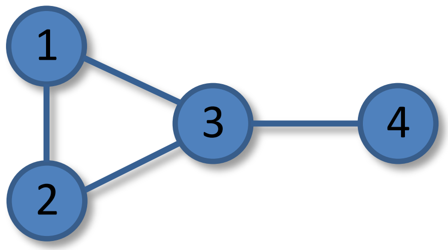
Graph

- *graph* is a pair (V, E) of two sets where
  - V = set of elements called *vertices* (singl. vertex)
  - E = set of pairs of vertices (elements of V) called *edges*

- Example: $G = (V, E)$ where

$$V = \{ 1, 2, 3, 4 \}$$
$$E = \{ \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\} \}$$



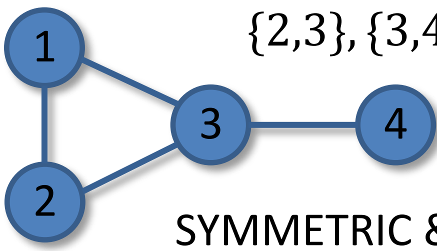
drawing of G

- Notation:
  - we write $G = (V, E)$ for a graph with vertex set V and edge set E
  - $V(G)$ is the vertex set of G, and $E(G)$ is the edge set of G

Types of graphs

- Undirected graph

$E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{3,4\}\}$



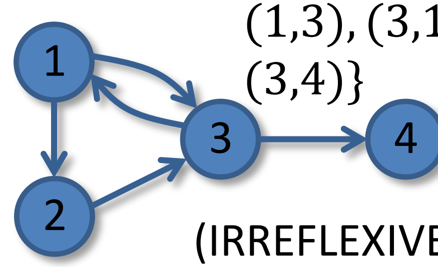
	1	2	3	4
1		1	1	
2	1		1	
3	1	1		1
4			1	

SYMMETRIC & IRREFLEXIVE

$E(G)$ = set of ***unordered*** pairs

- Directed graph

$E = \{(1,2), (2,3), (1,3), (3,1), (3,4)\}$



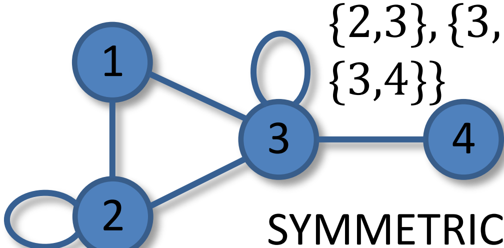
	1	2	3	4
1		1	1	
2			1	
3	1			1
4				

(IRREFLEXIVE) RELATION

$E(G)$ = set of ***ordered*** pairs

- Pseudograph (allows *loops*)

$E = \{\{1,2\}, \{1,3\}, \{2,2\}, \{2,3\}, \{3,3\}, \{3,4\}\}$



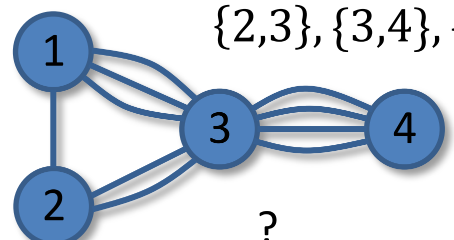
	1	2	3	4
1		1	1	
2	1	1	1	
3	1	1	1	1
4	0		1	

SYMMETRIC

loop = edge between vertex and itself

- Multigraph

$E = \{\{1,2\}, \{1,3\}, \{1,3\}, \{1,3\}, \{2,3\}, \{2,3\}, \{3,4\}, \{3,4\}, \{3,4\}, \{3,4\}\}$



?

	1	2	3	4
1		1	3	
2	1		2	
3	3	2		4
4			4	

$E(G)$ is a ***multiset***

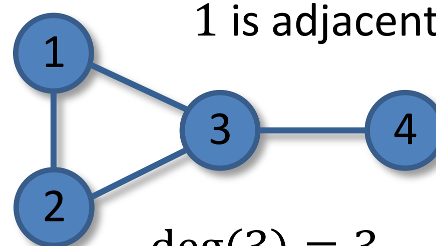
More graph terminology

- *simple* graph (undirected, no loops, no parallel edges)

for edge $\{u, v\} \in E(G)$ we say:

- u and v are **adjacent**
- u and v are **neighbours**
- u and v are **endpoints** of $\{u, v\}$
- we write $uv \in E(G)$ for simplicity

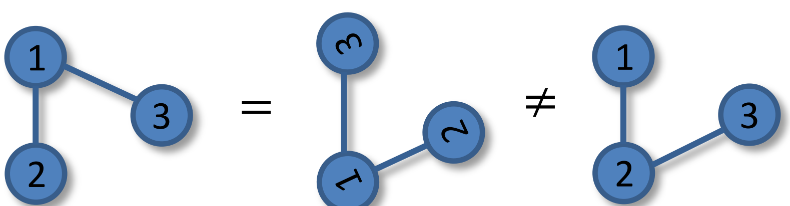
1 is adjacent to 2 and 3

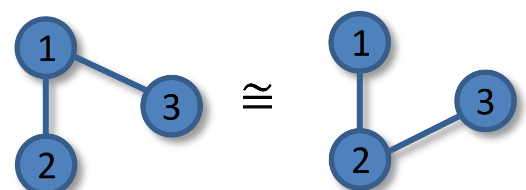


$\mathrm{N}(1) = \{2,3\}$
$\mathrm{N}(2) = \{1,3\}$
$\mathrm{N}(3) = \{1,2,4\}$
$\mathrm{N}(4) = \{3\}$

$\deg(3) = 3$
$\deg(4) = 1$
$\deg(1) = \deg(2) = 2$

- $\mathrm{N}(v)$ = set of **neighbours** of v in G
- $\deg(v)$ = **degree** of v is the number of its neighbours, ie. $|\mathrm{N}(v)|$

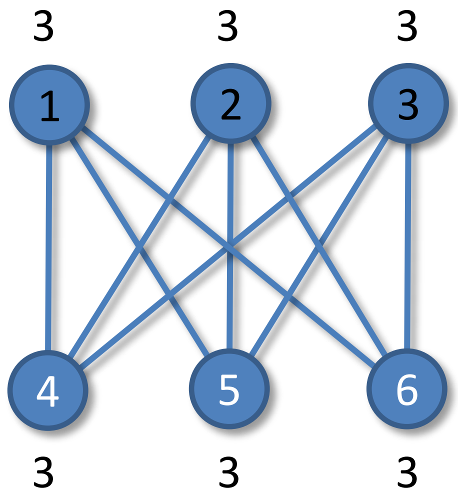




$f(1) = 2$
$f(2) = 3$
$f(3) = 1$

$G_1 = (V_1, E_1)$ is **isomorphic** to $G_2 = (V_2, E_2)$ if there exists a bijective mapping $f: V_1 \to V_2$ such that $uv \in E(G_1)$ if and only if $f(u)f(v) \in E(G_2)$

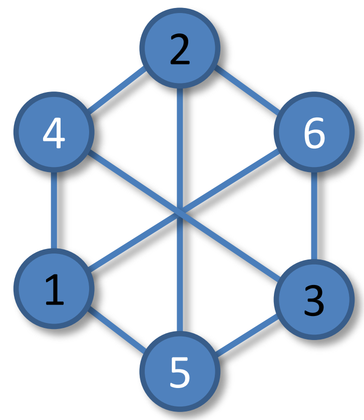
Isomorphism



(3,3,3,3,3,3)

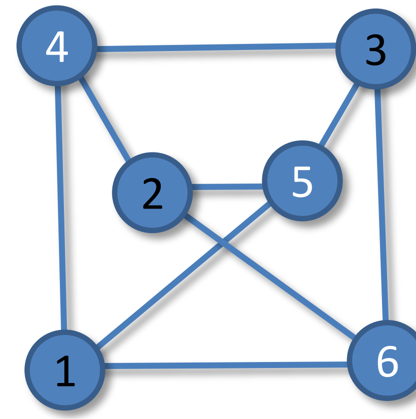
degree sequence

$\cong$



(3,3,3,3,3,3)

$\cong$

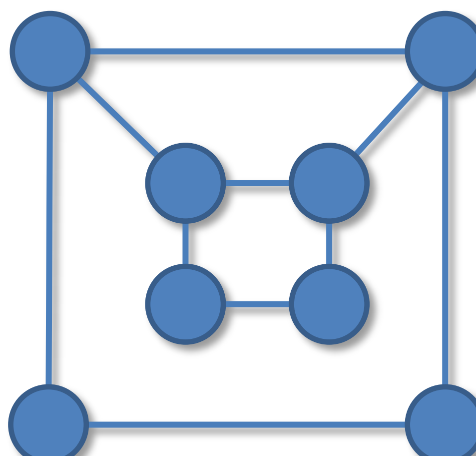


(3,3,3,3,3,3)

if f is an isomorphism

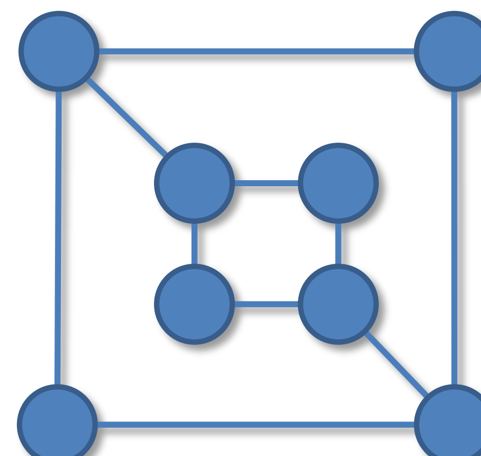
$\Rightarrow \quad \deg(u) = \deg(f(u))$

i.e., isomorphic graphs have
same degree sequences
only necessary not sufficient!!!



(3,3,3,3,2,2,2,2)

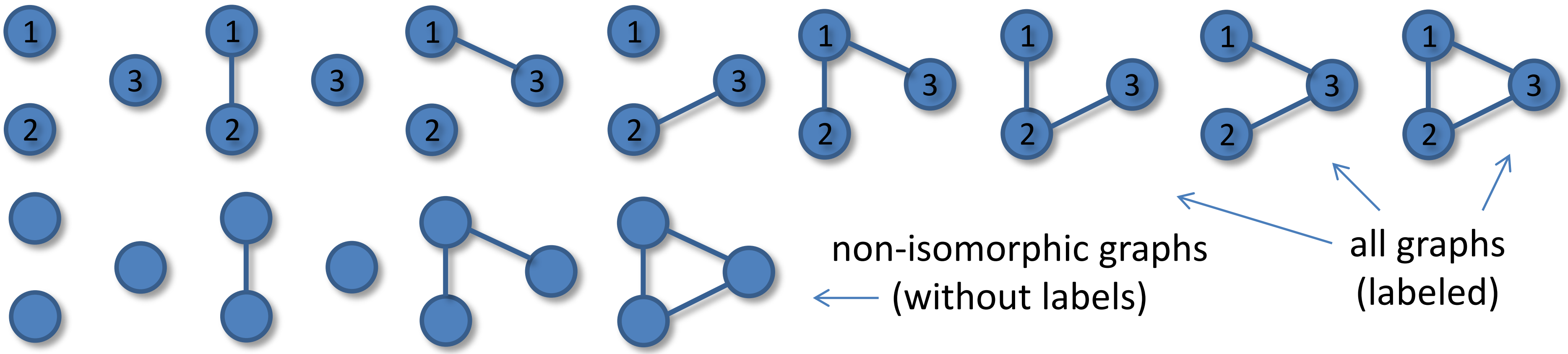
$\ncong$



(3,3,3,3,2,2,2,2)

Isomorphism

How many pairwise non-isomorphic graphs on n vertices are there?

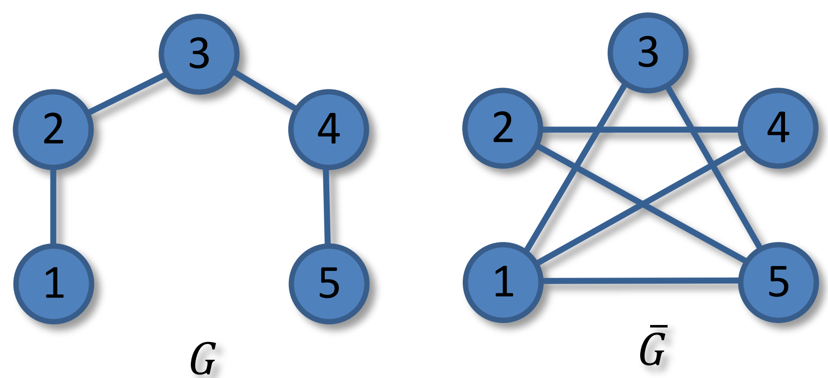


the **complement** of $G = (V, E)$ is the graph $\bar{G} = (V, \bar{E})$ where

$$\bar{E} = \{\, \{u, v\} \mid \{u, v\} \notin E \}$$

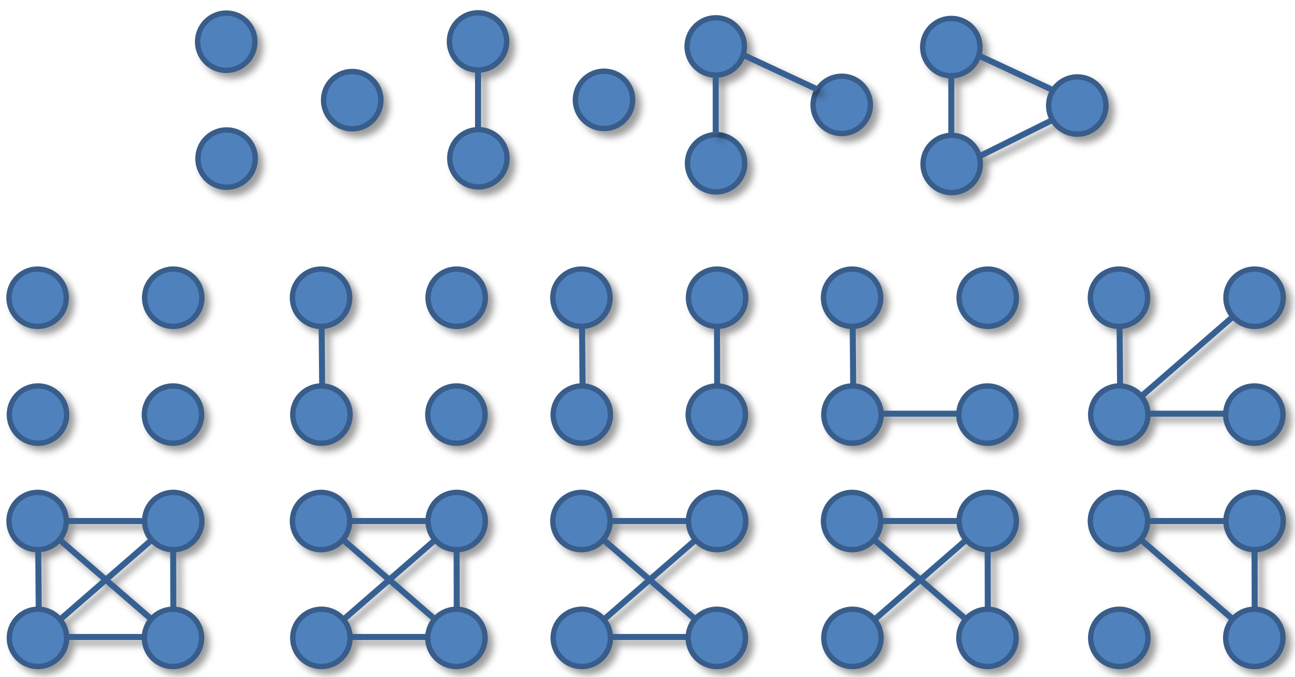
$E(G) = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}\}$

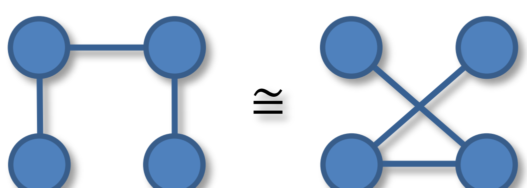
$E(\bar{G}) = \{\{1,3\}, \{1,4\}, \{1,5\}, \{2,4\},$
$\{2,5\}, \{3,5\}\}$



Isomorphism

How many pairwise non-isomorphic graphs on n vertices are there?



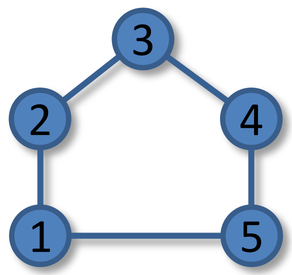


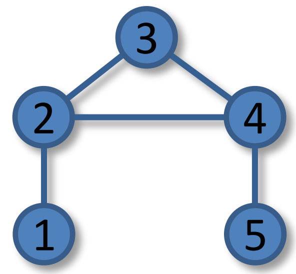
self-complementary graph

Isomorphism

- Are there self complementary graphs on 5 vertices ? Yes, 2

$1 \to 1, 2 \to 4, 3 \to 2$
$4 \to 5, 5 \to 3$





$1 \to 2, 2 \to 5, 3 \to 3$
$4 \to 1, 5 \to 4$

- ... 6 vertices ?

No, because in a self-complementary graph $G = (V, E)$

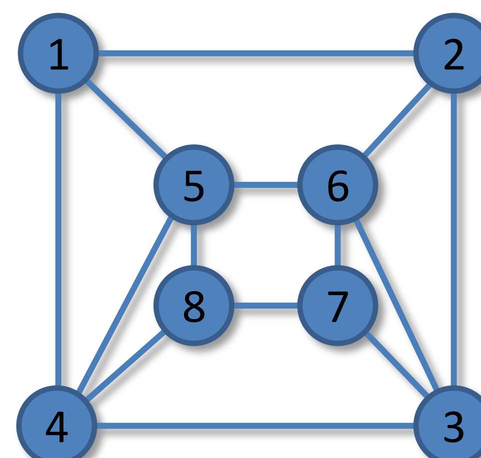
$|E| = |\bar{E}|$ and $|E| + |\bar{E}| = \binom{|V|}{2}$

but $\binom{6}{2} = 15$ is odd

- ... 7 vertices ? No, since $\binom{7}{2} = 21$
- ... 8 vertices ?

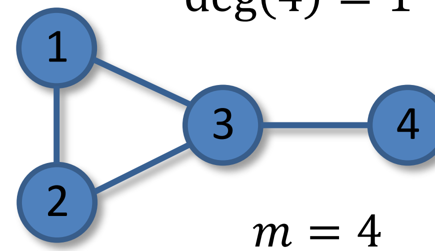
Yes, there are 10

$1 \to 4, 2 \to 6, 3 \to 1, 4 \to 7$
$5 \to 2, 6 \to 8, 7 \to 3, 8 \to 5$



Handshake lemma

$\deg(1) = 2$
$\deg(2) = 2$
$\deg(3) = 3$
$\deg(4) = 1$



$m = 4$

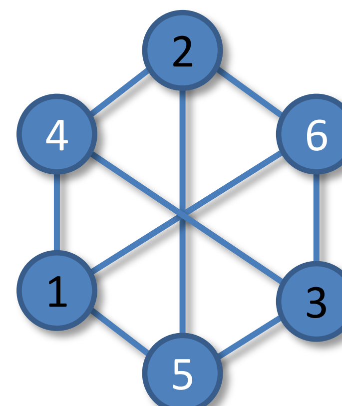
Lemma. Let $G = (V, E)$ be a graph with m edges.

$$\sum_{v \in V} \deg(v) = 2m$$

Proof. Every edge $\{u, v\}$ connects 2 vertices and contributes to exactly 1 to $\deg(u)$ and exactly 1 to $\deg(v)$. In other words, in $\sum_{v \in V} \deg(v)$ every edge is counted twice. □

How many edges has a graph with degree sequence:

- (3,3,3,3,3,3) ?
- (3,3,3,3,3) ?
- (0,1,2,3) ?



Corollary. In any graph, the number of vertices of odd degree is even.

Corollary 2. Every graph with at least 2 vertices has 2 vertices of the same degree.

Handshake lemma

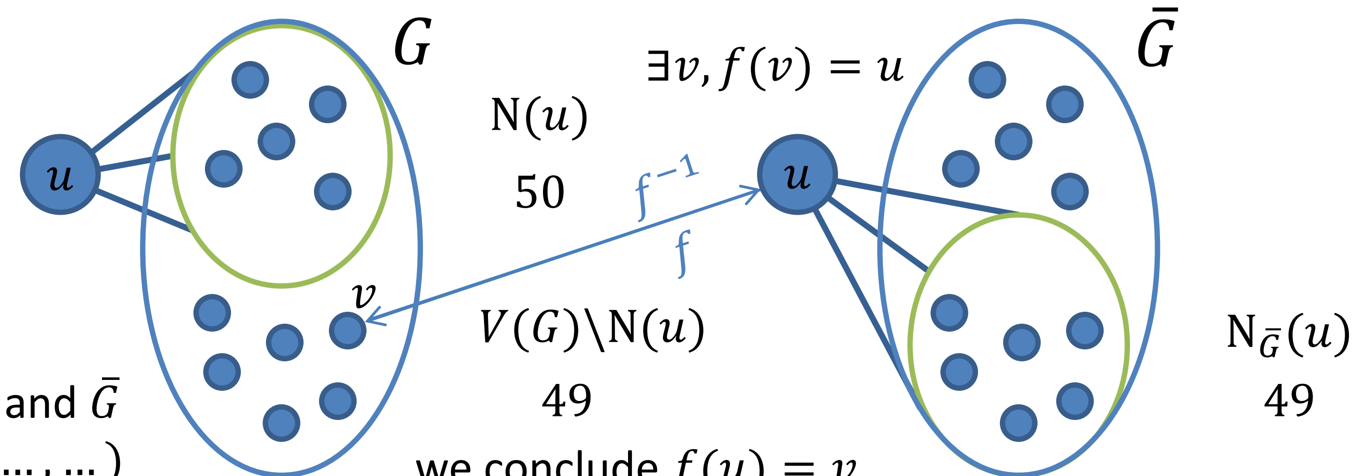
Is it possible for a self-complementary graph with 100 vertices to have exactly one vertex of degree 50 ?

Answer. No

f isomorphism between G and $\bar{G}$

v is a unique vertex of degree 49 in G

the degree seq. of G and $\bar{G}$
$(\dots, \dots, \dots, 50, 49, \dots, \dots, \dots)$



we conclude $f(u) = v$

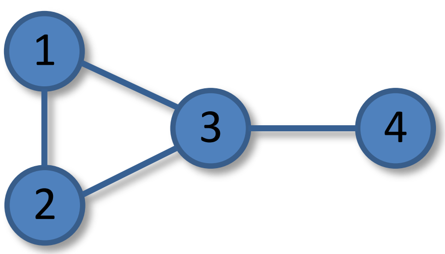
$\underbrace{uv \in E(G) \Leftrightarrow f(u)f(v) \in E(\bar{G})}_{\text{definition of isomorphism}}$ but $f(u)f(v) = \underbrace{vu \in E(\bar{G}) \Leftrightarrow uv \notin E(G)}_{\text{definition of complement}}$ □

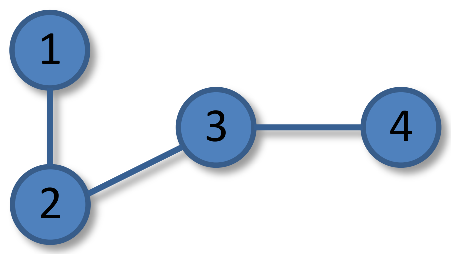
Subgraphs

Let $G = (V, E)$ be a graph

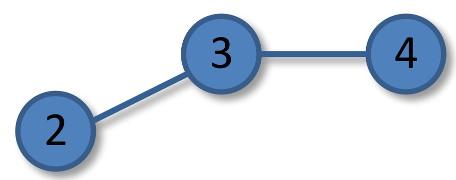
- a **subgraph** of G is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$
- an **induced subgraph** (a subgraph **induced** on $A \subseteq V$) is a graph

$$G[A] = (A, E_A) \text{ where } E_A = E \cap A \times A$$





subgraph



induced subgraph
on $A = \{2,3,4\}$

Walks, Paths and Cycles

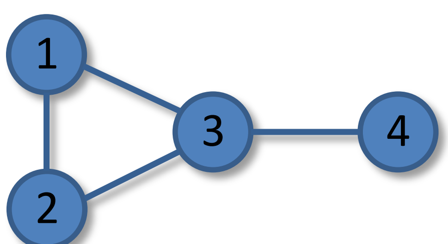
Let $G = (V, E)$ be a graph

- a **walk** (of length k) in G is a sequence

$$v_1, e_1, v_2, e_{2,} \dots, v_{k-1}, e_{k-1}, v_k$$

where $v_1, \dots, v_k \in V$ and $e_i = \{v_i, v_{i+1}\} \in E$ for all $i \in \{1..k-1\}$

- a **path** in G is a walk where $v_1, \dots, v_k$ are distinct
  (does not go through the same vertex twice)
- a **closed walk** in G is a walk with $v_1 = v_k$
  (starts and ends in the same vertex)
- a **cycle** in G is a closed walk where $k \geq 3$ and $v_1, \dots, v_{k-1}$ are distinct



1,{1,2},2,{2,1},1,{1,3},3
is a walk (not path)

1,{1,2},2,{2,3},3,{3,1},1
is a cycle

Special graphs

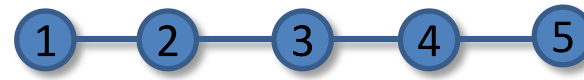
- P_n path on n vertices
- C_n cycle on n vertices
- K_n complete graph on n vertices
- B_n hypercube of dimension n

$$V(B_n) = \{0,1\}^n$$

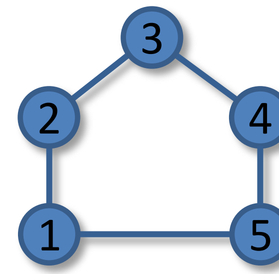
$(a_1, \ldots, a_n)$ adjacent to $(b_1, \ldots, b_n)$
if a_i's and b_i's differ in exactly once

$$\sum_{i=1}^{n} |a_i - b_i| = 1$$

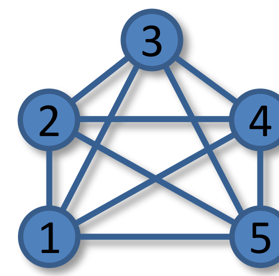
$(0,1,0,0)$ adjacent to $(0,1,0,1)$
not adjacent to $(0,1,1,1)$



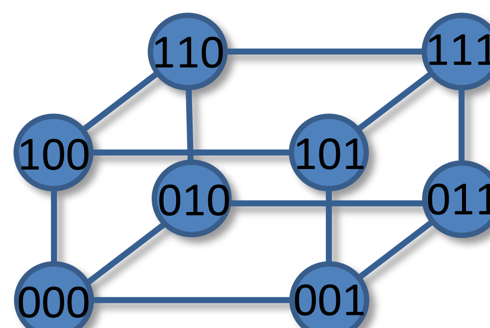
P_5



C_5



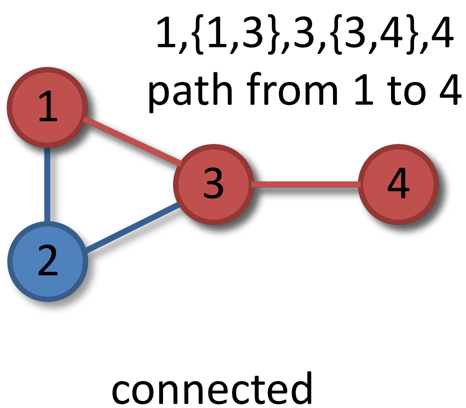
K_5

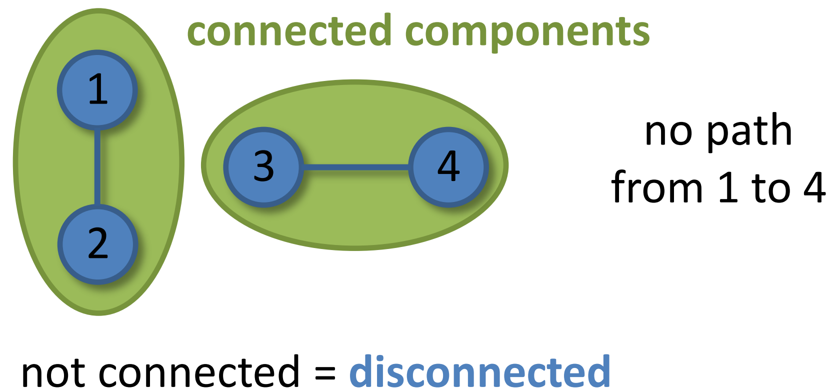


B_3

Connectivity

- a graph G is **connected** if for any two vertices u, v of G there exists a path (walk) in G starting in u and ending in v

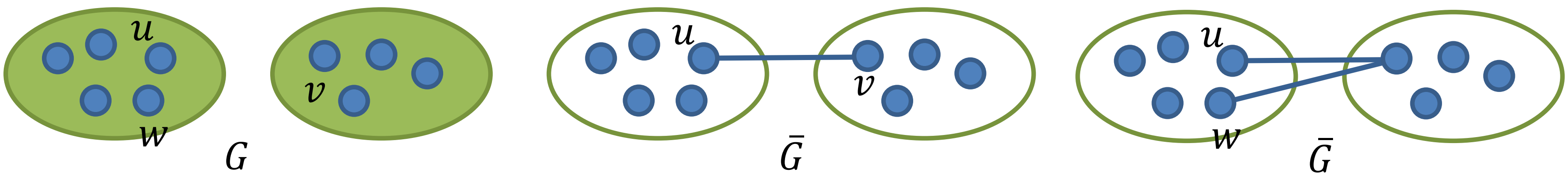




- a **connected component** of G is a maximal connected subgraph of G

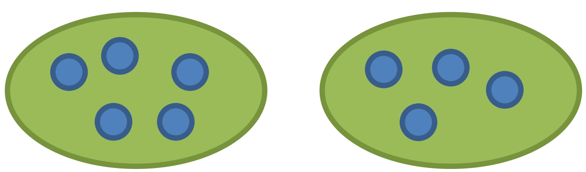
Connectivity

- Show that the complement of a disconnected graph is connected !



- What is the maximum number of edges in a disconnected graph ?

$\binom{k}{2}$



$\binom{n-k}{2}$

$$\max_k \binom{k}{2} + \binom{n-k}{2} = \binom{n-1}{2}$$

maximum when $k = 1$

- What is the minimum number of edges in a connected graph ?

path P_n on n vertices has $n - 1$ edges

cannot be less, why ? keep removing edges so long as you keep the graph connected $\Rightarrow$ a **(spanning) tree** which has $n - 1$ edges

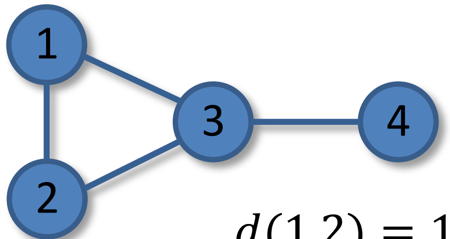
Distance, Diameter and Radius

recall walk (path) is a sequence of vertices and edges

$$v_1, e_1, v_2, e_2, \dots, v_{k-1}, e_{k-1}, v_k$$

if edges are clear from context we write $v_1, v_2, \dots, v_k$

- **length** of a walk (path) is the number of edges
- **distance** $d(u,v)$ from a vertex u to a vertex v is the length of a shortest path in G between u and v
- if no path exists define $d(u,v) = \infty$



$d(1,2) = 1$
$d(1,4) = 2$

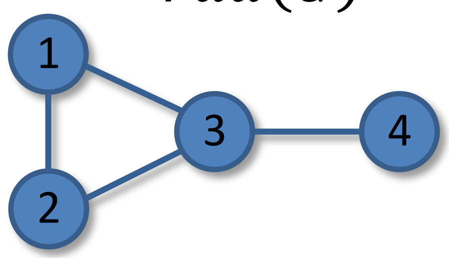
matrix of distances in G
distance matrix

$d(u,v)$	**1**	**2**	**3**	**4**
1	0	1	1	2
2	1	0	1	2
3	1	1	0	1
4	2	2	1	0

Distance, Diameter and Radius

- **eccentricity** of a vertex u is the largest distance between u and any other vertex of G; we write $ecc(u) = \max_v d(u, v)$
- **diameter** of G is the largest distance between two vertices
$$diam(G) = \max_{u,v} d(u, v) = \max_v ecc(v)$$
- **radius** of G is the minimum eccentricity of a vertex of G
$$rad(G) = \min_v ecc(v)$$

$rad(G) = 1$
(as witnessed by 3 called a **centre**)



$diam(G) = 2$
(as witnessed by 1,4 called a **diametrical pair**)

$ecc(1) = 2$
$ecc(2) = 2$
$ecc(3) = 1$
$ecc(4) = 2$

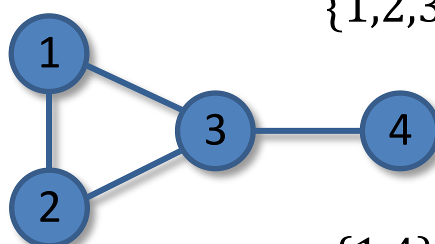
Find radius and diameter of graphs P_n, C_n, K_n, B_n

Prove that $rad(G) \leq diam(G) \leq 2 \cdot rad(G)$

Independent set and Clique

- **clique** = set of pairwise adjacent vertices
- **independent set** = set of pairwise non-adjacent vertices
- **clique number** $\omega(G)$ = size of largest clique in G
- **independence number** $\alpha(G)$ = size of largest independent set in G

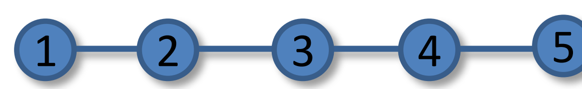
$\{1,2,3\}$ is a clique



$$\alpha(G) = 2$$
$$\omega(G) = 3$$

$\{1,4\}$ is an independent set

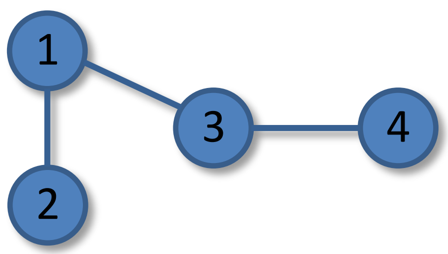
Find $\alpha(P_n), \alpha(C_n), \alpha(K_n), \alpha(B_n)$
$\omega(P_n), \omega(C_n), \omega(K_n), \omega(B_n)$



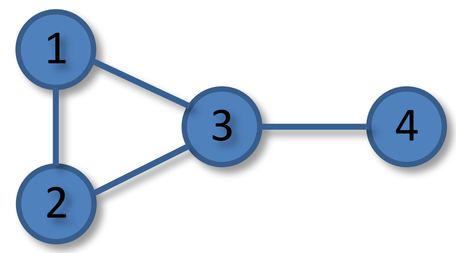
P_5

Bipartite graph

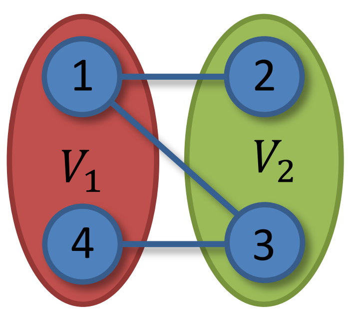
- a graph is **bipartite** if its vertex set V can be partitioned into two independent sets V_1, V_2;   i.e., $V_1 \cup V_2 = V$

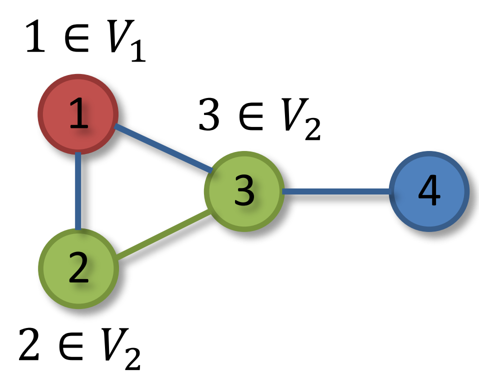


bipartite



not bipartite

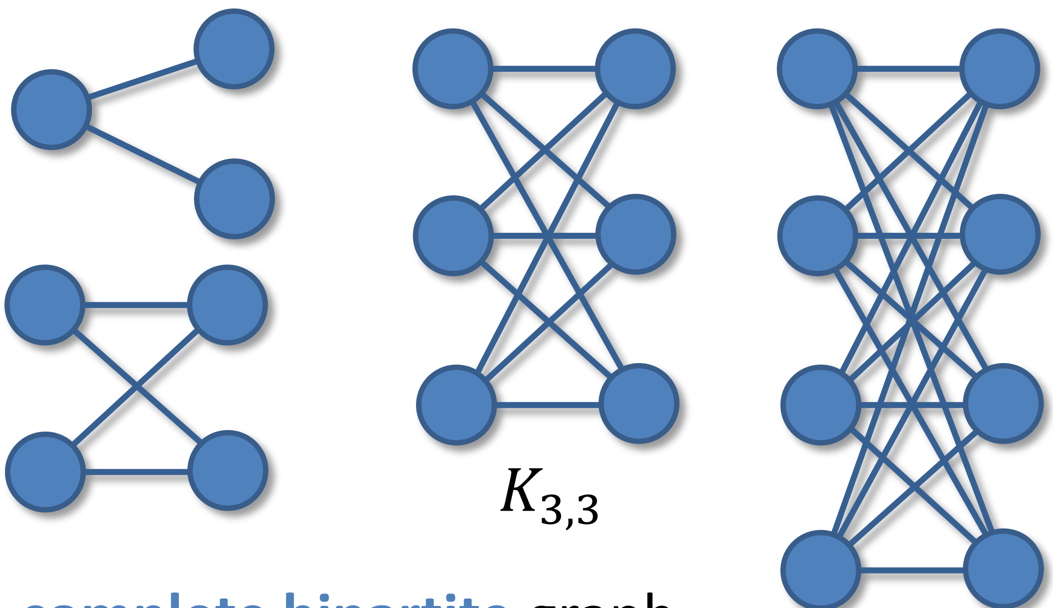




but now V_2 is not an independent set because $\{2,3\} \in E(G)$

Bipartite graph

- a graph is **bipartite** if its vertex set V can be partitioned into two independent sets V_1, V_2;   i.e., $V_1 \cup V_2 = V$



$K_{1,2}$ $K_{2,2}$ $K_{3,3}$ $K_{4,4}$

Which of these graphs are bipartite:
P_n, C_n, K_n, B_n ?

$K_{n,m}$ = **complete bipartite** graph

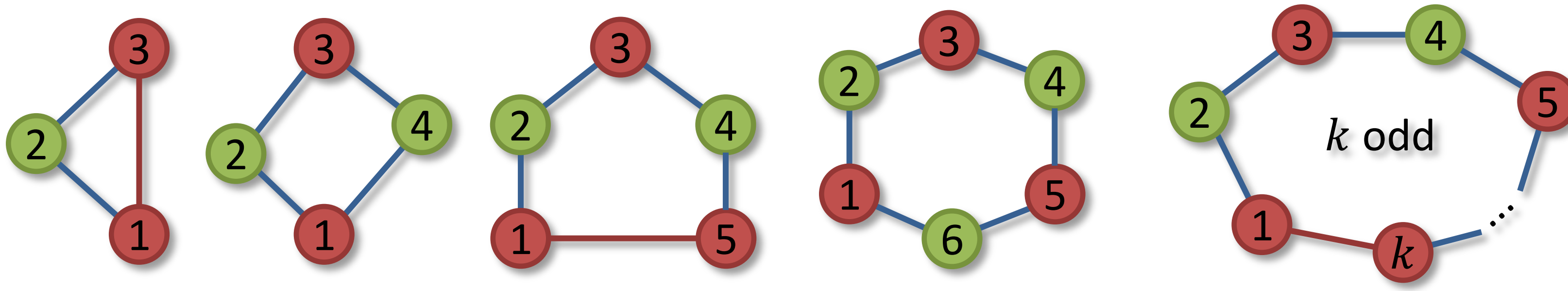
$$V(K_{n,m}) = \{a_1 \dots a_n , b_1 \dots b_m\}$$
$$E(K_{n,m}) = \{a_i b_j \mid i \in \{1..n\}, j \in \{1..m\}\}$$

Bipartite graph

Theorem. A graph is bipartite $\Leftrightarrow$ it has no cycle of odd length.

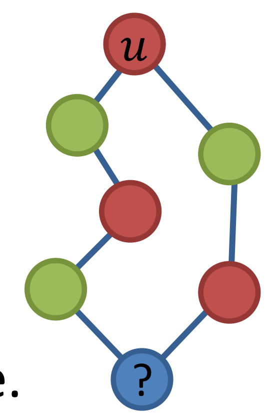
Proof. Let $G = (V, E)$. We may assume that G is connected.

"$\Rightarrow$" any cycle in a bipartite graph is of even length



"$\Leftarrow$" Assume G has no odd-length cycle. Fix a vertex u and put it in V_1.
Then repeat as long as possible:

- take any vertex in V_1 and put its neighbours in V_2, or
- take any vertex in V_2 and put its neighbours in V_1.



Afterwards $V_1 \cup V_2 = V$ because G is connected.

If one of V_1 or V_2 is not an independent set, then we find in G an odd-length closed walk $\Rightarrow$ cycle, impossible.

So, G is bipartite (as certified by the partition $V_1 \cup V_2 = V$) □