UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2017 EXAMINATIONS

CSC 438H1F/2404H1F

Duration - 3 hours
No Aids Allowed

There are 9 questions worth a total of 100 marks.
Answer all questions on the question paper, using backs of pages for scratch work.
Check that your exam book has 9 pages (including this cover page).

PLEASE COMPLETE THIS SECTION:

Name ______________________________

(Please underline your family name.)

Student Number ______________________________

FOR USE IN MARKING:

1. ________/12
2. ________/12
3. ________/10
4. ________/10
5. ________/10
6. ________/8
7. ________/8
8. ________/12
9. ________/18

Total:________/100

1. Let $\mathcal{L}$ be a predicate calculus language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let

$$\Sigma = \text{Th}(\mathcal{M}) = \{A \mid \text{A is an } \mathcal{L}\text{-sentence and } \mathcal{M} \models A\}$$

[7] a) Prove that Σ is a theory (i.e. prove that Σ is closed under logical consequence).

[5] b) Prove that Σ is a complete theory.

[12] 2. The following are the first two Peano Axioms:

P1: $\forall x(sx \neq 0)$

P2: $\forall x \forall y(sx = sy \rightarrow x = y)$

Is it true that P1, P2 $\models \forall x(x = 0 \vee \exists y(x = sy))$?

If true, give a suitable LK proof justifying this (see the next question for the equality axioms) **If false,** justify by giving a suitable structure.

[10] 3. Give an LK proof of the sequent

$$\forall x(x + 0 = x) \to \forall x \forall y(x + (y + 0) = x + y)$$

You do not need to put in weakenings or exchanges.

Here are the **LK** equality axioms:

EL1: $\to t = t$
EL2: $t = u \to u = t$
EL3: $t = u, u = v \to t = v$
EL4: $t_1 = u_1, ..., t_n = u_n \to ft_1...t_n = fu_1...u_n$, for each f in $\mathcal{L}$, where f is an n-ary function symbol.
EL5: $t_1 = u_1, ..., t_n = u_n, Pt_1...t_n \to Pu_1...u_n$, for each P in $\mathcal{L}$, where P is an n-ary predicate symbol.

[10] 4. Recall that **TA** (True Arithmetic) is the set of all sentences A in the language $\mathcal{L}_A = [0, s, +, \cdot; =]$ of arithmetic such that A is true in the standard model $\underline{\mathbb{N}}$. Suppose that $A(x)$ is a formula of $\mathcal{L}_A$ whose only free variable is x, such that $A(s^n0)$ is in **TA** for arbitrarily large $n \in \mathbb{N}$. Show that the infinite set of sentences

$$\mathbf{TA} \cup \{A(c), c \neq 0, c \neq s0, c \neq ss0, \cdots\}$$

is satisfiable, where c is a new constant.

[10] 5. Suppose that $A = range(f)$ for some computable unary function f. Give a primitive recursive relation $R(x, y)$ such that

$$A = \{x \mid \exists y R(x, y)\}$$

[8] 6. Let $f_1, f_1, f_2, ...$ be a list of all total computable functions $f : \mathbb{N} \to \mathbb{N}$. Define $F(x, y) = f_x(y)$. Prove that F is not computable.

[8] 7. Recall that there is a theorem in the Notes that states that every Δ_0 sentence in **TA** is in **RA**. Use this to prove that every $\exists\Delta_0$ sentence in **TA** is in **RA**.

8. Suppose that $A(x)$ is an $\exists\Delta_0$ formula which represents the r.e. set K in **PA**.

[2] (a) State what it means for $A(x)$ to represent K in **PA**.

[10] (b) Show that there is a consistent extension of Σ of **PA** such that $A(x)$ does *not* represent K in Σ. (Hint: Form Σ by adding a false axiom to **PA**.)

[18] 9. We say that a function $f : \mathbb{N} \to \mathbb{N}$ is *non-decreasing* if $f(x) \leq f(x+1)$ for all $x \in N$. Let

$$A = \{x \mid \{x\}_1 \text{ is nondecreasing}\}$$

Is A r.e.? Is A^c r.e.? Justify your answer. (Do not use Rice's Theorem).