Synopsis of CLR, Chap 27, pp 579–600

Network Flows

A *Flow network* $\mathcal{F} = (G, c, s, t)$ consists of a bi-directed graph $G = (V, E)$ (i.e. if $(u, v) \in E$ then $(v, u) \in E$), and for each edge $(u, v) \in E$ there is a nonnegative capacity function $c(u, v) : E \to \mathbb{R}^{\geq 0}$, and two distingushed vertices s (the source) and t (the sink or target). Note: The terminology "flow network" is pretty standard but in fact it should be called a "capacity network" (or perhaps to be more grammatical, a "capacitated network") to stress the fact that we start with capacities and not flows.

We can assume (without loss of generality) that $\forall v \in V$, there is a path $s \leadsto v \leadsto t$ and we can also assume that $\forall (u, v) \in E$ that either $c(u, v) > 0$ or $c(v, u) > 0$ (or both) and finally we assume that there are no self loops.

A *flow* in G is a function $f : E \to \mathbb{R}$ satisfying:

1. capacity constraint: $f(u, v) \leq c(u, v)$
2. skew symmetry: $f(u, v) = -f(v, u)$
3. flow conservation: $\forall u \neq s, t \quad \sum_{v \in V} f(u, v) = 0$

The *value* of a flow f is $|f| = \sum_{v \in V} f(s, v)$

Maximum flow problem
Given a flow network (G, c, s, t)
Find a flow of maximum value from s to t.

Residual Networks

Let $\mathcal{F}$ be a flow network, f a flow. For any $(u, v) \in E$, the *residual capacity* of (u, v) is $c_f(u, v) = c(u, v) - f(u, v)$.

The *residual graph* of G induced by f is

$$G_f = (V, E_f)$$

where

$$E_f = \{(u,v) \in E \mid c_f(u,v) > 0\}$$

The flow f also gives rise to the residual flow network $\mathcal{F}_f = (G, \bar{c}_f, s, t)$ where $\bar{c}_f(u,v) = c_f(u,v)$ for $(u,v) \in E_f$ and 0 otherwise (i.e. for $(u,v) \in E - E_f$).

A residual network $\mathcal{F}_f$ is itself a flow network with capacities $\bar{c}_f$.

Lemma 27.2 (relating a flow in the residual flow network to the given flow network) Let $\mathcal{F} = (G, c, s, t)$ be a flow network and let f be a flow in G. Let $\mathcal{F}_f$ be the residual network of induced by f and let f' be a flow in $\mathcal{F}_f$. Then the flow sum

$$(f + f')(u,v) \stackrel{d}{=} f(u,v) + f'(u,v)$$

is a flow in G with value

$$|f + f'| = |f| + |f'|.$$

Proof Check skew symmetry, capacity constraint, flow conservation

$(f + f')(u,v) = f(u,v) + f'(u,v) = -((f(v,u) + f'(v,u))$ skew symmetry

$(f + f')(u,v) = f(u,v) + f'(u,v) \leq f(u,v) + (c(u,v) - f(u,v)) = c(u,v)$ capacity contraint

Augmenting Paths

Given $G = (V, E)$ and a flow f. An *augmenting path* π is a simple path from s to t in the residual graph, G_f. Note that every edge in G_f has positive capacity.

The maximum amount of net flow we can ship along edges of π is called the *residual capacity* of π

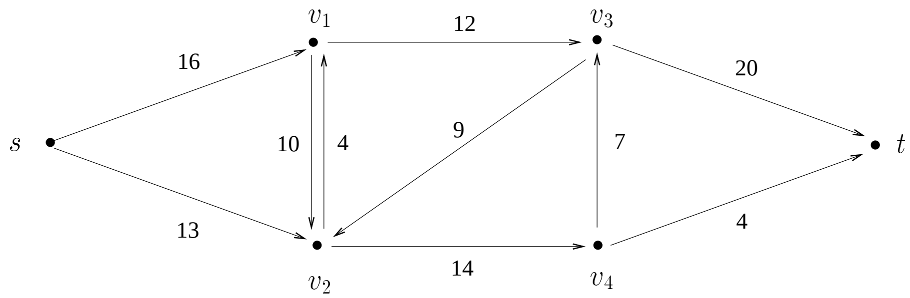
$$c_f(\pi) = \min\{c_f(u,v) \mid (u,v) \text{ is on } \pi\}$$

Lemma Fix $\mathcal{F} = (G, c, s, t)$, flow f, augmenting path π in G_f and define

$$f_\pi(u,v) = \begin{cases} c_f(\pi) & \text{if } (u,v) \in \pi \\ -c_f(\pi) & \text{if } (v,u) \in \pi \\ 0 & \text{otherwise} \end{cases}$$

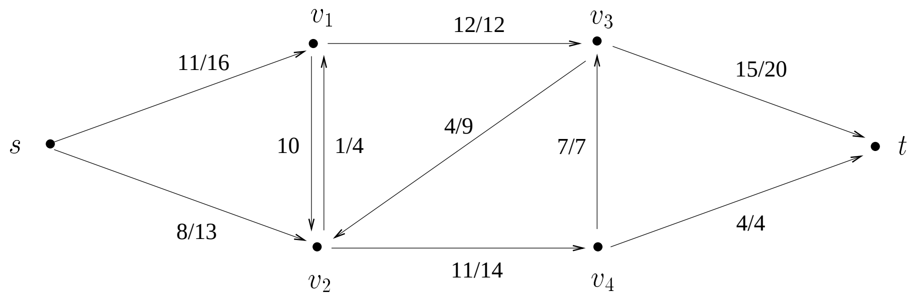
Then f_π is a flow in $\mathcal{F}_f$, $|f_\pi| = c_f(\pi) > 0$.

Example flow network:



The following graph is a flow network plus flow. The notation x/y on an edge (u,v) means

> x is the flow ($x = f(u,v)$)
> y is the capacity ($y = c(u,v)$)



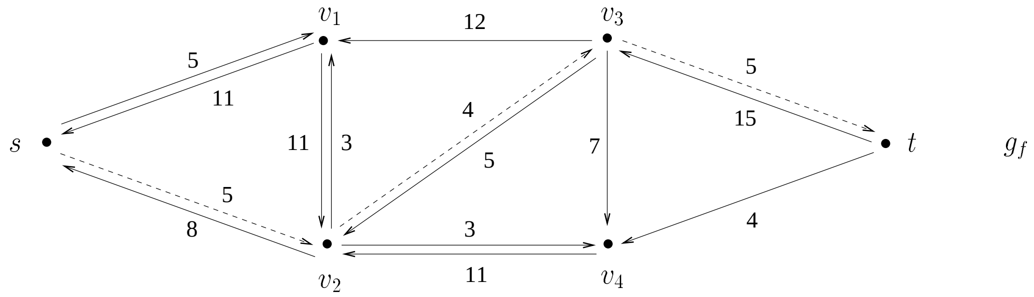
We write 10 on edge (v_1, v_2) rather than $-1/10$ (i.e. if $f(u,v) \leq 0$ then the edge is labelled only by its capacity)

Note that reverse flows (not indicated) are negative, i.e., $f(v_2, s) = -8$

Corollary Fix G, f, augmenting path π in G_f, and let f_π be as defined above. Let $f' = f + f_\pi$. Then f' is a flow in G, and

$$|f'| = |f| + |f_\pi| > |f|.$$

Example

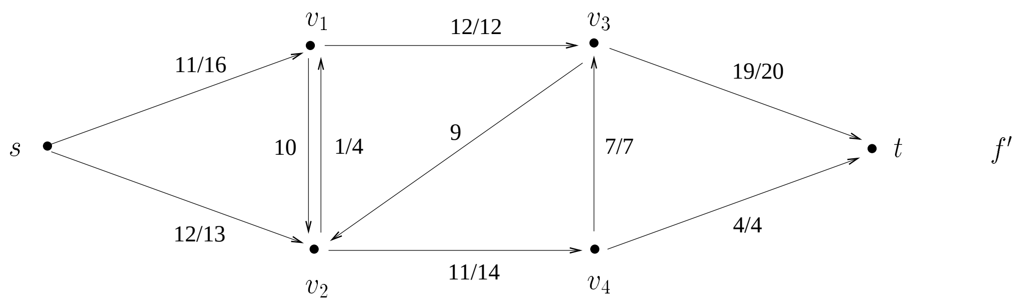


- - - denotes the path π

$|f| = 8 + 11 = 19$ (from previous picture)

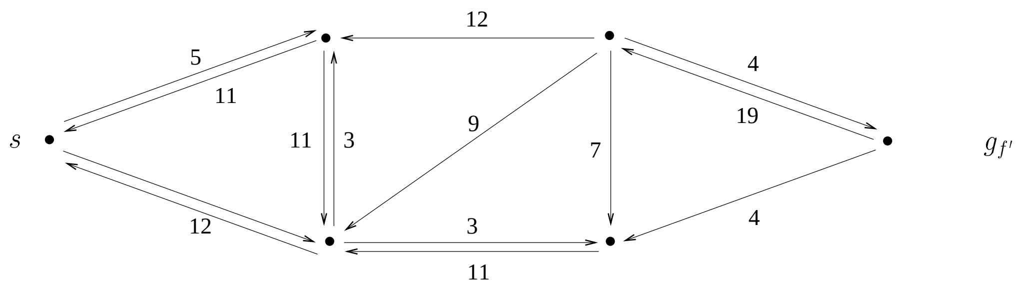
$c_f(\pi) = 4$

$|f_\pi| = 4$, $|f'| = |f| + |f_\pi| = 23$



FORD-FULKERSON METHOD (G, s, t)

1. Initialize flow f to 0
2. *While* there exists an augmenting path π in G_f
   DO augment flow f along π
3. return f



We want to prove that the algorithm works. We need to show that we found a max flow if and only if its residual network contains NO augmenting path.

Cuts of flow networks

A *cut* (S,T) of $G=(V,E)$ is a partition of V into S and $T=V-S$ such that $s \in S$, $t \in T$.

The *net flow* across (S,T) is

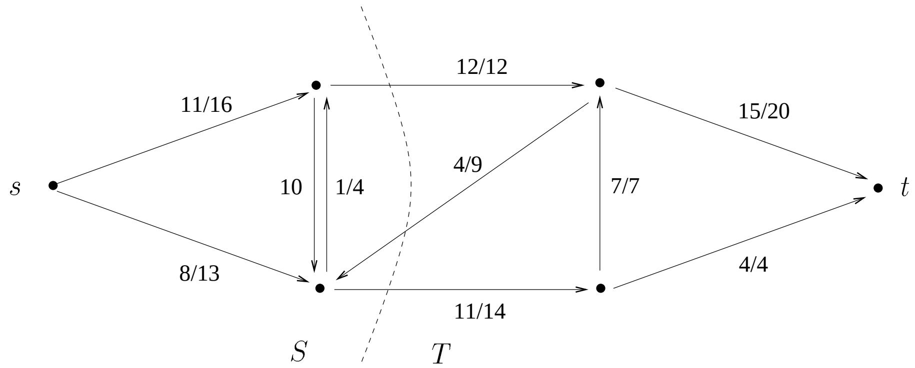
$$f(S,T) \stackrel{d}{=} \sum_{x\in S}\sum_{y\in T} f(x,y)$$

The *capacity* of the cut (S,T) is

$$c(S,T) \stackrel{d}{=} \sum_{x\in S}\sum_{y\in T} c(x,y)$$

Note that we are slightly abusing notation by extending the definition of flows and capacities to subsets of nodes.

Example



$f(S,T) = 12 - 4 + 11 = 19$
$c(S,T) = 12 + 14 = 26$

Lemma 27.5 Fix G, f, s, t and let (S,T) be a cut of G. Then the net flow across (S,T) is $f(S,T) = |f|$.

Proof by conservation of flow.

Corollary 27.6 The value of any flow f in G is at most the capacity of any cut in G.

Theorem 27.7 (MAX-FLOW, MIN-CUT THEOREM)
If f is a flow in $G = (V, E)$ with source s, sink t, then the following are equivalent

1. f is a max flow in G
2. G_f contains no augmenting paths
3. $|f| = c(S,T)$ for some cut (S,T) of G.

Proof

$(1) \Rightarrow (2)$ If G_f contains an augmenting path then f' is a larger flow than f

$(2) \Rightarrow (3)$ Suppose (2) holds.

Let $S = \{v \in V \mid \exists \text{ path from } s \text{ to } v \text{ in } G_f\}$
$T = V - S$

Clearly (S,T) is a cut, $s \in S$, $t \in T$ and $\forall (u,v)$ such that $u \in S$, $v \in T$
$f(u,v) = c(u,v)$ since otherwise $(u,v) \in E_f$ and v is in S.
$\therefore$ (by Lemma 27.5) $|f| = f(S,T) = c(S,T)$

(3) $\Rightarrow$ (1) By Corollary 27.6, $|f| \leq c(S,T)$ for any cut (S,T). The condition $|f| = c(S,T)$ thus
implies that f is a max flow.

FORD-FULKERSON (G, s, t)

For each $(u,v) \in E$
  DO $f[u,v] \leftarrow 0$
    $f[v,u] \leftarrow 0$
While there is a path π from $s \rightsquigarrow t$ in G_f
  DO $c_f(\pi) \leftarrow \min\{c_f(u,v) \mid (u,v) \in \pi\}$
    For each (u,v) in π
      DO $f[u,v] \leftarrow f[u,v] + c_f(\pi)$
        $f[v,u] \leftarrow -f[u,v]$

Choose augmenting path using BFS (breadth first search)
which will find a shortest (augmenting) path from s to t.

Claim Runtime is then $0(|V||E|^2)$. (If all edge capacities are integral, then the algorithm will always terminate with an optimal solution no matter how the augmenting paths are chosen but the algorithm may not run in polynomial time if the augmenting paths are chosen poorly.)