Duration: **75 minutes**
Aids Allowed: **NONE** (in particular, no calculator)

Student Number: |_|_|_|_|_|_|_|_|_|

Last (Family) Name(s): ______________________________

First (Given) Name(s): ______________________________

---

*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below *carefully*.

This term test consists of 2 questions on 8 pages (including this one), printed on both sides of the paper. *When you receive the signal to start, please make sure that your copy of the test is complete, fill in the identification section above, write your student number where indicated at the bottom of every odd-numbered page (except page 1), and write your name on the back of the last page.* Answer each question directly on the test paper, in the space provided. If you use the next page or the reverse side for your answer *state this clearly.*

The topic of this test is network flows and linear programming. All questions can be solved by either "reducing" (that is recasting) the problem to the max-flow (or min-cut) problem or to solving a linear program. For each problem you have to *prove* that your approach is correct, i.e. prove that the optimal answer to the max-flow, min-cut, or linear program yields the optimal answer to the problem you are asked to solve in each question.

In your answers, you may use without proof any result or theorem covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s)/theorem(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do.

If you are unable to answer a question, you will get 20% of the marks for that question if you write "I don't know" and nothing else — you will get 10% of the mark if your answer is completely blank. You will *not* get these marks if your answer contains contradictory statements (such as "I don't know" followed or preceded by parts of a solution that have not been crossed off).

Marking Guide

1: ______/15

2: ______/15

TOTAL: ______/30

Question 1. Another hiring problem [15 MARKS]

You are working in the human resources[1] of a company. You have n applicants who have applied for jobs in the company. There are m positions available at the company for which you are hiring applicants (at most one applicant for each position.) Each applicant is only compatible with some of the positions. The goal is to hire some of the applicants but there is one issue. The company has divided the applicants into k groups; we call these group 1, group 2, ..., group k. For legal reasons we need to hire at least d applicants from each group. (Each applicant is only in one group.) Your goal is to check if this is even possible!

To be more precise, you are given n, m, k and d, integers $g_1, \ldots, g_n$ where each $1 \le g_i \le k$, and sets $C_1, C_2, \ldots, C_n$ where each C_i is a subset of $\{1, \ldots, m\}$. You should think of C_i as the set of positions applicant i is compatible with and g_i as the group applicant i belongs to. The goal is to find an assignment of candidates to positions such that:

1. No applicant is assigned to more than one position. Some applicants can be assigned to no position in which case we will not hire them.
2. At most one candidate is assigned to each position.
3. Applicant i can only be assigned to positions which are in C_i.
4. For each group l ($1 \le l \le k$) we have to hire at least d candidates from group l, i.e. the number of candidates i such that $g_i = l$ and we hire candidate i for some position has to be at least d.

Use network flows to decide if this is possible and if it is output one such assignment (it does not matter which one.) Prove that your algorithm is correct.

Hint: Show how to construct a flow network from the inputs and how given the maximum flow of your network you can tell if this is possible or not.

Examples:

- If $n = 4, m = 3, k = 3, d = 1$, $C_1 = \{1, 2\}, C_2 = \{1, 2\}, C_3 = \{1\}, C_4 = \{1, 3\}$, $g_1 = 1, g_2 = 2, g_3 = 3, g_4 = 1$. Then the assignment is possible. One valid assignment is to hire applicant 2 for position 2, applicant 3 for position 1 and applicant 4 for position 3. That way we have hired one applicant from each group as required.

- If $n = 8, m = 6, k = 2, d = 3$, $C_1 = \{1, 2\}, C_2 = \{1, 2\}, C_3 = \{1\}, C_4 = \{2\}, C_5 = \{1, 2, 3, 4\}, C_6 = \{2, 3, 4, 5\}, C_7 = \{1, 4, 6\}, C_8 = \{4, 5, 6\}$, $g_1 = 1, g_2 = 1, g_3 = 1, g_4 = 1, g_5 = 2, g_6 = 2, g_7 = 2, g_8 = 2$. Then no assignment is possible. This is because we need to hire at least 3 people of group 1, i.e. three of the applicants 1, 2, 3, 4 have to be hired. But these applicants are only compatible with positions 1 and 2 so we can not hope to hire 3 of them.

- If $n = 4, m = 3, k = 2, d = 1$, $C_1 = \{1\}, C_2 = \{2\}, C_3 = \{3\}, C_4 = \{3\}$, $g_1 = 1, g_2 = 1, g_3 = 1, g_4 = 2$. Then the assignment is possible. One valid assignment is to hire applicant 2 for position 2 and applicant 4 for position 3. That way we have hired one applicant from each group as required. The following is *not* a valid assignment: Application 1 for position 1, applicant 2 for position 2 and applicant 3 for position 3.

- If $n = 5, m = 4, k = 2, d = 2$, $C_1 = \{1, 2\}, C_2 = \{2, 3\}, C_3 = \{3\}, C_4 = \{4, 1\}, C_5 = \{2\}$, $g_1 = 2, g_2 = 1, g_3 = 2, g_4 = 1, g_5 = 1$. Then the assignment is possible. We can hire applicant 1 for position 1, applicant 3 for position 3, applicant 4 for position 4 and applicant 5 for position 2.

[1] Human resources are the people in charge of hiring people.

- If $n = 4, m = 3, k = 2, d = 1$, $C_1 = \{1, 2\}, C_2 = \{1, 2\}, C_3 = \{1\}, C_4 = \{1, 3\}$, $g_1 = 1, g_2 = 2, g_3 = 2, g_4 = 1$. Then the assignment is possible. One valid assignment is to hire applicant 2 for position 2, applicant 3 for position 1 and applicant 4 for position 3. That way we have hired one applicant from each group as required.

[There's more space on the next page...]

Question 1. (CONTINUED)

[Use this page for the rest of your solution.]

Question 2. And numbers on a path yet again! [15 MARKS]

Formulate the following problem as a linear program. *Notice that the definition of deviation here is different from both Assignment 1 and Assignment 3!*

Given n real numbers $a_1, \ldots, a_n \in \mathbb{R}$. Find n real numbers $b_1, \ldots, b_n \in \mathbb{R}$ with the requirement that the b_i's form a non-decreasing sequence, i.e. $b_1 \leq b_2 \leq \cdots \leq b_n$ and they have the minimum *deviation*. The deviation of b_i's is the *sum* of the distances between an a_i and its corresponding b_i, i.e.,

$$\text{deviation} = |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n|.$$

Examples:

- If $n = 2$, $a_1 = 4$, $a_2 = 2$ then the minimum deviation is 2. An optimal answer is $b_1 = 2.2$, $b_2 = 2.2$. Another optimal answer is $b_1 = 3$, $b_2 = 3$.
- If $n = 4$, $a_1 = 5$, $a_2 = 6$, $a_3 = 7$, $a_4 = 6$ then the minimum deviation is 1. An optimal answer is $b_1 = 5$, $b_2 = 6$, $b_3 = 6.5$, $b_4 = 6.5$.
- If $n = 4$, $a_1 = 10$, $a_2 = 3$, $a_3 = 4$, $a_4 = 5$ then the minimum deviation is 7. An optimal answer is $b_1 = 3.5$, $b_2 = 3.5$, $b_3 = 4$, $b_4 = 5$.

[There's more space on the next page...]

Question 2. (CONTINUED)

[Use this page for the rest of your solution.]

On this page, please write nothing except your name.

Last (Family) Name(s): ____

First (Given) Name(s): ____