CS 137 - Graph Theory - Lecture 1
February 11, 2012

(further reading Rosen K. H.: *Discrete Mathematics and its Applications*, 5th ed., chapters 8.1, 8.2, 8.3)

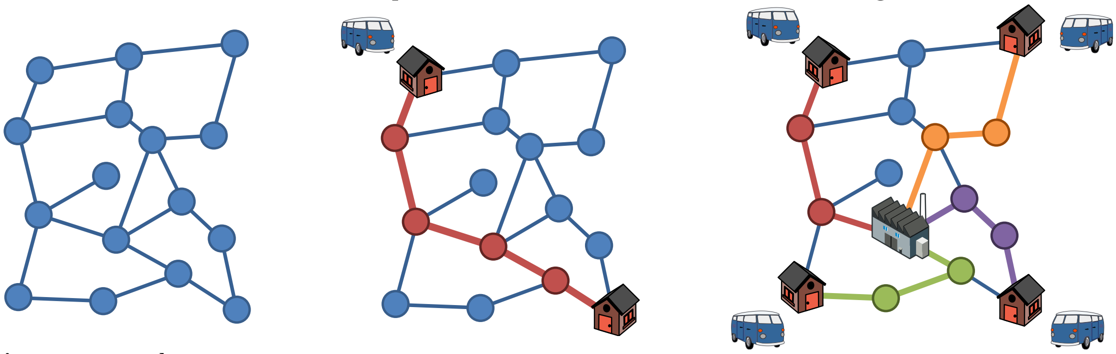
1.1. Summary

- Intuition (What?/Why?)
- Basic terminology/notation
- Basic Counting
- Graph isomorphism

1.2. Intuition

A *graph* is a collection of points and lines between the points.

For instance, think of a road network - points are cities and lines are roads connecting the cities.



Questions we can ask:

- is there a road connecting two cities?
- how many cities must we go through when we want to travel from x to y?
- can we continuously travel through all cities without going through the same city twice?

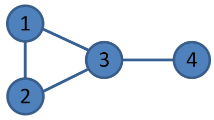
Note: the answers to these questions do not depend on the shape of the roads or positions of the cities – all we need to know is which cities are connected by roads.

A *graph* is a mathematical abstraction/model of connections/relations.
- simple model (yet powerful)
- practical applications (Computing, Management Science, Engineering, and much more)
- fun (solving problems by doodling ;-))

1.3. Definition

A graph G is a pair (V, E) where
- V = set of elements (called *vertices*, singl. vertex, or *nodes*) e.g. think $\{1, \ldots, n\}$ or a set of points
- E = set of 2-element subsets of V (called *edges*)

Example: $G = (V, E)$
$V = \{1, 2, 3, 4\}$
$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}\}$



(Intuitively) A *drawing* of a graph G consists of
- points corresponding to vertices V
- lines/curves between points corresponding to edges in E

Notes:
- a drawing of G is not G itself
- $V(G)$ vertices of G
- $E(G)$ edges of G
- we can write $uv \in E(G)$ instead of $\{u, v\} \in E(G)$

1.4. Basic Terminology

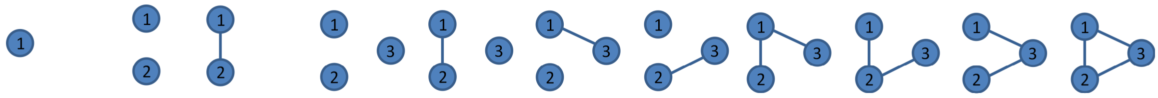
Let G be a graph. If $e = \{u, v\}$ is a pair (edge) in $E(G)$, then

- u and v are *adjacent*
- u and v are *neighbours*
- u (v) is an *endpoint* of e
- u (v) is *incident* to e
- $N(v)$ = the set of all neighbours of v (the *neighbourhood*)
- $deg(v)$ = *degree of* v is the number of neighbours of v, i.e. the size of $N(v)$

- *empty graph* = has no edges
- *complete graph* = has all possible edges (relative to its vertex set)

1.5. Isomorphism

Question: How many different graphs with the vertex set $\{1, ..., n\}$?



	1	2	3	4	...	n
# of graphs on $\{1, \ldots, n\}$	1	2	8	64	...	$2^{\binom{n}{2}}$

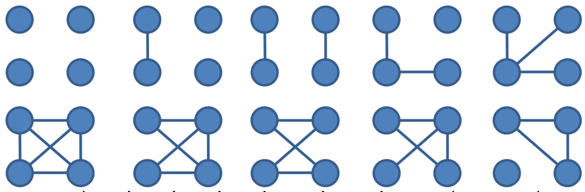
Note: the answer is the same as long as the vertex set has n elements

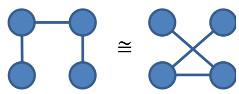
Two graphs G_1 and G_2 are *isomorphic* if there exists a bijective mapping $f : V(G_1) \to V(G_2)$ such that

$$\{u, v\} \in E(G_1) \text{ if and only if } \{f(u), f(v)\} \in E(G_2)$$

We write $G_1 \simeq G_2$. The mapping f is called an *isomorphism* of the graphs G_1 and G_2.

Question: How many different non-isomorphic graphs with n vertices ?





	1	2	3	4	5	6	7	8	9	10	11	12
# of non-iso graphs on $\{1, \ldots, n\}$	1	1	2	4	11	34	156	1044	12346	274668	12005168	1018997864

see http://oeis.org/A000088 (The On-Line Encyclopedia of Integer Sequences)

Note: often the properties we discuss are the same for isomorphic graphs – we say that the graphs we consider are *unlabelled* (i.e. when drawing the graphs we do not need to specify the labels of points which is often convenient)

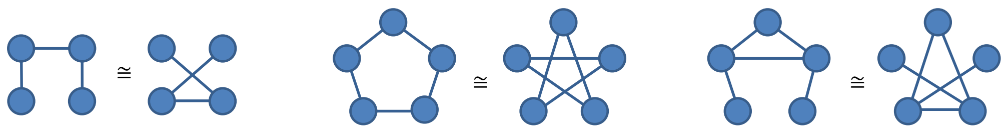
The *complement* of a graph G is the graph $\overline{G}$ where

$$V(\overline{G}) = V(G) \qquad \text{and} \qquad E(\overline{G}) = \Big\{\{u, v\} \;\Big|\; u \neq v \text{ and } \{u, v\} \notin E(G)\Big\}$$

Notes:

- the complement of $\overline{G}$ is G itself, i.e. $\overline{(\overline{G})} = G$
- the complement of an empty graph is a complete graph

A graph G is *self-complementary* if G is isomorphic to $\overline{G}$.



Question: How many self-complementary graphs on n vertices ?

...for $n = 6$? if G is a self-complementary graph on n vertices, then G and $\overline{G}$ are isomorphic and thus have the same number of edges. Note that $|E(G)| + |E(\overline{G})| = \binom{n}{2}$ by definition. Therefore $2|E(G)| = \binom{n}{2}$ which is odd for $n = 6$, for $n = 7$, and generally whenever $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 17	18	19
#	1	0	0	1	2	0	0	10	36	0	0	720	5600	0	0	703760	11220000	0

see http://oeis.org/A000171