CSC 438/2404 Week 2

- HW1 out! Due Fri Sept 27th
  at start of Tutorial

Last class:

1. Intro
2. Propositional logic

    syntax / semantics

    Resolution: proof system for propositional logic

    - soundness
    - completeness

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1. Intro
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   syntax / semantics

   Resolution: proof system for propositional logic

   - soundness
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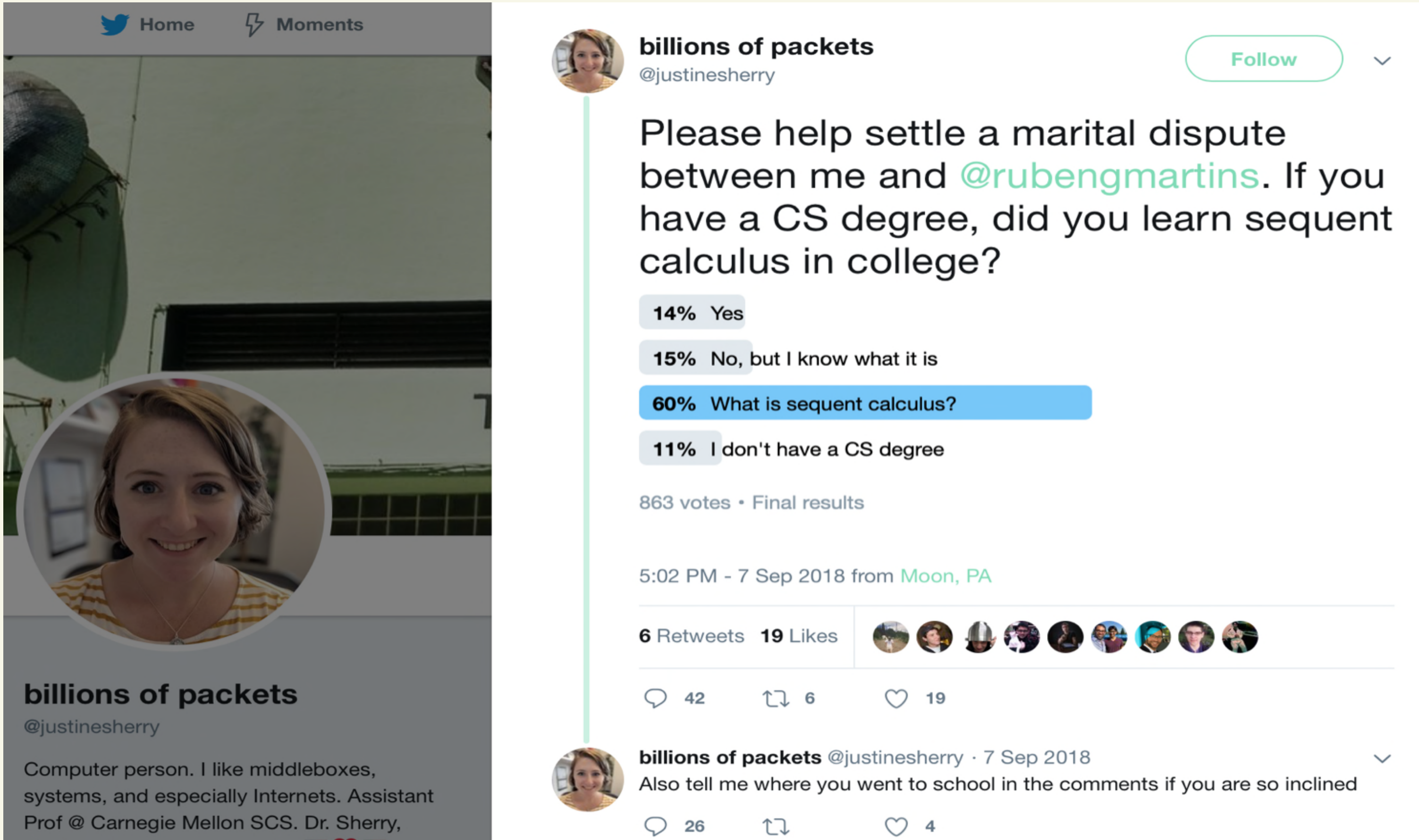
Pages 1-9 of Lecture Notes, plus supplementary notes on Resolution

Today

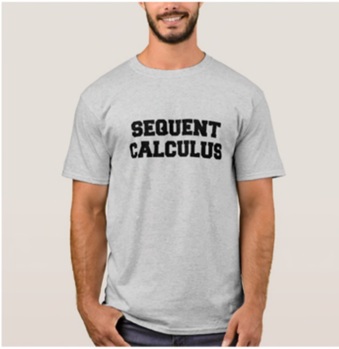
- Another proof system for propositional logic: PK
  - Soundness of PK
  - Completeness of PK
- Propositional Compactness Theorem
- Derivational Soundness/Completeness of PK

Pages 9 – 17 of Lecture Notes

Sequent Calculus goes viral on Twitter




Home
Moments
4
17
Pierce Darragh @_pdarragh · 7 Sep 2018
Pretty sure it's the calculus of how to put shiny plastic things on clothing
1
8
John Regehr @johnregehr · 7 Sep 2018
YES!!!
2
kat 猫 ✨🐙✨ @wirehead2501 · 7 Sep 2018
Replying to @justinesherry @rubengmartins
@chrisamaphone no; i have a CS BS and comp eng MS and only had to take Calc I and II. don't think i've ever heard the term "sequent"
2
Chris Martens @chrisamaphone · 8 Sep 2018
But surely you have heard its plural, as in, “a sequents of events” 😬
1
6
jason reed @jcreed · 8 Sep 2018
if you are against bedazzling: sequin't
1
3
jason reed @jcreed · 8 Sep 2018
oh wait I have confused sequins with rhinestones, my bad
3
ions of packets
tinesherry
puter person. I like middleboxes,
ems, and especially Internets. Assistant
@ Carnegie Mellon SCS. Dr. Sherry,



Sequent Calculus Standard Shirt

$18.95

15% Off with code GANGSALLHERE

Sequent Calculus Fitted Shirt

$17.95

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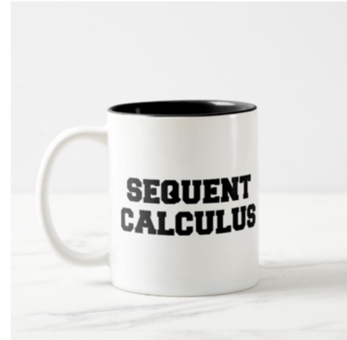

Sequent Calculus Mug

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Sequent Calculus Tote Bag

$9.95

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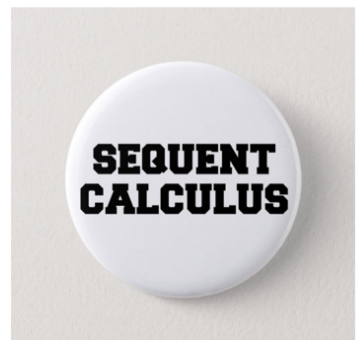
German Fitted Sequent Calculus Shirt

German Sequent Calculus Shirt



Sequent Calculus Pin

Gentzen's PK proof system

Lines in a PK proof are sequents

$$\underbrace{A_1, \ldots, A_k}_{\text{antecedent}} \rightarrow \underbrace{B_1, \ldots, B_r}_{\text{succedent}}$$

$A_1, \ldots, A_k, B_1, \ldots, B_r$ are propositional formulas

$\rightarrow$ is a new symbol (NOT part of language of propositional logic)

Gentzen's PK proof system

Lines in a PK proof are sequents

$$A_1, \ldots, A_k \longrightarrow B_1, \ldots, B_r$$

Semantics:

$$A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_r$$

the conjunction of the A_i's implies the disjunction of the B_i's

Gentzen's PK proof system

Lines in a PK proof are **sequents**

$$A_1, \ldots, A_k \longrightarrow B_1, \ldots, B_r \qquad \stackrel{d}{=} S$$

Semantics:

$$A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_r \qquad \stackrel{d}{=} A_S$$

the conjunction of the A_i's implies the disjunction of the B_i's

Gentzen's PK proof system

Lines in a PK proof are **sequents**

$$A_1, \ldots, A_k \rightarrow$$

0 or False

$$\rightarrow B_1, \ldots, B_q$$

1 or True

Convention: Empty conjunction (antecedent empty) $\rightarrow 1$

Empty disjunction (succedent empty) $\rightarrow 0$

PK Rules

Intuitively: Structural Rules (cedents are sets)

Logical Rules (define the boolean connectives $\wedge, \vee, \neg$)

Cut Rule

STRUCTURAL RULES

	Left	Right
Weakening	$\dfrac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$
Exchange	$\dfrac{\Gamma_1, A, B, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, B, A, \Gamma_2 \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta_1, A, B, \Delta_2}{\Gamma \Rightarrow \Delta_1, B, A, \Delta_2}$
Contraction	$\dfrac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$

LOGICAL RULES

	Left	Right
$\neg$ Intro	$\dfrac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$	$\dfrac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$
$\wedge$ Intro	$\dfrac{A, B, \Gamma \Rightarrow \Delta}{(A \wedge B), \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, (A \wedge B)}$
$\vee$ Intro	$\dfrac{A, \Gamma \Rightarrow \Delta, \quad B, \Gamma \Rightarrow \Delta}{(A \vee B), \Gamma \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, (A \vee B)}$

CUT RULE

$$\frac{\Gamma \to \Delta, A \qquad \Gamma, A \to \Delta}{\Gamma \to \Delta}$$

$$\frac{\neg A \to \underbrace{\neg\Gamma, \Delta} \qquad A \to \underbrace{\neg\Gamma, \Delta}}{}$$

$$\frac{\neg A \to \Gamma' \qquad A \to \Gamma'}{\to \Gamma'}$$

Axiom

$$A \to A$$

<u>Example:</u> A PK proof of a formula A is a PK proof of $\to A$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P \to P}{P, \neg Q \to P}}{P, \neg P, \neg Q \to} \qquad \dfrac{\dfrac{Q \to Q}{Q, \neg P \to Q}}{Q, \neg P, \neg Q \to}}{(P \vee Q), \neg P, \neg Q \to}}{(P \vee Q), \neg P \wedge \neg Q \to}}{(\neg P \wedge \neg Q) \to \neg(P \vee Q)}$$

Weakening

$\neg$ - Left

OR LEFT

AND - Left

$\neg$ Right

PK SOUNDNESS : Every sequent provable in PK is VALID

As in the propositional case, we first verify the soundness of all rules + then prove PK soundness by induction

Lemma (Soundness of Rules)
For every rule of PK, if all top sequents are valid, then the bottom sequent is valid
also the axiom is valid

PK soundness If S has a PK proof, then A_S is valid

PK COMPLETENESS: Every valid propositional sequent has a PK proof

Main idea: again we will give an algorithm that will produce a PK proof for any valid sequent

Algorithm: Write sequent at bottom (of proof)

1. Repeatedly: pick an outermost connective in a formula in a leaf sequent of current proof & apply the rule for that connective (in reverse)
2. Continue until all leaf sequents consist of just atoms

Show: If we run algorithm on a valid sequent $\Gamma \Rightarrow \Delta$, then at end, all leaf sequents must contain an atom occuring both on left + right — ie $A, B, C \Rightarrow A, D$

Then can finish proof by applying weakening (in reverse)

ie.
$$\frac{A \Rightarrow A}{A, B, C \Rightarrow A, D}$$

PK completeness (cont'd)

Key Property is the **INVERSION PRINCIPLE**:

each PK rule except weakening has the property that $\forall$ truth assignments τ, if τ satisfies bottom sequent, then τ satisfies both upper sequents

* called inversion since it is the reverse direction of what we needed to prove soundness: $\forall \tau$ if τ satisfies both upper sequents, then τ satisfies lower sequent

PK completeness

- If $\Gamma \rightarrow \Delta$ is valid, by Inversion Property, all leaf sequents generated in step 1 of Algorithm are VALID, and have one less connective than sequent below

- Thus eventually step 1 halts, where each leaf sequent involves only atoms and each leaf sequent is valid
  $\therefore$ each leaf sequent looks like $A, \Gamma \rightarrow A, \Delta$
  ie has an atom A on both Left & Rt sides

PK completeness

If not: Say $A, B \to C, D, E$ is a leaf. Then this sequent is not valid!

Cut-Elimination Theorem for PK

If $\Gamma \rightarrow \Delta$ has a PK proof, then it has a proof with no use of the cut rule.

Derivational Soundness & Completeness of PK

Theorem Let Φ be a set of (possibly infinite) sequents. Then $\Phi \vdash S$ iff S has a (finite) PK-Φ proof

PK-Φ proof of S : is a PK proof of S from Φ and axioms of PK.

Propositional Compactness

Theorem (Form 2, see notes for 2 other equivalent forms)

Let Φ be a set of (possibly infinite) formulas

$\Phi \vdash A$ iff A is a logical consequence of a finite subset of Φ

Proof (Derivational Soundness/completeness)

By compactness, it suffices to prove the case where Φ is finite

- Let $\Phi = \{s_1, \dots, s_k\}$, and suppose $\Gamma \Rightarrow \Delta$ is a logical consequence of $\{s_1, \dots, s_k\}$. Thus

$$(*) \quad \Gamma, A_{s_1}, \dots, A_{s_k} \Rightarrow \Delta \quad \text{is valid}$$

- Thus by PK completeness, $(*)$ has a PK proof
- Derive $\Gamma \Rightarrow \Delta$ from $(*)$ and $\Rightarrow A_{s_1}, \dots, \Rightarrow A_{s_k}$

Derive $\Gamma \to \Delta$ from $\{\to A_{s_1}, \to A_{s_2}, \to A_{s_3}, (*)\}$

$$\frac{\to A_{s_1}}{\Gamma, A_{s_2}, A_{s_3} \to \Delta, A_{s_1}} \quad \text{(weakening)}$$

$$\frac{\Gamma, A_{s_1}, A_{s_2}, A_{s_3} \to \Delta \qquad \Gamma, A_{s_2}, A_{s_3} \to \Delta, A_{s_1}}{\Gamma, A_{s_2}, A_{s_3} \to \Delta} \quad \text{(cut)}$$

$$\frac{\to A_{s_2}}{\Gamma, A_{s_3} \to \Delta, A_{s_2}} \quad \text{(weakening)}$$

$$\frac{\Gamma, A_{s_2}, A_{s_3} \to \Delta \qquad \Gamma, A_{s_3} \to \Delta, A_{s_2}}{\Gamma, A_{s_3} \to \Delta} \quad \text{(cut)}$$

$$\frac{\to A_{s_3}}{\Gamma \to A_{s_3}, \Delta} \quad \text{(weakening)}$$

$$\frac{\Gamma, A_{s_3} \to \Delta \qquad \Gamma \to A_{s_3}, \Delta}{\Gamma \to \Delta} \quad \text{(cut)}$$

Proof (Propositional Compactness)

Suppose $\Phi \vdash A$. Then $\overbrace{\Phi, \neg A}^{\Psi}$ is unsatisfiable

Show: If Ψ is UNSAT, then some finite subset of Ψ is UNSAT (Form 1)

Pf sketch Assume the set of underlying atoms in Ψ is countable: $P_1, P_2, \ldots$

- Make decision tree T that queries P_1 at layer 1, then P_2 at layer 2, etc.

- Each path in T corresponds to a complete truth assignment

- Prune T to T':

  For every node v of T, remove subtree rooted below v if partial truth assignment from root to v falsifies some formula $f \in \Psi$. Label v by f

- Every path in T' is finite (since Ψ unsat, so $\forall$truth ass to all vars, some $f \in \Psi$ is falsified, and each $f \in \Psi$ is finite)

- By König's Lemma, T' is finite

König's Lemma If T' is a rooted binary tree, where every branch/path of T' is finite, Then T' is finite.

- Thus, the formulas $\psi' \subseteq \Psi$ labelling the leaves of T' form a finite subset of Ψ, and thus ψ' is UNSAT & finite subset of Ψ.