Bichain graphs: geometric model and universal graphs

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Abstract

Bichain graphs form a bipartite analog of split permutation graphs, also known as split graphs of Dilworth number 2. Unlike graphs of Dilworth number 1 that enjoy many nice properties, split permutation graphs are substantially more complex. To better understand the global structure of split permutation graphs, in the present paper we study their bipartite analog. We show that bichain graphs admit a simple geometric representation and have a universal element of quadratic order, i.e. an n-universal bichain graph with n^2 vertices. The latter result improves a recent cubic construction of universal split permutation graphs.

Keywords: intersection graph, universal graph, split permutation graph

1. Introduction

In this paper, we introduce the class of bichain graphs, which is a generalization of chain graphs. The latter class has appeared in the literature under various names, such as difference graphs [8] or bisplit graphs [6], and has been extensively studied by many researchers, because graphs in this class enjoy many nice properties. In particular, chain graphs admit a simple characterization in terms of forbidden induced subgraphs: these are precisely $2K_2$-free bipartite graphs. Many algorithmic problems that are generally NP-hard admit polynomial-time solutions when restricted to the class of chain graphs (see e.g. [9]), which is partially due to the fact that chain graphs have bounded clique-width [4]. Also, graphs in this class are well-quasi-ordered by the induced subgraph relation [16] and they have a small universal element [12], i.e. a graph with $2n$ vertices containing all n-vertex graphs from the class as induced subgraphs. The class of chain graphs also plays a critical role in the study of the speed of hereditary graph properties. In particular, this is one the nine minimal hereditary classes whose speed (i.e. asymptotic growth) is factorial [2].

Another minimal class with factorial speed is the class of threshold graphs. This has also received considerable attention in the literature (see e.g. [13]) and it has many attractive properties including bounded clique-width, well-quasi-orderability by induced subgraphs and small universal graphs [7]. The similarity between these two classes is no surprise, as they are closely related. To reveal this relationship, observe that every threshold graph is a split graph, i.e. its vertices can be partitioned into a clique and an independent set. If we remove the edges from the clique part of a threshold graph, then what is left is a chain graph. Conversely, inserting all the edges in one part of the bipartition of a chain graph yields a threshold graph. So, in a sense, the class of chain graphs is the bipartite analog of threshold graphs.

Both classes, chain graphs and threshold graphs, have many important generalizations. One of them is known as bipartite permutation graphs (generalizes chains graphs) and one as split permutation graphs (generalizes threshold graphs). The class of split permutation graphs contains all threshold graphs, because threshold graphs are graphs of

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Dilworth number 1, while split permutation graphs are split graphs of Dilworth number at most 2 [3]. This small jump from 1 to 2 changes the situation dramatically. In particular, the clique-width of split permutation graphs is unbounded and graphs in this class are not well-quasi-ordered by induced subgraphs, which was recently shown in [10]. Moreover, in the same paper it was conjectured that split permutation graphs constitute a *minimal* hereditary class of unbounded clique-width. This property is quite rare and to date only two classes possessing this property are known: bipartite permutation graphs and unit interval graphs [11]. In both cases, the proof of minimality exploits the idea of universal graphs, because universal graphs describe a typical structure of graphs in the class. It is known that not every class admits a universal element. For the two minimal classes of unbounded clique-width, bipartite permutation graphs and unit interval graphs, an n-universal graph exists and for both of them it has n^2 vertices. However, for split permutation graphs even the existence of a universal element was an open question until recently. In [1], this question was answered affirmatively by constructing a split permutation graph with $4n^3$ vertices containing all split permutation graphs with n vertices as induced subgraphs. However, this construction is complicated and tells us very little about the typical structure of split permutation graphs.

To better understand the global structure of split permutation graphs, in the present paper we reduce the problem to their bipartite analog. Let us repeat that a split graph is a graph partitionable into a clique and an independent set. The edges inside the clique part of a split graph are irrelevant for the purposes of our study, because all complications occur between the two parts. By removing the edges from the clique part of a split permutation graph we obtain a bipartite graph which we call a bichain graph.

We formally introduce the class of bichain graphs and derive some useful properties of these graphs in Section 3. Then in Section 4 we propose a geometric model to represent bichain graphs. Finally, in Section 5 we construct an n-universal bichain graph with n^2 vertices. By inserting the edges in one of its parts, we obtain an n-universal split graph with n^2 vertices, thus improving the construction proposed in [1] from cubic to quadratic. Section 6 concludes the paper by discussing some open problems related to bichain and split permutation graphs. All preliminary information related to the topic of the paper can be found in Section 2.

2. Preliminaries

A graph $G = (V, E)$ has vertex set $V(G)$ and edge set $E(G)$. We write uv for an edge $\{u, v\} \in E(G)$. We denote by $N(u)$ the set of neighbours of u in G, and write $N[u]$ for the set $N(u) \cup \{u\}$. For a set $S \subseteq V(G)$, we write $N[S]$ to denote the set $\bigcup_{u \in S} N[u]$, and write $N(S)$ for the set $N[S] \setminus S$.

For disjoint sets $X \subseteq V(G)$ and $Y \subseteq V(G)$, we say that X is *complete* to Y if $xy \in E(G)$ for all $x \in X$ and all $y \in Y$. We say that X is *anticomplete* to Y if $xy \notin E(G)$ for all $x \in X$ and all $y \in Y$. We denote by $G[X]$ the subgraph of G induced by X, and we write $G - X$ for the graph $G[V(G) \setminus X]$.

By $2K_2$ we denote the disjoint union of two edges.

In a graph, an independent set is a subset of vertices no two of which are adjacent. A graph G is said to be *bipartite* if $V(G)$ can be partitioned into two independent sets A, B. We say that (A, B) is a *bipartition* of G and write $G = (A, B, E)$ to denote a bipartite graph with a bipartition (A, B) and edge set E.

A *permutation graph* is the intersection graph of line segments whose endpoints lie on two parallel lines.

We say that a set of vertices in a graph forms a *chain* if their neighbourhoods form a chain with respect to set inclusion, i.e. if for any two vertices in the set, the neighbourhood of one of them includes (not necessarily properly) the neighbourhood of the other.

2.1. Chain graphs and alternating sequences

A bipartite graph such that each part in its bipartition forms a chain is called a *chain graph*. It is well-known (and not difficult to see) that a bipartite graph is a chain graph if and only if it is $2K_2$-free, i.e. it does not contain $2K_2$ as an induced subgraph. From this it follows, in particular, that a chain in one part of a bipartite graph implies a chain in the other part, i.e. a bipartite graph is a chain graph if and only if at least one of its parts forms a chain. Below we provide an alternative characterization of chain graphs. To this end, we introduce the following definition.

A sequence of vertices $u_1, u_2, \ldots, u_t$ in a bipartite graph $G = (A, B, E)$ is called an *alternating sequence* if $u_i \in A$ and $u_i u_{i+1} \in E$ for all odd i, and $u_i \in B$ and $u_i u_{i+1} \notin E$ for all even i. We say that an alternating sequence $u_1, \ldots, u_t$ consists of edges $u_i u_{i+1}$ for odd i, and non-edges $u_{i+1} u_i$ for even i.

An alternating sequence $u_1, \ldots, u_t, u_{t+1}$ is *closed* if $u_1 = u_{t+1}$. This implies that t is even (since $u_1 \in A$ and so $u_{t+1} = u_1 \in A$), and thus $t \geq 4$ (since $u_1 u_2$ is an edge while $u_{t+1} u_t$ is not).

For instance, for the graph in Figure 1, the sequence a_1, b_2, a_4, b_3, a_1 is a closed alternating sequence, where $a_1 b_2, a_4 b_3$ are the edges of this sequence, and $a_4 b_2, a_1 b_3$ are the non-edges of this sequence.

Lemma 1. *A bipartite graph G is a chain graph if and only if G contains no closed alternating sequence.*

Proof. ($\Leftarrow$) We prove the contrapositive. Let G is a bipartite graph with bipartition (A, B). Suppose that G is not a chain graph. Then G contains an induced $2K_2$, i.e. G contains vertices $u, z \in A$ and $v, w \in B$ such that $uv, zw \in E(G)$ and $uw, vz \notin E(G)$. We see that u, v, z, w, u is a closed alternating sequence of G.

($\Rightarrow$) For contradiction, assume that G is a chain graph but contains a closed alternating sequence. Let $u_1, \ldots, u_t, u_{t+1}$ be a shortest such sequence. Recall that $u_1 = u_{t+1}$ and $u_i \in A$, $u_i u_{i+1} \in E(G)$ if i is odd, while $u_i \in B$, $u_i u_{i+1} \notin E(G)$ if i is even. Moreover, t is even and $t \geq 4$. If $u_1 u_4 \notin E(G)$, then the vertices u_1, u_2, u_3, u_4 are all distinct and induce a $2K_2$ in G. Indeed, we have $u_1, u_3 \in A$, $u_2, u_4 \in B$, and $u_1 u_2, u_3 u_4 \in E(G)$ while $u_2 u_3, u_1 u_4 \notin E(G)$. This is impossible, since G is a chain graph. So we conclude $u_1 u_4 \in E(G)$. This implies that $t > 4$, since $u_1 u_t = u_{t+1} u_t \notin E(G)$. Thus $t \geq 6$, since t is even. But now $u_1, u_4, u_5, \ldots, u_{t+1}$ is also a closed alternating sequence of G, contradicting the minimality of $u_1, \ldots, u_{t+1}$. $\square$

3. Bichain graphs

In this section, we introduce the main notion of the paper, bichain graphs, and derive a number of properties of these graphs.

We say that a bipartite graph is a *bichain graph* if each part in its bipartition can be split into at most two chains. In other words, $G = (A, B, E)$ is a bichain graph if A can be partitioned into A_1, A_2 and B can be partitioned into B_1, B_2 such that for each $i \in \{1, 2\}$, both $G - A_i$ and $G - B_i$ are chain graphs. We say that (A_1, A_2, B_1, B_2) is a *bichain partition* of G. Figure 1 represents an example of a bichain graph with the bichain partition $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3, a_4\}$, $B_1 = \{b_1, b_2\}$, $B_2 = \{b_3, b_4\}$.

From the definition it follows that the class of bichain graphs forms an extension of chain graphs. Moreover, Figure 1 shows that this extension is proper.

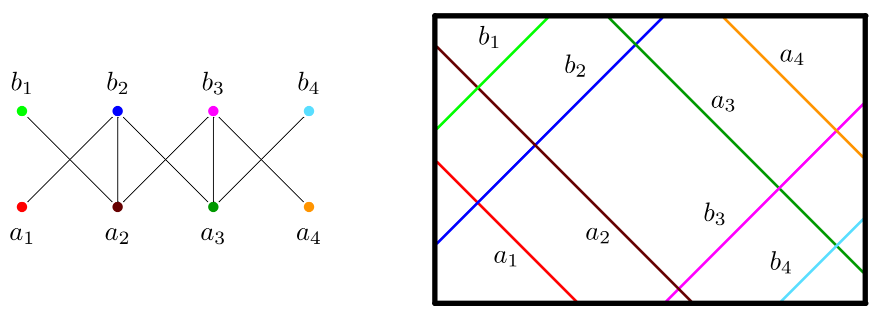


Figure 1: Example of a bichain graph and its diagonal representation.

A bichain partition (A_1, A_2, B_1, B_2) of G is *special* if

($\star$) for all $u \in A_1$, $v \in A_2$ and all $x \in N(u) \setminus N(v)$ and $y \in N(v) \setminus N(u)$, we have $x \in B_1$ and $y \in B_2$.

(Note that this simply excludes the possibility that $x \in B_2$ and $y \in B_1$.)

Lemma 2. *If G is a bichain graph, then there exists a special bichain partition of G.*

Proof. Suppose that $G = (A, B, E)$ is a bichain graph, and let (A_1, A_2, B_1, B_2) be an arbitrary bichain partition of G.

For vertices $u \in A_1$, $v \in A_2$, and $x \in N(u) \setminus N(v)$, $y \in N(v) \setminus N(u)$, we say (u, v, x, y) is a *parallel* $2K_2$ if $x \in B_1$ and $y \in B_2$, and we say that (u, v, x, y) is a *crossing* $2K_2$ if $x \in B_2$ and $y \in B_1$.

For each $i \in \{1, 2\}$, let C_i denote the set of all vertices of B_i that belong to a crossing $2K_2$. Let D_i denote the set of all vertices of B_i that belong to a parallel $2K_2$. We have the following property.

Claim 2.1: $C_i \cap D_i = \emptyset$ *for each* $i \in \{1, 2\}$.

By symmetry, it suffices to prove the claim for $i = 1$. Suppose that there exists $x \in C_1 \cap D_1$. This means that there are vertices $u, u' \in A_1$, $v, v' \in A_2$, and $y, x' \in B_2$ (possibly $u = u'$ or $v = v'$ or $y = x'$) such that (u, v, x, y) is a parallel $2K_2$ while (u', v', x', x) is a crossing $2K_2$. Namely, we have $x \in N(u) \setminus N(v)$ and $x \in N(v') \setminus N(u')$ while $y \in N(v) \setminus N(u)$ and $x' \in N(u') \setminus N(v')$. Observe that $u \neq u'$ and $v \neq v'$, since $x \in N(u) \setminus N(u')$ and $x \in N(v') \setminus N(v)$. If $vx' \in E(G)$, then the vertices v, v', x, x' induce a $2K_2$ in $G - A_1$. To see this, note that $v \neq v'$ and $v'x \in E(G)$, while $vx, v'x' \notin E(G)$. This is impossible, so we conclude that $vx' \notin E(G)$. Similarly, if $ux' \notin E(G)$, then the vertices u, u', x, x' induce a $2K_2$ in $G - A_2$. Thus we conclude that $ux' \in E(G)$. This implies that $y \neq x'$, since $u \in N(x') \setminus N(y)$ and $v \in N(y) \setminus N(x')$. But that also means that the vertices u, v, x', y induce a $2K_2$ in $G - B_1$, a contradiction. This proves Claim 2.1.

Now we define a new partition of $V(G)$ as follows. Let $B'_1 = (B_1 \setminus C_1) \cup C_2$, and let $B'_2 = (B_2 \setminus C_2) \cup C_1$. We show that (A_1, A_2, B'_1, B'_2) is a bichain partition of G and it satisfies $(\star)$. This will imply the lemma.

Recall that for each $i \in \{1, 2\}$, both $G - A_i$ and $G - B_i$ are chain graphs.

Claim 2.2: $G - B'_i$ *is a chain graph for each* $i \in \{1, 2\}$.

By symmetry, it suffices to prove the claim for $i = 1$. Suppose that $G - B'_1$ contains an induced $2K_2$ on vertices u, v, x, y. Namely, we have $u, v \in A$ and $x, y \in B'_2$ where $x \in N(u) \setminus N(v)$ and $y \in N(v) \setminus N(u)$. If $x, y \in B_1$ or $x, y \in B_2$, then $G - B_2$ or $G - B_1$ is not a chain graph, impossible. Thus by symmetry, we may assume that $x \in B_1$ and $y \in B_2$. Since $x, y \notin B'_1 = (B_1 \setminus C_1) \cup C_2$, this means that $x \in C_1$ and $y \in B_2 \setminus C_2$.

Now, if $u, v \in A_1$ or $u, v \in A_2$, then $G - A_2$ or $G - A_1$ is not a chain graph, impossible. If $u \in A_1$ and $v \in A_2$, then (u, v, x, y) is a parallel $2K_2$. This means that $x \in D_1$, but we assumed $x \in C_1$ which contradicts Claim 2.1, Therefore, $u \in A_2$ and $v \in A_1$, which means that (v, u, y, x) is a crossing $2K_2$. This implies that $y \in C_2$, but we assumed that $y \in B_2 \setminus C_2$, impossible. This proves Claim 2.2.

From Claim 2.2 we deduce that (A_1, A_2, B'_1, B'_2) is indeed a bichain partition of G. It remains to show that it satisfies $(\star)$. Suppose not. Then there are vertices $u \in A_1$, $v \in A_2$, $x \in N(u) \setminus N(v)$, $y \in N(v) \setminus N(u)$ such that $x \in B'_2$ and $y \in B'_1$. First, we show that $x \notin C_1$. Suppose that $x \in C_1$. If $y \in B_1$, then $G - B_2$ is not a chain graph because of the $2K_2$ on vertices u, v, x, y. Thus $y \in B_2$ which means that (u, v, x, y) is a parallel $2K_2$. This implies $x \in D_1$, but we assumed $x \in C_1$, contradicting Claim 2.1. Therefore $x \notin C_1$ and hence $x \in B_2 \setminus C_2$, since we assumed $x \in B'_2 = (B_2 \setminus C_2) \cup C_1$. If also $y \in B_2$, then $G - B_1$ is not a chain graph. So $y \in B_1$ which means that (u, v, x, y) is a crossing $2K_2$. This implies $x \in C_2$ but we assumed $x \in B_2 \setminus C_2$, a contradiction. Therefore, no such vertices u, v, x, y exist which proves that (A_1, A_2, B'_1, B'_2) indeed satisfies $(\star)$.

That concludes the proof. ☐

3.1. *Characterizing special bichain partitions*

We say that an edge (non-edge) uv of G is *a crossing edge (non-edge)* with respect to a bichain partition (A_1, A_2, B_1, B_2) of G if $u \in A_1$ and $v \in B_2$, or if $u \in A_2$ and $v \in B_1$.

We say that a closed alternating sequence of G is *bad* with respect to a bichain partition (A_1, A_2, B_1, B_2) if the sequence contains at least as many crossing edges as crossing non-edges (with respect to this partition).

For instance, consider the graph in Figure 1 and the bichain partition $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3, a_4\}$, $B_1 = \{b_1, b_2\}$, $B_2 = \{b_3, b_4\}$. The sequence a_1, b_2, a_4, b_3, a_1 contains 2 crossing non-edges with respect to this partition, namely a_4b_2, a_1b_3, and has no crossing edges (both its edges a_1b_2, a_4b_3 are not crossing).

Lemma 3. *Let G be bichain graph. A bichain partition of G is special if and only if no closed alternating sequence of G is bad with respect to this partition.*

Proof. ($\Leftarrow$) We prove the contrapositive. Consider a bichain partition (A_1, A_2, B_1, B_2) of G and assume that it is not special. This means that there exist $u \in A_1$, $v \in A_2$ and $x \in N(u) \setminus N(v)$, $y \in N(v) \setminus N(u)$ such that $x \notin B_1$ or $y \notin B_2$. Note that $x, y \in B_1 \cup B_2$ and the vertices u, x, v, y are all distinct and induce a $2K_2$ in G. Thus if $x, y \in B_1$, then u, x, v, y induce a $2K_2$ in $G - B_2$, and if $x, y \in B_2$, then u, x, v, y induce a $2K_2$ in $G - B_1$. Since neither is possible (because (A_1, A_2, B_1, B_2) is a bichain partition) and since $x \notin B_1$ or $y \notin B_2$, we conclude that $x \in B_2$ and $y \in B_1$. This implies that u, x, v, y, u is a closed alternating sequence of G that contains two crossing edges ux, vy with respect to (A_1, A_2, B_1, B_2), while neither of its non-edges uy, vx is crossing. Therefore G contains a closed alternating sequence that is bad with respect to (A_1, A_2, B_1, B_2).

($\Rightarrow$) Consider a special bichain partition (A_1, A_2, B_1, B_2) of G. For a contradiction, we assume that G contains a closed alternating sequence that is bad (with respect to this partition). Let $u_1, \ldots, u_t, u_{t+1}$ be a shortest such sequence. Recall that $u_1 = u_{t+1}$ and $u_i \in A_1 \cup A_2$, $u_i u_{i+1} \in E(G)$ if i is odd, while $u_i \in B_1 \cup B_2$, $u_i u_{i+1} \notin E(G)$ if i is even. Moreover, t is even and $t \geq 4$. Since $u_1, \ldots, u_{t+1}$ is bad, it follows by Lemma 1 that the sequence contains at least one vertex in each of the sets A_1, A_2, B_1, B_2. Thus at least one edge or non-edge of the sequence is crossing, and hence, by the definition of a bad sequence, it contains at least one crossing edge. Without loss of generality (up to cyclically renaming the vertices in the sequence), we may assume that the edge $u_1 u_2$ is a crossing edge. We show that the minimality of $u_1, \ldots, u_{t+1}$ implies that actually every edge in this sequence is crossing.

Claim 3.3: *Every edge of the sequence $u_1, \ldots, u_{t+1}$ is a crossing edge with respect to (A_1, A_2, B_1, B_2).*

For a contradiction, let i be the smallest index such that $u_i u_{i+1}$ is an edge that is not crossing. Thus i is odd and $i \geq 3$, since $u_1 u_2$ is a crossing edge. Without loss of generality (by symmetry), we may assume that $u_i \in A_1$. Thus $u_{i+1} \in B_1$, since $u_i u_{i+1}$ is not crossing. Note that $i - 2 \geq 1$ and so the vertices u_{i-2} and u_{i-1} exist. Thus we have that $u_{i-2} \in A_1 \cup A_2$ and $u_{i-2} u_{i-1} \in E(G)$ while $u_{i-1} \in B_1 \cup B_2$ and $u_{i-1} u_i \notin E(G)$, since $i-2$ is odd and $i-1$ is even. This shows that $u_{i-2}, u_{i-1}, u_i, u_{i+1}$ are all distinct, and $u_{i-2} u_{i-1}$ is a crossing edge, by the minimality of i.

Suppose first that $u_{i-2} u_{i+1}$ is not an edge. Then the vertices $u_{i-2}, u_{i-1}, u_i, u_{i+1}$ induce a $2K_2$ in G. Thus if $u_{i-2} \in A_1$, then $u_{i-2}, u_{i-1}, u_i, u_{i+1}$ induce a $2K_2$ in $G - A_2$, while if $u_{i-2} \in A_2$, then $u_{i-1} \in B_1$, since $u_{i-2} u_{i-1}$ is a crossing edge. But then $u_{i-2}, u_{i-1}, u_i, u_{i+1}$ induce a $2K_2$ in $G - B_2$. Neither of these is possible, because (A_1, A_2, B_1, B_2) is a bichain partition. Therefore, we conclude that $u_{i-2} u_{i+1} \in E(G)$.

This shows that $u_1, \ldots, u_{i-2}, u_{i+1}, \ldots, u_{t+1}$ is a closed alternating sequence of G. Recall that $u_{i-2} \in A_1 \cup A_2$. Suppose first that $u_{i-2} \in A_1$. Since $u_{i+1} \in B_1$, the edge $u_{i-2} u_{i+1}$ is non-crossing. On the other hand, $u_{i-2} u_{i-1}$ is a crossing edge, and thus $u_{i-1} \in B_2$. This means that $u_{i-1} u_i$ is a crossing non-edge, since $u_i \in A_1$. Finally, recall that $u_i u_{i+1}$ is a non-crossing edge. This shows that the sequence $u_1, \ldots, u_{t+1}$ contains exactly one more crossing edge and one more crossing non-edge than the sequence $u_1, \ldots, u_{i-2}, u_{i+1}, \ldots, u_{t+1}$. But then the sequence $u_1, \ldots, u_{i-2}, u_{i+1}, \ldots, u_{t+1}$ is bad because $u_1, \ldots, u_{t+1}$ is, which contradicts the minimality of $u_1, \ldots, u_{t+1}$.

We may therefore assume that $u_{i-2} \in A_2$. This implies that $u_{i-2} u_{i+1}$ is a crossing edge, and since $u_{i-2} u_{i-1}$ is also crossing, we deduce $u_{i-1} \in B_1$. Thus $u_{i-1} u_i$ is a non-crossing non-edge, since $u_i \in A_1$, and we recall that $u_i u_{i+1}$ is a non-crossing edge. Thus the sequence $u_1, \ldots, u_{i-2}, u_{i+1}, \ldots, u_{t+1}$ contains the same number of crossing edges and non-edges as $u_1, \ldots, u_{t+1}$, again contradicting the minimality of $u_1, \ldots, u_{t+1}$.

This proves Claim 3.3.

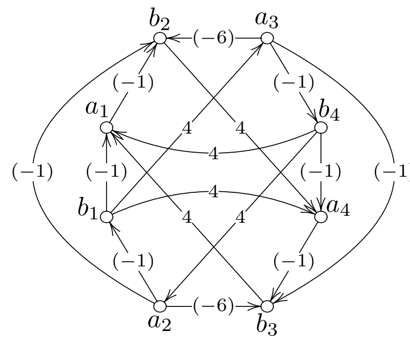
Using this claim, we derive a contradiction. Recall that t is even, that $u_1 = u_{t+1}$, and that $u_t u_{t+1} \notin E(G)$. Thus $u_1 u_t \notin E(G)$ and we can let i be the smallest even index in $\{1, \ldots, t\}$ such that $u_1 u_i \notin E(G)$. Note that $u_i \in B_1 \cup B_2$, since i is even. Without loss of generality (by symmetry), we may assume that $u_i \in B_1$.

Observe that $i > 2$, since $u_1 u_2 \in E(G)$. Thus $i \geq 4$, since i is even. In particular, $i - 2 \geq 2$ which implies that the vertex u_{i-2} exists and it is distinct from u_1. From the minimality of i, we further deduce that $u_1 u_{i-2} \in E(G)$. Also, we recall that $u_{i-2} \in B_1 \cup B_2$ and $u_{i-2} u_{i-1} \notin E(G)$, since $i-2$ is even, while $u_1, u_{i-1} \in A_1 \cup A_2$ and $u_{i-1} u_i \in E(G)$, since 1 and $i-1$ are odd. Thus the vertices $u_1, u_{i-2}, u_{i-1}, u_i$ are all distinct and induce a $2K_2$ in G. By Claim 3.3, the edge $u_{i-1} u_i$ is a crossing edge. Thus $u_{i-1} \in A_2$, since $u_i \in B_1$. Consequently, if $u_1 \in A_2$, then $u_1, u_{i-2}, u_{i-1}, u_i$ induce a $2K_2$ in $G - A_1$, and if $u_{i-2} \in B_1$, then $u_1, u_{i-2}, u_{i-1}, u_i$ induce a $2K_2$ in $G - B_2$. Again, neither is possible, since (A_1, A_2, B_1, B_2) is a bichain partition of G. So we conclude that $u_1 \in A_1$ and $u_{i-2} \in B_2$. But now we contradict ($\star$) for the vertices $u_1 \in A_1$, $u_{i-1} \in A_2$, $u_{i-2} \in N(u_1) \setminus N(u_{i-1})$, and $u_i \in N(u_{i-1}) \setminus N(u_1)$. Therefore, the bichain partition (A_1, A_2, B_1, B_2) is not special, which is a contradiction.

That concludes the proof. $\square$

$$\alpha = 5 \qquad A_1 = \{a_1, a_2\} \quad A_2 = \{a_3, a_4\} \quad B_1 = \{b_1, b_2\} \quad B_2 = \{b_3, b_4\}$$

$$\begin{array}{ll}
z_{a_1} - z_{b_1} \le -1 & \qquad z_{a_1} - z_{b_3} \le 4 \\
z_{b_2} - z_{a_1} \le -1 & \qquad z_{a_1} - z_{b_4} \le 4 \\
z_{b_1} - z_{a_2} \le -1 & \qquad z_{a_2} - z_{b_4} \le 4 \\
z_{b_2} - z_{a_2} \le -1 & \qquad z_{a_3} - z_{b_1} \le 4 \\
z_{a_4} - z_{b_4} \le -1 & \qquad z_{a_4} - z_{b_1} \le 4 \\
z_{b_3} - z_{a_4} \le -1 & \qquad z_{a_4} - z_{b_2} \le 4 \\
z_{b_4} - z_{a_3} \le -1 & \qquad z_{b_3} - z_{a_2} \le -6 \\
z_{b_3} - z_{a_3} \le -1 & \qquad z_{b_2} - z_{a_3} \le -6
\end{array}$$



Solution: $\quad z_{a_1} = 5 \quad z_{a_2} = 9 \quad z_{a_3} = 9 \quad z_{a_4} = 5 \quad z_{b_1} = 6 \quad z_{b_2} = 2 \quad z_{b_3} = 3 \quad z_{b_4} = 7$

Figure 2: The inequalities and directed graph H for the graph from Figure 1.

Next we present a technical lemma that will be useful in providing the geometric intersection model for bichain graphs in Section 4.

Lemma 4. *If (A_1, A_2, B_1, B_2) is a special bichain partition of G, then there exists an integer $0 \le \alpha \le |V(G)|$ such that the following system $(\triangle)$ of inequalities in variables $\{z_u\}_{u \in V(G)}$ has a solution:*

$$\begin{array}{ll}
z_b - z_a \le -1 & \text{for all } a \in A_i \text{ and } b \in B_i \text{ where } i \in \{1,2\} \text{ and } ab \in E(G), \\
z_a - z_b \le -1 & \text{for all } a \in A_i \text{ and } b \in B_i \text{ where } i \in \{1,2\} \text{ and } ab \notin E(G), \\
z_b - z_a \le -\alpha - 1 & \text{for all } a \in A_i \text{ and } b \in B_j \text{ where } i \ne j \text{ and } ab \in E(G), \\
z_a - z_b \le \alpha - 1 & \text{for all } a \in A_i \text{ and } b \in B_j \text{ where } i \ne j \text{ and } ab \notin E(G).
\end{array} \qquad (\triangle)$$

In fact, there exists a solution $z_u = z_u^*$, $u \in V(G)$, such that each z_u^* is a positive integer and $z_u^* \le |V(G)| \cdot (\alpha + 1)$.

Proof. Let (A_1, A_2, B_1, B_2) be a special bichain partition of G. We show that the system $(\triangle)$ has a solution for $\alpha = |V(G)|$. To this end, we construct the following directed graph H (see Figure 2 for illustration):

- the vertices of H are $V(G)$, and
- there is a directed arc e of weight $w(e) = \beta$ from u to v if the system $(\triangle)$ contains the inequality $z_v - z_u \le \beta$.

In order to find a solution to $(\triangle)$, we show that no directed cycle of H has negative total weight. For contradiction, suppose that H contains a directed cycle C on vertices $u_1, \ldots, u_t$ where $e_t = (u_t, u_1)$ and $e_i = (u_i, u_{i+1})$ for $i \in \{1, \ldots, t-1\}$ are the arcs of C and where $\sum_{i=1}^{t} w(e_i) < 0$. Define $u_{t+1} = u_1$.

Observe that $t \le |V(G)| = \alpha$, since all vertices on C are distinct. By examining the system of inequalities, note that each e_i is an arc either from a vertex of $A_1 \cup A_2$ to a vertex of $B_1 \cup B_2$, or from a vertex of $B_1 \cup B_2$ to a vertex of $A_1 \cup A_2$. In particular, if $e_i = (u_i, u_{i+1})$ is an arc of H from $A_1 \cup A_2$ to $B_1 \cup B_2$, then $u_i u_{i+1} \in E(G)$, while if $e_i = (u_i, u_{i+1})$ is an arc of H from $B_1 \cup B_2$ to $A_1 \cup A_2$, then $u_i u_{i+1} \notin E(G)$. Without loss of generality (by the symmetry of the cycle C), we may assume that $u_1 \in A_1 \cup A_2$.

From this it follows that $u_1, u_2, \ldots, u_t, u_{t+1}$ is a closed alternating sequence of G. Moreover, observe that if $u_i u_{i+1}$ is an edge of G, and it is a crossing edge with respect to (A_1, A_2, B_1, B_2), then $w(e_i) = -\alpha - 1$, while if it is a non-crossing edge, then $w(e_i) = -1$. Similarly, if $u_i u_{i+1}$ is a not an edge of G and it is a crossing non-edge with respect to (A_1, A_2, B_1, B_2), then $w(e_i) = \alpha - 1$, while if it is non-crossing, then $w(e_i) = -1$.

Since (A_1, A_2, B_1, B_2) is a special bichain partition of G, we conclude by Lemma 3 that $u_1, \ldots, u_t, u_{t+1}$ is not a bad sequence with respect to (A_1, A_2, B_1, B_2). This means that the number γ_e of crossing edges of this sequence is strictly smaller than the number γ_n of crossing non-edges of this sequence. Therefore we can calculate:

$$\sum_{i=1}^{t} w(e_i) = \sum_{\substack{u_i u_{i+1} \in E(G) \\ \text{is crossing}}} w(e_i) + \sum_{\substack{u_i u_{i+1} \notin E(G) \\ \text{is crossing}}} w(e_i) + \sum_{\substack{u_i u_{i+1} \\ \text{is not crossing}}} w(e_i) =$$

$$= \gamma_e(-\alpha - 1) + \gamma_n(\alpha - 1) - (t - \gamma_e - \gamma_n) = \alpha(\gamma_n - \gamma_e) - t \geq \alpha - t \geq 0$$

This shows that the total weight of the cycle C is non-negative, a contradiction. Thus no such a cycle C exists in H, and so H indeed contains no directed cycle of negative total weight, as claimed.

Using this fact, we construct a solution to the inequalities ($\triangle$). To this end, we add an new "source" vertex s to H and add an arc of weight M (to be chosen later) from s to every other vertex. Then for each $u \in V(G)$, the value z^*_u is defined as the distance (the length of a shortest walk) from s to u in this augmented graph. Clearly, since H contains no cycles of negative weight, the values z^*_u are well-defined real numbers (each shortest walk is in fact a shortest path – does not repeat vertices). Moreover, for every arc $e = (u, v)$ of H, the triangle inequality for the distance (when we travel to v by going to u and then taking the edge e) yields $z^*_v \leq z^*_u + w(e)$; in other words, $z^*_v - z^*_u \leq w(e)$. This shows that the values z^*_u constructed this way indeed form a solution $z_u = z^*_u$ to ($\triangle$).

Finally, by taking $M = |V(G)| \cdot (\alpha + 1)$, we make sure that $1 \leq z^*_u \leq |V(G)| \cdot (\alpha + 1)$ for each $u \in V(G)$, since no shortest walk in H repeats vertices and smallest negative weight of an arc in H is $-(\alpha + 1)$. Moreover, since the weights of all arcs in H are integers, it follows that each z^*_u is also an integer. This completes the proof. $\square$

4. Diagonal representations

In this section, we introduce a geometric intersection model for bichain graphs, which we call diagonal representation.

A *diagonal representation* of G is an intersection representation that assigns to each vertex of G a segment connecting two points on the boundary of a fixed axis-parallel rectangle R so that

(i) all segments are distinct,
(ii) every segment is parallel either to the line $y = -x$, or to the line $y = x$,
(iii) no segment connects points on opposite sides of the rectangle R, and
(iv) two vertices of G are adjacent if and only if the corresponding segments cross each other.

See Figure 1 for an illustration of this representation. We show that this representation characterizes bichain graphs.

Theorem 5. *Let G be a graph. Then the following are equivalent.*

(i) *G is a bichain graph.*
(ii) *G admits a diagonal representation.*

Proof. **(ii)⇒(i)**: Suppose that G admits a diagonal representation in a rectangle $R = [x_1, x_2] \times [y_1, y_2]$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Let A denote those vertices whose segments are parallel to the line $y = -x$. Let B denote the remaining vertices (those parallel to the line $y = x$). Split A into A_1 and A_2 where A_1 are the vertices whose segments connect the bottom side of R with the left side of R, and A_2 are the vertices whose segments connect the top side of R with the right side of R. Likewise, split B into B_1 and B_2 where B_1 are the vertices whose segments connect the top and the left side of R while B_2 are the vertices whose segments connect the bottom and the right side of R. For instance, for the graph in Figure 1, we have $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$, where $A_1 = \{a_1, a_2\}$, $A_2 = \{a_3, a_4\}$, $B_1 = \{b_1, b_2\}$, and $B_2 = \{b_3, b_4\}$.

We show that (A_1, A_2, B_1, B_2) is a bichain partition of G which will imply that G is a bichain graph. First, note that A is an independent set. Indeed, the segments assigned to the vertices in A are all parallel to the line $y = -x$ and are all distinct. So they do not pairwise intersect. Similarly, we see that B is an independent set.

It remains to verify for each $i \in \{1, 2\}$ that $G - A_i$ and $G - B_i$ are chain graphs. By symmetry (between A and B and between A_1 and A_2, resp. B_1 and B_2), it suffices to check this for $G - A_1$.

Suppose that $G - A_1$ contains an induced $2K_2$ on vertices u, v, p, q with edges uv and pq where $u, p \in A$ and $v, q \in B$. Since $u, p \in A$, the segments representing u and v are parallel to the line $y = -x$. Thus there are $\delta_u, \delta_p \in \mathbb{R}$ such that the segment representing u lies on the line $y = -x + \delta_u$ and the segment representing p lies on the line $y = -x + \delta_p$. Similarly, the segments representing $v, q \in B$ are parallel to the line $y = x$. So there exist $\delta_v, \delta_q \in \mathbb{R}$ such that the segment representing v lies on the line $y = x + \delta_v$ and the segment representing q lies on the line $y = x + \delta_q$. By symmetry, we may assume that $\delta_u \geq \delta_p$. (If not, we exchange u with p, and v with q.) Further, since

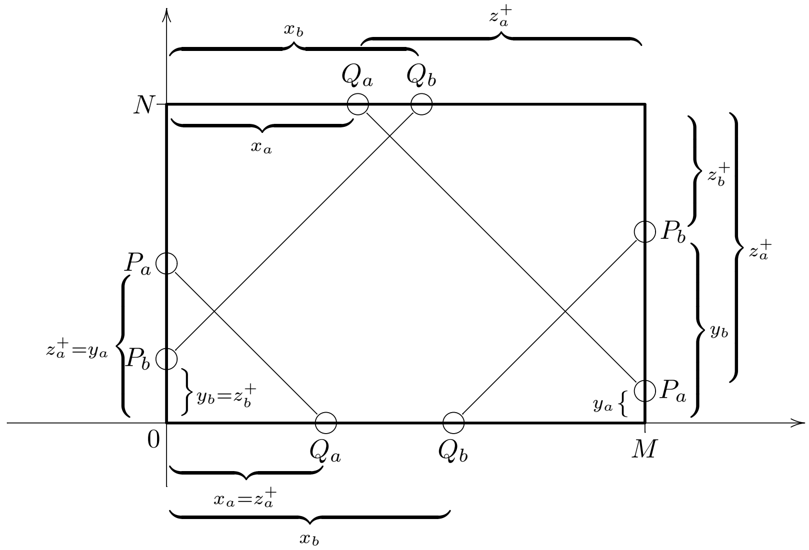


Figure 3: Illustration of the construction.

$u, p \in A$ and $G - A_1$ contains no vertices of A_1, we conclude $u, p \in A_2$. Therefore the segments representing u and p intersect the top and right sides of the rectangle R. In other words, the lines $y = -x + \delta_u$ and $y = -x + \delta_p$ intersect the sets $[x_1, x_2] \times \{y_2\}$ and $\{x_2\} \times [y_1, y_2]$. This gives the following inequalities (recall that $\delta_p \leq \delta_u$):

$$x_1 + y_2 \leq \delta_p \leq \delta_u \leq x_2 + y_2 \qquad\qquad x_2 + y_1 \leq \delta_p \leq \delta_u \leq x_2 + y_2$$

Since $uv \in E(G)$, the segments representing u and v intersect. In other words, the intersection point of the lines $y = -x + \delta_u$ and $y = x + \delta_v$ lies inside the rectangle $R = [x_1, x_2] \times [y_1, y_2]$. Note that the point where these two lines intersect has coordinates $x = (\delta_u - \delta_v)/2$ and $y = (\delta_u + \delta_v)/2$. This yields the following:

$$x_1 \leq \frac{\delta_u - \delta_v}{2} \leq x_2 \qquad y_1 \leq \frac{\delta_u + \delta_v}{2} \leq y_2$$

Similarly, note that the intersection point of the lines $y = -x + \delta_p$ and $y = x + \delta_v$ has coordinates $x = (\delta_p - \delta_v)/2$ and $y = (\delta_p + \delta_v)/2$. We can bound these coordinates using the above inequalities as follows:

$$x_2 \geq \frac{\delta_u - \delta_v}{2} \geq \frac{\delta_p - \delta_v}{2} \geq \frac{x_1 + y_2 - \delta_v}{2} \geq \frac{x_1 + \frac{\delta_u + \delta_v}{2} - \delta_v}{2} \geq \frac{x_1 + \frac{\delta_u - \delta_v}{2}}{2} \geq \frac{x_1 + x_1}{2} = x_1$$
$$y_2 \geq \frac{\delta_u + \delta_v}{2} \geq \frac{\delta_p + \delta_v}{2} \geq \frac{x_2 + y_1 + \delta_v}{2} \geq \frac{\frac{\delta_u - \delta_v}{2} + y_1 + \delta_v}{2} \geq \frac{\frac{\delta_u + \delta_v}{2} + y_1}{2} \geq \frac{y_1 + y_1}{2} = y_1$$

This shows that the lines $y = -x + \delta_p$ and $y = x + \delta_v$ intersect inside the rectangle R. But this means that the segments representing v and p intersect. However, we have $vp \notin E(G)$, a contradiction.

So we conclude that no such vertices u, v, p, q exist and thus $G - A_1$ is indeed a chain graph. By the same token (by symmetry), also $G - A_2$, $G - B_1$, and $G - B_2$ are chain graphs. Therefore, G is indeed a bichain graph.

This proves (ii)⇒(i).

(i)⇒(ii): Suppose that G is a bichain graph. Then by Lemma 2, there exists a special bichain partition of G. Let us denote this partition as (A_1, A_2, B_1, B_2). Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$.

We apply Lemma 4 to the partition (A_1, A_2, B_1, B_2). This yields an integer $\alpha \geq 0$ for which the system ($\triangle$) has a solution $z_u = z_u^*$, $u \in V(G)$ where each z_u^* is a positive integer.

Let $n = |V(G)|$. We fix an ordering of $V(G)$ and denote it $u_1, u_2, \ldots, u_n$. We further define the following:

– $N = 1 + n \cdot \max\limits_{u \in V(G)} z_u^*$,
– $M = N + n \cdot \alpha$,

– for each $k = 1, \ldots, n$, define $z^+_{u_k} = n \cdot z^*_{u_k} + k - n$.

Note that $1 \le z^+_u \le N - 1$ for all $u \in V(G)$, which follows from the definition of N and the fact that $1 \le z^*_u$. In fact, for distinct $k, \ell \in \{1, \ldots, n\}$, we have that $z^+_{u_k} \ne z^+_{u_\ell}$, since $z^*_{u_k}, z^*_{u_\ell}$ are positive integers and $1 \le k, \ell \le n$. Moreover, we observe the following property.

Claim 5.1: *Let $u_k \in A_i$ and $u_\ell \in B_j$ where $i, j \in \{1, 2\}$.*

(a) *If $i = j$, then $u_k u_\ell \in E(G)$ if and only if $z^+_{u_\ell} \le z^+_{u_k}$.*
(b) *If $i \ne j$, then $u_k u_\ell \in E(G)$ if and only if $z^+_{u_\ell} + n \cdot \alpha \le z^+_{u_k}$.*

To prove this, consider $u_k \in A_i$ and $u_\ell \in B_j$. Suppose first that $i = j$. Then, if $u_k u_\ell \in E(G)$, we have $z^*_{u_\ell} - z^*_{u_k} \le -1$, since the values z^*_u form a solution to ($\triangle$). Thus since $k, \ell \in \{1, \ldots, n\}$, we deduce that $z^+_{u_\ell} < z^+_{u_k}$ as follows:

$$z^+_{u_\ell} = n \cdot z^*_{u_\ell} + \ell - n \le n \cdot z^*_{u_\ell} \le n \cdot (z^*_{u_k} - 1) < n \cdot z^*_{u_k} + k - n = z^+_{u_k}.$$

Similarly, if $u_k u_\ell \notin E(G)$, then $z^*_{u_k} - z^*_{u_\ell} \le -1$ and we deduce that $z^+_{u_k} < z^+_{u_\ell}$. This shows that $u_k u_\ell \in E(G)$ if and only if $z^+_{u_\ell} \le z^+_{u_k}$, as claimed.

Now, assume that $i \ne j$. Then, if $u_k u_\ell \in E(G)$, we have $z^*_{u_\ell} - z^*_{u_k} \le -\alpha - 1$, since the values z^*_u form a solution to the system ($\triangle$). From this we deduce that $z^+_{u_\ell} < z^+_{u_k} - n \cdot \alpha$ as follows:

$$z^+_{u_\ell} = n \cdot z^*_{u_\ell} + \ell - n \le n \cdot z^*_{u_\ell} \le n \cdot (z^*_{u_k} - \alpha - 1) < n \cdot z^*_{u_k} + k - n - n \cdot \alpha = z^+_{u_k} - n \cdot \alpha.$$

Similarly, if $u_k u_\ell \notin E(G)$, then $z^*_{u_k} - z^*_{u_\ell} \le \alpha - 1$ in which case we deduce that $z^+_{u_k} < z^+_{u_\ell} + n \cdot \alpha$. Together, we conclude, as required, that $u_k u_\ell \in E(G)$ if and only if $z^+_{u_\ell} + n \cdot \alpha \le z^+_{u_k}$. This proves Claim 5.1.

Now we are ready to describe the construction. We construct a diagonal representation of G as follows. The underlying rectangle is chosen to have corner points $(0, 0)$, $(M, 0)$, $(0, N)$, and (M, N). For $a \in A_1$, we represent a as the segment connecting the point $P_a = (0, z^+_a)$ to the point $Q_a = (z^+_a, 0)$. For $a \in A_2$, we represent a as the segment connecting the point $P_a = (M, N - z^+_a)$ to the point $Q_a = (M - z^+_a, N)$. For $b \in B_1$, the segment representing b goes from $P_b = (0, z^+_b)$ to $Q_b = (N - z^+_b, N)$, while for $b \in B_2$, the segment for b goes from $P_b = (M, N - z^+_b)$ to $Q_b = (z^+_b + M - N, 0)$. See Figure 3 for a detailed illustration of this construction.

We verify that the segments form a diagonal representation of G. Clearly, the segments connect points on consecutive sides of the rectangle, where the segments for the vertices in A are all parallel to the line $y = -x$, while the segments for the vertices in B are parallel to the line $y = x$. Further, note that the segments are all distinct. Namely, the segments representing the vertices in A are all distinct, and the segment representing the vertices in B are all distinct. In particular, for $u_k, u_\ell \in A$ where $k \ne \ell$, the segments representing u_k and u_ℓ are distinct, because $z^+_{u_k} \ne z^+_{u_\ell}$ and $z^+_{u_\ell} < N \le M$. For $u_k, u_\ell \in B$ where $k \ne \ell$, the two segments for u_k and u_ℓ are distinct, because again $z^+_{u_k} \ne z^+_{u_\ell}$ and $1 \le z^+_{u_\ell}$. Thus all segments in the representation are indeed distinct. Consequently, the segments representing the vertices in A are pairwise non-intersecting, since they are all parallel to the line $y = -x$. Likewise, the segments representing the vertices in B are pairwise non-intersecting, since they are all parallel to $y = x$.

To conclude that the representation is indeed a diagonal representation of G, it remains to verify that for u_k, u_ℓ where $u_k \in A$ and $u_\ell \in B$, the segments representing u_k and u_ℓ intersect if and only if $u_k u_\ell \in E(G)$.

Suppose first that $u_k \in A_1$ and $u_\ell \in B_1$. Then it follows that the segment representing u_k lies on the line $y = -x + z^+_{u_k}$ while the segment representing u_ℓ lies on the line $y = x + z^+_{u_\ell}$. The intersection point of the two lines is (x^*, y^*) where $x^* = \frac{1}{2} z^+_{u_k} - \frac{1}{2} z^+_{u_\ell}$ and $y^* = \frac{1}{2} z^+_{u_k} + \frac{1}{2} z^+_{u_\ell}$. Therefore, the two segments representing u_k and u_ℓ intersect if and only if (x^*, y^*) lies in the rectangle. Note that $x^* < M$ and $0 < y^* < N$, since $0 < z^+_{u_k}, z^+_{u_\ell} < N \le M$. Thus (x^*, y^*) lies in the rectangle if and only if $x^* \ge 0$. In other words, if and only if $z^+_{u_\ell} \le z^+_{u_k}$. By Claim 5.1a, this happens if and only if $u_k u_\ell \in E(G)$, since $u_k \in A_1$ and $u_\ell \in B_1$. Put together, we conclude that the two segments representing u_k and u_ℓ intersect if and only if $u_k u_\ell \in E(G)$, as required.

We proceed similarly in all the other cases. If $u_k \in A_2$ and $u_\ell \in B_2$, then the point of (possible) intersection of the segments is (x^*, y^*) where $x^* = M - \frac{1}{2} z^+_{u_k} + \frac{1}{2} z^+_{u_\ell}$ and $y^* = N - \frac{1}{2} z^+_{u_k} - \frac{1}{2} z^+_{u_\ell}$. Thus the point lies in the rectangle if and only if $x^* \le M$ which is if and only if $z^+_{u_\ell} \le z^+_{u_k}$ which is if and only if $u_k u_\ell \in E(G)$ by Claim 5.1a.

If $u_k \in A_1$ and $u_\ell \in B_2$, then the (potential) intersection point of the segments is (x^*, y^*) where $x^* = \frac{1}{2} z^+_{u_k} + \frac{1}{2} z^+_{u_\ell} + \frac{1}{2} n \cdot \alpha$ and $y^* = \frac{1}{2} z^+_{u_k} - \frac{1}{2} z^+_{u_\ell} - \frac{1}{2} n \cdot \alpha$. This implies that the point lies in the rectangle if and only if $y^* \ge 0$ if and only if $z^+_{u_\ell} + n \cdot \alpha \le z^+_{u_k}$ if and only if $u_k u_\ell \in E(G)$ by Claim 5.1b. Finally, if $u_k \in A_2$ and $u_\ell \in B_1$, then

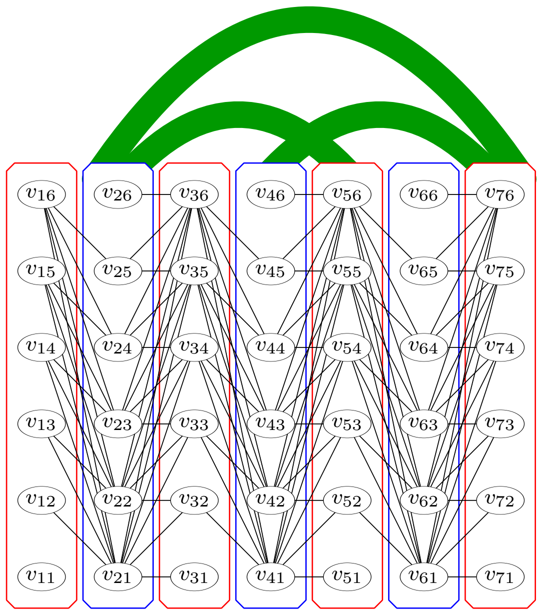


Figure 4: The Z-grid $Z_{n,m}$ with $n = 7$ columns and $m = 6$ rows, where the edges of type (c) are shown on top of the figure.

we have $x^* = M - \frac{1}{2}z^+_{u_k} - \frac{1}{2}z^+_{u_\ell} - \frac{1}{2}n \cdot \alpha$ and $y^* = N - \frac{1}{2}z^+_{u_k} + \frac{1}{2}z^+_{u_\ell} + \frac{1}{2}n \cdot \alpha$. Thus the point (x^*, y^*) lies in the rectangle if and only if $y^* \leq N$ if and only if $z^+_{u_\ell} + n \cdot \alpha \leq z^+_{u_k}$ if and only if $u_k u_\ell \in E(G)$ by Claim 5.1b.

This completes all cases and so we can conclude that the segments indeed form a diagonal representation of G. This proves (i)⇒(ii) and concludes the proof. $\square$

The reader can wonder how the class of bichain graphs compares to other classes admitting a geometric intersection model, for instance, permutation graphs. We claim that the two classes are incomparable, i.e. none of them contains the other. Moreover, the class of bichain graphs is incomparable even with *bipartite* permutation graphs. For instance, the chordless path on 7 vertices is a bipartite permutation graph, but it is not a bichain graph, which can be easily seen by definition. On the other hand, the graph in Figure 1 is not a permutation graph, while it is a bichain graph as the figure shows.

On the other hand, both classes (bichain graphs and permutation graphs) are subclasses of k-polygon graphs (i.e. the intersection graphs of straight-line chords inside a convex k-gon) for each $k \geq 4$. Since k-polygon graphs form a subclass of circle graphs (i.e. the intersection graphs of straight-line chords inside a circle) [5], we conclude that both classes are subclasses of circle graphs. For permutation graphs, this is a well-known fact. For bichain graphs, this conclusion follows from our geometric representation proposed in Theorem 5. We formally state this conclusion as a corollary from the theorem.

Corollary 6. *Evey bichain graph is a circle graph.*

5. Universal bichain graphs

In this section, we construct a universal graph for bichain graphs. We start with the description of our construction, which we call the Z-grid.

The *Z-grid* $Z_{n,m}$ is the graph defined as follows:

- vertex set is $V(Z_{n,m}) = \left\{ v_{ij} \,\middle|\, i \in \{1, \ldots, n\} \text{ and } j \in \{1, \ldots, m\} \right\}$
- vertex v_{ij} is adjacent to vertex $v_{i'j'}$ if and only if
  (a) i is even, $i' = i + 1$ and $j' \geq j$,
  (b) i is odd, $i' = i + 1$ and $j' < j$,
  (c) i is even and $i' \geq i + 3$ is odd.

An example of the Z-grid is represented in Figure 4. Following the depiction therein, we shall speak of rows and columns of a Z-grid. Namely, the set of vertices $\{v_{ij} \mid j \in \{1, \ldots, m\}\}$ is the *i-th column* of $Z_{n,m}$, while the vertices $\{v_{ij} \mid i \in \{1, \ldots, n\}\}$ form the *j-th row* of $Z_{n,m}$.

The main goal of this section is to prove that the Z-grid $Z_{n,n}$ is an n-universal bichain graph, i.e. it is a bichain graph containing every bichain graph with n vertices as an induced subgraph. We start by showing that the Z-grid itself is a bichain graph.

Lemma 7. *For any positive integers n, m, the Z-grid $Z_{n,m}$ is a bichain graph.*

Proof. We define a partition of the vertices of $Z_{n,m}$ and show that it is a bichain partition. This will imply that $Z_{n,m}$ is a bichain graph. We define the sets as follows:

$$A_1 = \{v_{ij} \mid i \equiv 1 \pmod 4 \text{ and } j \in \{1, \ldots, m\}\} \qquad B_1 = \{v_{ij} \mid i \equiv 2 \pmod 4 \text{ and } j \in \{1, \ldots, m\}\}$$

$$A_2 = \{v_{ij} \mid i \equiv 3 \pmod 4 \text{ and } j \in \{1, \ldots, m\}\} \qquad B_2 = \{v_{ij} \mid i \equiv 0 \pmod 4 \text{ and } j \in \{1, \ldots, m\}\}$$

We show that (A_1, A_2, B_1, B_2) is a bichain partition of $Z_{n,m}$. Note that by definition, if the vertex v_{ij} is adjacent to $v_{i'j'}$, then i is odd and i' even, or i is even and i' odd. This implies that $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ are independent sets. It remains to show that $Z_{n,m} - A_i$ and $Z_{n,m} - B_i$ are chain graph for $i = 1, 2$. By symmetry, it suffices to check that $Z_{n,m} - A_1$ is a chain graph.

Suppose otherwise, and let x, y, z, w denote vertices in $Z_{n,m}$ that induce a $2K_2$ with edges xy, zw such that $x, y, z, w \notin A_1$. Thus $x, y, z, w \in A_2 \cup B_1 \cup B_2$. In particular, since $B_1 \cup B_2$ is an independent set and xy is an edge, it follows that one of x, y belongs to A_2. By the same token, one of z, w belongs to A_2. By symmetry, we may assume that $x, z \in A_2$. Thus $y, w \in B_1 \cup B_2$. Since x, y, z, w denote vertices in $Z_{n,m}$, we have that $x = v_{i_1 j_1}$, $y = v_{i_2 j_2}$, $z = v_{i_3 j_3}$, and $w = v_{i_4 j_4}$ for some indices $i_1, \ldots, i_4, j_1, \ldots, j_4$. In particular, since $x, z \in A_2$ and $y, w \in B$, we deduce that i_1 and i_3 are odd, while i_2, i_4 are even. Since xy, zw are edges while xw, yz are non-edges, we deduce:

$$i_2 \leq i_1 + 1 \quad i_4 \leq i_3 + 1 \quad i_1 \leq i_4 + 1 \quad i_3 \leq i_2 + 1.$$

From this we deduce that $|i_1 - i_3| \leq 2$, since $-2 \leq i_2 - i_3 - 1 \leq i_1 - i_3 \leq i_4 - i_3 + 1 \leq 2$. This allows us to conclude that $i_1 = i_3$, since $(i_1 - i_3) \equiv 0 \pmod 4$ because $x, z \in A_2$. Therefore $i_2, i_4 \in \{i_1 - 1, i_1 + 1\}$, since i_2, i_4 are even and $i_1 = i_3$ while $i_3 - 1 \leq i_2 \leq i_1 + 1$ and $i_1 - 1 \leq i_4 \leq i_3 + 1$. This implies, by the definition of $Z_{n,m}$, that $j_2 \leq j_1$ and $j_4 \leq j_3$, because xy, zw are edges, while it also implies that $j_1 \leq j_4$ and $j_3 \leq j_2$ because xw, yz are non-edges. Put together, we have $j_1 \leq j_4 \leq j_3 \leq j_2 \leq j_1$. Therefore $j_1 = j_3$ but since also $i_1 = i_3$, we deduce that $x = z$, impossible. We conclude that no such vertices x, y, z, w exist, which yields the claim. $\square$

In order to prove the main result of this section, we need a particular decomposition of bichain graphs. The starting point of the decomposition is described in the following lemma.

Lemma 8. *If G is a bichain graph, then there is a special bichain partition (A_1, A_2, B_1, B_2) of G such that*

($\star\star$) there exists a non-empty set $X \subseteq A_1$ such that $N(X) \subseteq B_1$.

Proof. By Lemma 2, let (A_1, A_2, B_1, B_2) be a special bichain partition of G, i.e. a partition satisfying ($\star$). Let W denote the set of all vertices $y \in B_2$ such that $N(y) \supseteq A_1$ (possibly $W = \emptyset$).

We construct a new partition of $V(G)$ as follows. Let $B'_1 = B_1 \cup W$. Let $B'_2 = B_2 \setminus W$. We claim that (A_1, A_2, B'_1, B'_2) is a bichain partition of G and it satisfies ($\star$). Recall that (A_1, A_2, B_1, B_2) is a bichain partition of G satisfying ($\star$). Namely for each $i \in \{1, 2\}$, both $G - A_i$ and $G - B_i$ are chain graphs.

Claim 8.1: *$G - B'_i$ is a chain graph for each $i \in \{1, 2\}$.*

Clearly, $G - B'_1$ is a chain graph, since it is an induced subgraph of $G - B_1$ which itself is a chain graph. Suppose that $G - B'_2$ contains an induced $2K_2$. Namely, suppose that there are vertices $u, v \in A$ and $x, y \in B \setminus B'_2$ such that $x \in N(u) \setminus N(v)$ and $y \in N(v) \setminus N(u)$. If $u, v \in A_1$ or $u, v \in A_2$, then $G - A_2$ or $G - A_1$ is not a chain graph, impossible. Thus, by symmetry, we may assume that $u \in A_1$ and $v \in A_2$. Therefore, since (A_1, A_2, B_1, B_2) satisfies ($\star$), we conclude that $x \in B_1$ and $y \in B_2$. Now, recall that $y \notin B'_2 = B_2 \setminus W$. Thus $y \in W$, since $y \in B_2$. This means that $N(y) \supseteq A_1$. However, $u \in A_1$ and $uy \notin E(G)$, a contradiction. This proves Claim 8.1.

Claim 8.2: (A_1, A_2, B'_1, B'_2) *satisfies ($\star$).*

Suppose that there exist $u \in A_1$, $v \in A_2$, and vertices $x \in N(u) \setminus N(v)$ and $y \in N(v) \setminus N(u)$ such that $x \notin B'_1$ or $y \notin B'_2$. If $x, y \in B'_1$ or $x, y \in B'_2$, then $G - B'_1$ or $G - B'_2$ is not a chain graph, contradicting Claim 8.1. Thus we conclude that $x \in B'_2$ and $y \in B'_1$. Recall that $B'_2 = B_2 \setminus W$. Since (A_1, A_2, B_1, B_2) satisfies ($\star$), we deduce $x \in B_1$ and $y \in B_2$. Thus $y \in B_2 \cap B'_1 = W$. This means that $N(y) \supseteq A_1$. However, $u \in A_1$ and $uy \notin E(G)$, a contradiction. Therefore, no such vertices u, v, x, y exist. This proves Claim 8.2.

From Claims 8.1 and 8.2, we deduce that (A_1, A_2, B'_1, B'_2) is indeed a bichain partition of G and it indeed satisfies ($\star$). We now show that it also satisfies ($\star\star$) which will imply the lemma. Let X be the set of all vertices $u \in A_1$ such that $N(u) \subseteq B'_1$. Suppose that $X = \emptyset$. Since $G - B'_1$ is a chain graph by Claim 8.1, there exist $y \in B'_2$ such that $N(y) \supseteq N(b)$ for all $b \in B'_2$. Since $y \in B'_2$ and $B'_2 = B_2 \setminus W$, we conclude that $y \notin W$, namely $N(y) \not\supseteq A_1$. Thus there exists $u \in A_1$ such that $u \notin N(y)$. If $N(u) \subseteq B'_1$, then $u \in X$ and hence $X \neq \emptyset$. But we assumed $X = \emptyset$. So there exists $b \in N(u) \cap B'_2$. By the choice of y, we have $N(y) \supseteq N(b)$. However, this is impossible, since $u \in N(b) \setminus N(y)$. We therefore conclude $X \neq \emptyset$. Thus X is a non-empty subset of A_1, and we have $N(X) \subseteq B'_1$, since $N(u) \subseteq B'_1$ for all $u \in X$. This shows that (A_1, A_2, B'_1, B'_2) indeed satisfies ($\star\star$).

That concludes the proof. $\square$

Now we are in a position to prove the main result of the section stating that every bichain graph G with n vertices is contained in the grid $Z_{n,n}$ as an induced subgraph. Moreover, we will prove a stronger result stating that $Z_{n,n}$ contains an induced copy of G such that every row contains exactly one vertex of G. We call such a copy *row-sparse*.

Theorem 9. *Let G be an n-vertex bichain graph. Then G is isomorphic to a row-sparse induced subgraph of $Z_{n,n}$.*

Proof. Consider an n-vertex bichain graph G. We show how to find a row-sparse copy of G in $Z_{n,n}$. Let (A_1, A_2, B_1, B_2) be a bichain partition of G satisfying both ($\star$) and ($\star\star$). Such a partition is guaranteed by Lemma 8.

We iteratively define the following sets: for each $i = 0, 1, 2, \ldots$ in turn, having defined $W_1, W_2, \ldots, W_{4i}$, we define the sets $W_{4i+1}, W_{4i+2}, W_{4i+3}, W_{4i+4}$ as follows:

- W_{4i+1} is the set of all $u \in A_1 \setminus (W_1 \cup W_5 \cup \cdots \cup W_{4i-3})$ such that $N(u) \subseteq B_1 \cup (W_4 \cup W_8 \cup \cdots \cup W_{4i})$
- W_{4i+2} is the set of all $u \in B_1 \setminus (W_2 \cup W_6 \cup \cdots \cup W_{4i-2})$ such that $N(u) \supseteq A_1 \setminus (W_1 \cup W_5 \cup \cdots \cup W_{4i+1})$
- W_{4i+3} is the set of all $u \in A_2 \setminus (W_3 \cup W_7 \cup \cdots \cup W_{4i-1})$ such that $N(u) \subseteq B_2 \cup (W_2 \cup W_6 \cup \cdots \cup W_{4i+2})$
- W_{4i+4} is the set of all $u \in B_2 \setminus (W_4 \cup W_8 \cup \cdots \cup W_{4i})$ such that $N(u) \supseteq A_2 \setminus (W_3 \cup W_7 \cup \cdots \cup W_{4i+3})$

Recall that the condition ($\star\star$) holds for (A_1, A_2, B_1, B_2). This gives us $W_1 \neq \emptyset$. Also, observe that $W_{4i+1} \subseteq A_1$ and $W_{4i+2} \subseteq B_1$ while $W_{4i+3} \subseteq A_2$ and $W_{4i+4} \subseteq B_2$, for all $i \geq 0$. Thus, by construction, all these sets are pairwise disjoint. In the following claim, we show that they completely cover (and thus partition) G.

Claim 9.3: $V(G) = \bigcup_{i=1}^{\infty} W_i$

Define the following sets:

$$C_1 = A_1 \setminus \Big(\bigcup_{i=0}^{\infty} W_{4i+1} \Big) \quad D_1 = B_1 \setminus \Big(\bigcup_{i=0}^{\infty} W_{4i+2} \Big) \quad C_2 = A_2 \setminus \Big(\bigcup_{i=0}^{\infty} W_{4i+3} \Big) \quad D_2 = B_2 \setminus \Big(\bigcup_{i=0}^{\infty} W_{4i+4} \Big)$$

First, suppose that $C_1 \neq \emptyset$. Since $G - A_2$ is a chain graph, so is $G' = G - A_2 - (A_1 \setminus C_1)$. Thus there exists $u \in C_1$ such that $N_{G'}(u) \subseteq N_{G'}(a)$ for all $a \in C_1$. In fact, $N_G(u) \subseteq N_G(a)$ for all $a \in C_1$, since $N_G(u) \subseteq B_1 \cup B_2 \subseteq V(G')$. Since $u \in C_1 = A_1 \setminus (\bigcup_{i=0}^{\infty} W_{4i+1})$, we have $u \notin W_{4i+1}$ for all $i \geq 0$. This implies that u has a neighbour in $B_2 \setminus (W_4 \cup W_8 \cup \cdots \cup W_{4i})$ for all $i \geq 0$, for otherwise we would have included it in W_{4i+1} for some i. Thus u has a neighbour in $B_2 \setminus (\bigcup_{i=0}^{\infty} W_{4i+4}) = D_2$. Let $x \in D_2$ be any such neighbour. Since $x \in D_2$, note that $x \notin W_{4i+4}$ for all $i \geq 0$. So x has a non-neighbour in $A_2 \setminus (W_3 \cup W_7 \cup \cdots \cup W_{4i+3})$ for all $i \geq 0$, otherwise we would have included it in W_{4i+4} for some i. Thus x has a non-neighbour in $A_2 \setminus (\bigcup_{i=0}^{\infty} W_{4i+3}) = C_2$. Let $v \in C_2$ be any such non-neighbour. Since $v \in C_2$, we have that $v \notin W_{4i+3}$ for all $i \geq 0$. So v has a neighbour in $B_1 \setminus (W_2 \cup W_6 \cup \cdots \cup \cdots W_{4i+2})$ for all $i \geq 0$, otherwise we would have included it in W_{4i+3} for some i. Thus v has a neighbour in $B_1 \setminus (\bigcup_{i=0}^{\infty} W_{4i+2}) = D_1$. Let $y \in D_1$ be any such neighbour. Since $y \in D_1$, we have that $y \notin W_{4i+2}$ for all $i \geq 0$. So y has a non-neighbour in $A_1 \setminus (W_1 \cup W_5 \cup \cdots \cup W_{4i+1})$ for all $i \geq 0$, otherwise we would have included it in W_{4i+2} for some i. Thus y has a non-neighbour in $A_1 \setminus (\bigcup_{i=0}^{\infty} W_{4i+1}) = C_1$. Let $a \in C_1$

be any such non-neighbour. Recall that also $u \in C_1$, and by the choice of u, we have $N(u) \subseteq N(a)$. Therefore also u is a non-neighbour of y.

Altogether, we have $u \in A_1$, $v \in A_2$, $x \in B_2$ and $y \in B_1$ where $x \in N(u) \setminus N(v)$ and $y \in N(v) \setminus N(u)$. This means that ($\star$) fails for (A_1, A_2, B_1, B_2). But we assume that ($\star$) holds for (A_1, A_2, B_1, B_2).

Therefore, we conclude that $C_1 = \emptyset$. We show that this implies that also each of C_2, D_1, D_2 is empty. Indeed, if there exists $y \in D_1$, then (repeating the argument from the above paragraph) we conclude that y has a non-neighbour in C_1. However, C_1 is empty. Thus we deduce that also D_1 is empty. Next, if there exists $v \in C_2$, we conclude that v has a neighbour in D_1. But D_1 is empty, so also C_2 must be. Finally, if there is $x \in D_2$, then x has a non-neighbour in C_2, but C_2 is empty. Thus D_2 is empty. This proves that each of the sets C_1, C_2, D_1, D_2 is empty, and hence $V(G) = \bigcup_{i=1}^{\infty} W_i$ as promised.

This proves Claim 9.3.

We further notice that the way the vertices are assigned to earliest possible sets W_i, the construction guarantees the following useful properties. See Figure 5 for a depiction of these properties.

Claim 9.4: *For all $k \in \{1, 2, \ldots\}$:*

(a) *each $x \in W_{2k+1}$ has a neighbour in W_{2k},*
(b) *each $y \in W_{2k}$ has a non-neighbour in W_{2k-1},*
(c) *W_{2k+1} is complete to $W_2 \cup W_4 \cup \cdots \cup W_{2k-2}$,*
(d) *W_{2k} is anticomplete to $W_1 \cup W_3 \cup \cdots \cup W_{2k-3}$.*

To prove (a), consider $x \in W_{2k+1}$ where $k \geq 1$.

Suppose first that k is even, i.e. $k = 2i$ for some $i \geq 1$. In other words, $x \in W_{4i+1}$ and the definition of W_{4i+1} gives us that $x \in A_1 \setminus (W_1 \cup W_5 \cup \cdots \cup W_{4i-3})$ and $N(x) \subseteq B_1 \cup (W_4 \cup W_8 \cup \cdots \cup W_{4i})$. Since x was not put in any of the sets $W_1, W_5, \ldots, W_{4i-3}$, it follows that $N(x) \not\subseteq B_1 \cup (W_4 \cup W_8 \cup \cdots \cup W_{4i'})$ for all $i' < i$. From this we conclude that x has a neighbour in W_{4i}. So x has a neighbour in W_{2k}, since $k = 2i$.

A similar argument works if k is odd, i.e. if $k = 2i + 1$ for some $i \geq 0$. Here $x \in W_{4i+3}$ which implies that $x \in A_2 \setminus (W_3 \cup W_7 \cup \cdots \cup W_{4i-1})$ and $N(x) \subseteq B_2 \cup (W_2 \cup W_6 \cup \cdots \cup W_{4i+2})$. Since x was not put in any of the sets $W_3, W_7, \ldots, W_{4i-1}$, we conclude that $N(x) \not\subseteq B_2 \cup (W_2 \cup W_6 \cup \cdots \cup W_{4i'+2})$ for all $i' < i$. We conclude that x has a neighbour in W_{4i+2}. So, as required, x has a neighbour in is W_{2k}, since $k = 2i + 1$. This proves (a).

The proof of (b) is analogous. Consider $y \in W_{2k}$ where $k \geq 1$. If $k = 2i+1$ for some $i \geq 0$, then $y \in W_{4i+2}$ and so $y \in B_1 \setminus (W_2 \cup W_6 \cup \cdots \cup W_{4i-2})$ and $N(y) \supseteq A_1 \setminus (W_1 \cup W_5 \cup \cdots \cup W_{4i+1})$. Since y is not in any of the sets $W_2, W_6, \ldots, W_{4i-2}$, we conclude that $N(y) \not\supseteq A_1 \setminus (W_1 \cup W_5 \cup \cdots \cup W_{4i'+1})$ for all $i' < i$. Thus y must have a non-neighbour in $W_{4i+1} = W_{2k-1}$, as required. Similarly, if $k = 2i + 2$ for $i \geq 0$, then $y \in B_2 \setminus (W_4 \cup W_8 \cup \cdots \cup W_{4i})$ and $N(y) \supseteq A_2 \setminus (W_3 \cup W_7 \cup \cdots \cup W_{4i+3})$. Thus $N(y) \not\supseteq A_2 \setminus (W_3 \cup W_7 \cup \cdots \cup W_{4i'+3})$ for all $i' < i$, and so y must have a non-neighbour in $W_{4i+3} = W_{2k-1}$, as required. This proves (b).

To prove (c), consider smallest index k for which W_{2k+1} is not complete to $W_2 \cup W_4 \cup \cdots \cup W_{2k-2}$. Then there exists $u \in W_{2k+1}$ and $v \in W_{2j}$ where $j \leq k - 1$ such that $uv \notin E(G)$. Clearly, $k \geq 2$. By (a) and since $k \geq 1$, we deduce that u has a neighbour w in W_{2k}. Similarly, by (b), w has a non-neighbour z in W_{2k-1}. Recall that $v \in W_{2j}$.

Suppose first that j is odd. Then $v \in B_1$ and $N(v) \supseteq A_1 \setminus (W_1 \cup W_5 \cup \cdots \cup W_{2j-1})$ by the definition of W_{2j}. Since $u \notin N(v)$ and $u \in W_{2k+1}$ where $k \geq j + 1$, we deduce that $u \in A_2$. This implies that k is odd. So $w \in B_1$ and $z \in A_1$, since $w \in W_{2k}$ and $z \in W_{2k-1}$. Now, since both j and k are odd while $j \leq k - 1$, we deduce that $j \leq k - 2$. Thus the minimality of k implies that W_{2k-1} is complete to W_{2j}. In particular, we have that z is adjacent to v. However, then we have $v, w \in B_1$, $z \in A_1$, $u \in A_2$ where $uw, vz \in E(G)$ while $uv, zw \notin E(G)$. In addition, $uz, vw \notin E(G)$, since $A_1 \cup A_2$ and B_1 are independent sets. This shows that the vertices u, v, z, w induce a copy of $2K_2$ in $G - B_2$ which is therefore not a chain graph. But then (A_1, A_2, B_1, B_2) is not a bichain partition.

Similarly, if j is even, then $v \in B_2$ and $N(v) \supseteq A_2 \setminus (W_3 \cup W_7 \cup \cdots \cup W_{2j-1})$. Thus $u \in A_1$, since $u \notin N(v)$ and $u \in W_{2k+1}$ where $k \geq j + 1$. This implies that k is even and so $w \in B_2$ and $z \in A_2$, since $w \in W_{2k}$ and $z \in W_{2k-1}$. In addition, we deduce $j \leq k - 2$, since both j and k are even. Thus $zv \in E(G)$ by the minimality of k. We conclude that $v, w \in B_2$, $u \in A_1$, $z \in A_2$, and so u, v, z, w induce a copy of $2K_2$ in $G - B_1$, a contradiction.

This proves (c).

The proof of (d) is analogous. Consider smallest k for which W_{2k} is not anticomplete to $W_1 \cup W_3 \cup \cdots \cup W_{2k-3}$. Then there exists $v \in W_{2k}$ adjacent to some $u \in W_{2j-1}$ where $j \leq k - 1$. Clearly, $k \geq 2$. By (b), v is non-adjacent to

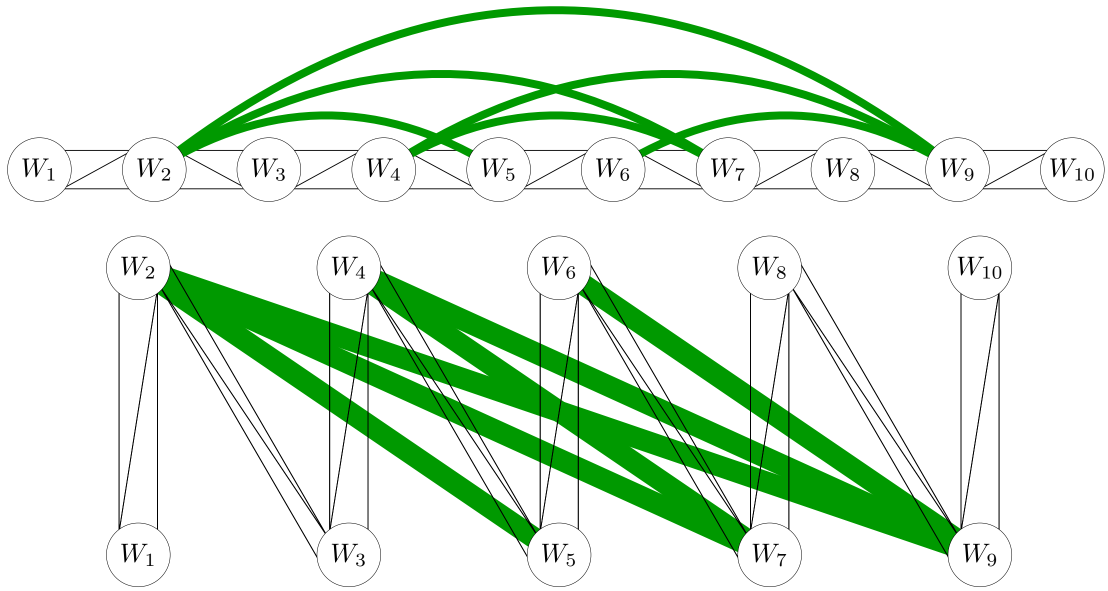


Figure 5: Decomposition of a bichain graph (two ways of drawing it).

some $w \in W_{2k-1}$, and by (a), w is adjacent to some $z \in W_{2k-2}$. Recall that $u \in W_{2j-1}$. Suppose first that j is even. Then $u \in A_2$ and $N(u) \subseteq B_2 \cup (W_2 \cup W_6 \cup \cdots \cup W_{2j-2})$. This yields that $v \in B_2$, since $v \in N(u)$ and $v \in W_{2k}$ where $k \geq j+1$. Therefore, k is even, and hence, $w \in A_2$ and $z \in B_1$, since $w \in W_{2k-1}$ and $z \in W_{2k-2}$. Moreover, $j \leq k-2$, since both j and k are even. Thus $uz \notin E(G)$ by the minimality of k. Together, we have $u, w \in A_2$, $z \in B_1$, $v \in B_2$, and u, v, w, z induce a $2K_2$ in $G - A_1$, a contradiction. Similarly if j is odd. In that case, $u \in A_1$ and $N(u) \subseteq B_1 \cup (W_4 \cup W_8 \cup \cdots \cup W_{2j-2})$. Thus $v \in B_1$, since $v \in N(u)$ and $v \in W_{2k}$ where $k \geq j+1$. It follows that k is odd. So $w \in A_1$, $z \in B_2$, and $j \leq k-2$, since also j is odd. Therefore, $uz \notin E(G)$ by the minimality of k. Together, $u, w \in A_1$, $v \in B_1$, $z \in B_2$, and u, v, z, w induce a $2K_2$ in $G - A_2$, a contradiction.

This proves Claim 9.4.

Claim 9.5: $V(G) = W_1 \cup W_2 \cup \cdots \cup W_n$

To see this, note first that, by Claim 9.3, we have $V(G) = \bigcup_{i=1}^{\infty} W_i \supseteq W_1 \cup W_2 \cup \cdots \cup W_n$. Thus, for contradiction, suppose that $\bigcup_{i=1}^{n} W_i$ is a proper subset of $V(G)$. In other words, assume that $n > |\bigcup_{i=1}^{n} W_i| = \sum_{i=1}^{n} |W_i|$. This implies that there exists $k \in \{1, \ldots, n\}$ such that W_k is empty. We claim that W_j for each $j \geq k+1$ is also empty.

For contradiction, consider smallest $j \geq k+1$ such that W_j is non-empty, i.e. W_j contains some vertex x. Note that $j \geq 2$, since $k \geq 1$. Thus if j is odd, then we deduce, by Claim 9.4(a), that x has a neighbour in W_{j-1}. In particular, W_{j-1} is non-empty. Similarly, if j is even, then x has a non-neighbour in W_{j-1} by Claim 9.4(b), and so W_{j-1} is non-empty. Thus we conclude that $j - 1 \neq k$, since W_k is empty. This implies that $j - 1 \geq k + 1$ and W_{j-1} is non-empty, which contradicts the minimality of j.

So we conclude that no such index j exists, and hence, $V(G)$ is equal to $\bigcup_{i=1}^{k} W_i$ which is a subset of $\bigcup_{i=1}^{n} W_i$, since $k \in \{1, \ldots n\}$. But we assume that $\bigcup_{i=1}^{n} W_i$ is a proper subset of $V(G)$, a contradiction.

This proves Claim 9.5.

We are ready to describe how to define an isomorphism of G to $Z_{n,n}$. To this end, we consider the partition $W_1, \ldots, W_n$ of G as described above. Recall that (A_1, A_2, B_1, B_2) is a special bichain partition. Thus by Lemma 4, there exists $0 \leq \alpha \leq n$ such that the system ($\triangle$) has a solution $z_u = z_u^*$, $u \in V(G)$.

In order to show that G is isomorphic to an induced subgraph of $Z_{n,n}$, we map, for each i, the vertices of W_i to the i-th column of $Z_{n,n}$. The position inside the columns will be dictated by the values z_u^*.

For each $i \in \{1, \ldots, n\}$ and each $u \in W_i$, we define the height h_u of u as follows:

$$h_u = z_u^* + (n - \lceil i/2 \rceil) \cdot \alpha.$$

We order the vertices of $V(G)$ based on their height h_u (ties broken arbitrarily). In other words, we fix an ordering $u_1, u_2, \ldots, u_n$ of $V(G)$ in which $h_{u_j} \leq h_{u_{j'}}$ whenever $j \leq j'$. Using this ordering, we define a mapping f of $V(G)$ into $V(Z_{n,n})$ as follows: for each $i \in \{1, \ldots, n\}$, we consider each $u_j \in W_i$ and define $f(u_j) = v_{ij}$.

Clearly, the mapping f is a well-defined mapping into $V(Z_{n,n})$, since $i, j \in \{1, \ldots, n\}$. Moreover, f is an injective mapping, since each vertex u_j is mapped to the j-th row of $Z_{n,n}$. In particular, the image of f induces in $Z_{n,n}$ a row-sparse subgraph of $Z_{n,n}$. Thus to finish the proof, it remains to show that f is an isomorphism. In other words, it remains to show that for distinct j, j', we have $u_j u_{j'} \in E(G)$ if and only if $f(u_j)f(u_{j'}) \in E(Z_{n,n})$.

Consider $j, j' \in \{1, \ldots, n\}$ where $j \neq j'$. Let i, i' be indices such that $u_j \in W_i$ and $u_{j'} \in W_{i'}$. Then we have $f(u_j) = v_{ij}$ and $f(u_{j'}) = v_{i'j'}$. Thus we need to that $u_j u_{j'} \in E(G)$ if and only if $v_{ij} v_{i'j'} \in E(Z_{n,n})$.

Suppose first that i, i' are both odd or both even. Then $u_j, u_{j'} \in A_1 \cup A_2$ or $u_j, u_{j'} \in B_1 \cup B_2$, and $v_{ij}, v_{i'j'}$ are either in the same column, or in two odd-numbered columns, or in two even-numbered columns of $Z_{n,n}$. By the definition of $Z_{n,n}$, there are no edges in and between such columns. Therefore, we conclude that $v_{ij} v_{i'j'} \notin E(Z_{n,n})$. Moreover, both $A_1 \cup A_2$ and $B_1 \cup B_2$ are independent sets. Thus we have that $u_j u_{j'} \notin E(G)$, as required.

Therefore, by symmetry, we may assume that i is odd and i' is even. If $i' \leq i - 3$, then $u_j u_{j'} \in E(G)$ by Claim 9.4(c), since i is odd while i' is even. For this reason, also $v_{ij} v_{i'j'} \in E(Z_{n,n})$ by the definition of $Z_{n,n}$. Similarly, if $i' \geq i + 3$, then $u_j u_{j'} \notin E(G)$ by Claim 9.4(d), and $v_{ij} v_{i'j'} \notin E(Z_{n,n})$ by the definition of $Z_{n,n}$.

Thus we may assume that $i' = i \pm 1$. First suppose that $i' = i + 1$. This implies that $u_j \in A_r$ and $u_{j'} \in B_r$ for some $r \in \{1, 2\}$. Moreover, since i' is even and i is odd, we have $i'/2 = \lceil i/2 \rceil$. Suppose that $u_j u_{j'} \in E(G)$. Then $z^*_{u_{j'}} - z^*_{u_j} \leq -1$, because the values z^*_u are a solution to ($\triangle$). So $z^*_{u_{j'}} < z^*_{u_j}$ which implies that $h_{u_{j'}} < h_{u_j}$, since $\lceil i'/2 \rceil = i'/2 = \lceil i/2 \rceil$. Therefore, from the definition of the ordering $u_1, \ldots, u_n$, we deduce that $j' < j$. Therefore, $v_{ij} v_{i'j'} \in E(Z_{n,n})$ by the definition of $Z_{n,n}$, since $i' = i + 1$ and i is odd while i' is even.

Conversely, suppose that $u_j u_{j'} \notin E(G)$. Then $z^*_{u_j} - z^*_{u_{j'}} \leq -1$, because the values z^*_u are a solution to ($\triangle$). So $z^*_{u_j} < z^*_{u_{j'}}$ which implies $h_{u_j} < h_{u_{j'}}$. Thus $j < j'$ and so $v_{ij} v_{i'j'} \notin E(Z_{n,n})$, as required.

It remains to consider $i' = i - 1$. Since i is odd while i' is even, this implies that $u_j \in A_r$ and $u_{j'} \in B_s$ for $r, s \in \{1, 2\}$ where $r \neq s$. Moreover, we have $i'/2 = \lceil i/2 \rceil - 1$. Suppose first that $u_j u_{j'} \in E(G)$. Then $z^*_{u_{j'}} - z^*_{u_j} \leq -\alpha - 1$, since the values z^*_u are a solution to ($\triangle$). Thus $z^*_{u_{j'}} < z^*_{u_j} - \alpha$ and so $h_{u_{j'}} < h_{u_j}$ as follows:

$$h_{u_{j'}} = z^*_{u_{j'}} + (n - i'/2) \cdot \alpha < z^*_{u_j} + (n - i'/2 - 1) \cdot \alpha = z^*_{u_j} + (n - \lceil i/2 \rceil) \cdot \alpha = h_{u_j}$$

Thus $j' < j$ and we conclude that $v_{ij} v_{i'j'} \in E(Z_{n,n})$ by the definition of $Z_{n,n}$.

Conversely, suppose that $u_j u_{j'} \notin E(G)$. Then $z^*_{u_j} - z^*_{u_{j'}} \leq \alpha - 1$, since the values z^*_u satisfy ($\triangle$). Thus $z^*_{u_j} < z^*_{u_{j'}} + \alpha$ and so $h_{u_j} < h_{u_{j'}}$, since $h_{u_j} = z^*_{u_j} + (n - \lceil i/2 \rceil) \cdot \alpha < z^*_{u_{j'}} + (n - \lceil i/2 \rceil + 1) \cdot \alpha = h_{u_{j'}}$. We conclude that $j < j'$ and hence $v_{ij} v_{i'j'} \notin E(Z_{n,n})$, as required.

This completes the proof of Theorem 9. □

6. Concluding remarks

In the present paper, we proved a number of results about bichain graphs, the bipartite analog of split permutation graphs. In particular, we developed a geometric representation for bichain graphs and constructed a quadratic n-universal graph for this class. Among various open problems related to bichain and split permutation graphs, let us mention the conjecture from [10] asking whether split permutation (and hence bichain) graphs constitute a *minimal* hereditary class of graphs of unbounded clique-width. The results obtained in this paper suggest the following approach to the above question.

In [15], it was shown that graphs in a hereditary class have bounded clique-width if and only if they have bounded rank-width. Also, in [14] it was shown that bipartite graphs of large rank-width contain a large universal bipartite permutation graph as a vertex minor. Vertex minors are defined in terms of vertex deletions and local complementations. Local complementation is the operation of complementing the edges in the neighbourhood of a vertex. The importance of this operation is due to the fact that it does not change the rank-width of a graph. Therefore, a possible approach to proving minimality of bichain graphs could be to transform a universal bichain graph into a universal bipartite permutation graph via a sequence of local complementations. While bipartite graphs are not closed under local complementation, circle graphs are. Both bichain graphs are circle graphs (by Corollary 6) and it is well-known that permutation graphs are circle graphs, so the sequence of local complementations from a universal bichain graph to a universal bipartite permutation will all happen within the class of circle graphs. Moreover, for circle graphs the operation of local complementation has a nice geometric interpretation: the local complementation applied at a vertex x of a circle graph corresponds to cutting the circle along the chord representing x and turning over one of the semicircles along this chord. This may suggest a geometric approach to transforming bichain graphs into bipartite permutation graphs and vice versa. A more challenging task is to show that this transformation is possible within 4-polygon graphs, as both bichain graphs and bipartite permutation graphs are subclasses of this class. We leave this challenging task for future research.

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