is a flow satisfying the conditions of the original problem.*
Conversely, if the original network had a feasible flow f'' then the flow f''' (defined as follows) is a valid flow in the new network with value $\sum_e lower(e)$. For any edge $e \in E$ of the original network f''' is defined as, $f'''(e_1) = f''(e) - lower(e)$, $f'''(e_2) = f'''(e_3) = lower(e)$.

Proof. Presented in the tutorial. □