CSC 438F/2404F **Midterm Study Problems** **Fall 2019**

Below are some suggested exercises to help prepare for the midterm. Many of these we went over in tutorial or class, or were given as homework problems, but some are new. To get the most out of doing practice problems, I suggest that you first carefully read and understand the corresponding course notes/lectures, including the definitions and examples.

1. Exercises 6,11,14 Notes on Propositional Calculus
2. Exercises 2,3,4,5,8,10,15 Notes on Predicate Calculus
3. Exercises 3,5,6,7 of Notes on LK Completeness
4. Exercises 5,7,9,10,11 Notes on Herbrand, Equality, Compactness
5. Given an example of a propositional formula that is satisfiable but that is not a tautology.
6. Prove the equivalence of all three forms of propositional compactness. Try to get a good sense of when it may be useful to use compactness theorem (propositional or first order) to solve a problem.
7. Understand the proofs of completeness for Resolution, PK and LK. Practice running the underlying algorithm in each case. That is, given an unsatisfiable CNF formula, run the algorithm given in the proof of completeness of Resolution to obtain a Resolution refutation. Similarly, run the PK/LK algorithms on a valid formulas in order to obtain a proof.
8. Give an LK-Φ proof of the sequent $\exists y \forall x Pxy \rightarrow \forall x \exists y Pxy$.
9. The language of successor is $\mathcal{L}_s = [0, s, =]$. The theory of successor, $Th(s)$ is the set of all sentences over $\mathcal{L}_s$ that are logical consequences of the following set of axioms, Ψ:

   P1) $\forall x(sx \neq 0)$
   P2) $\forall x \forall y(sx = sy \supset x = y)$
   P3) $\forall x(x = 0 \vee \exists y(x = sy))$
   S1) $\forall x(sx \neq x)$
   S2) $\forall x(ssx \neq x)$
   S3) $\forall x(ssx \neq x)$
   .
   .
   .

Prove that there is no finite set Γ of sentences in $Th(s)$ such that every sentence in $Th(s)$ is a logical consequence of Γ. Note that the sentences in Γ are not necessarily among the original set Ψ of axioms.

10. What does it mean for a proof system for predicate logic (or propositional logic) to be sound ? to be complete?

11. Using the completeness proof for Resolution, construct a Resolution refutation for the following formula. (That is, pick an ordering of the variables, create a decision tree as in the proof of completeness, and then relabel the vertices of the decision tree in order to obtain a tree-like Resolution refutation.)

$$(P_1)(\neg P_1 \vee P_2)(\neg P_2 \vee P_3)(\neg P_3 \vee P_4)(P_5)(\neg P_5 \vee \neg P_4)$$

[5] 12. Explain in a few sentences why any Resolution refutation of a 2CNF formula has size polynomial in the number of underlying variables. (Recall a 2CNF formula is a CNF formula where all clauses contain at most 2 literals.)

13. Give a sentence $\mathcal{A}$ which has an infinite model but no finite model. Your sentence should involve only one binary predicate symbol and no function symbols. You should specify an infinite model for $\mathcal{A}$ but you do not need to prove that $\mathcal{A}$ has no finite model.

NOTE: you will be given the following rules for PK (so you do not need to memorize them).

- Structural rules: Exchange, weakening
- OR right: From $\Gamma \to \Delta, A, B$ derive $\Gamma \to \Delta, A \vee B$.
- OR left: From $A, \Gamma \to \Delta$ and $B, \Gamma \to \Delta$, derive $A \vee B, \Gamma \to \Delta$.
- AND right: From $\Gamma \to \Delta, A$ and $\Gamma \to \Delta, B$ derive $\Gamma \to \Delta, A \wedge B$.
- AND left: From $A, B, \Gamma \to \Delta$ derive $A \wedge B, \Gamma \to \Delta$.
- NEG right: From $A, \Gamma \to \Delta$ derive $\Gamma \to \Delta, \neg A$
- NEG left: From $\Gamma \to \Delta, A$ derive $\neg A, \Gamma \to \Delta$.
- CUT: From $A, \Gamma \to \Delta$ and $\Gamma \to \Delta, A$ derive $\Gamma \to \Delta$.

The two addition LK rules are as follows.

- $\forall$ left: From $A(t), \Gamma \to \Delta$ derive $\forall x A(x), \Gamma \to \Delta$.

- $\forall$ right: From $\Gamma \to \Delta, A(b)$ derive $\Gamma \to \Delta, \forall x A(x)$.
- $\exists$ left: From $A(b), \Gamma \to \Delta$ derive $\exists x A(x), \Gamma \to \Delta$.
- $\exists$ right: From $\Gamma \to \Delta, A(t)$ derive $\Gamma \to \Delta, \exists x A(x)$.

The free variable b must not occur in the conclusion in $\forall$ right and $\exists$ left, and t is a proper term (free variables only).