CS438/2404 Lecture 4

- HW 2: OUT! Due Oct 18

TODAY

First Order Logic

Quick Review - syntax, semantics

LK completeness

Consequences of completeness

Lecture Notes: completeness p. 31-38

FIRST ORDER LOGIC (Review)

Underlying language $\mathcal{L}$ specified by:

1. $\forall n \in \mathbb{N}$ a set of n-ary function symbols (ie.: $f, g, h, +, \cdot$)

   0-ary function symbols are called constants

2. $\forall n \in \mathbb{N}$ a set of n-ary predicate symbols (i.e. $P, Q, R, <, \leq$)

Plus:

- Variables : $x, y, z, \ldots a, b, c, \ldots$
- $\neg, \vee, \wedge, \exists, \forall$
- parenthesis (,)

} Built in symbols

Terms over $\mathcal{L}$

(1) Every variable is a term

(2) If f is an n-ary function symbol, and $t_1, \ldots, t_n$ terms, then $f t_1 \ldots t_n$ is a term

Terms over $\mathcal{L}$

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Examples of terms (0, s, f, $+$, $\bullet$)

0-ary   unary   binary

$f0sss0$, $+xfyz$, $\bullet + ab\, ss0$

$f(0sss, 0)$ $x + f(y, z)$ $(a+b) * ss0$

FIRST ORDER FORMULAS OVER $\mathcal{L}$

(1) $Pt_1 .. t_n$ is an atomic $\mathcal{L}$-formula, where P is an n-ary predicate in $\mathcal{L}$, and $t_1 .. t_n$ are terms over $\mathcal{L}$

(2) If A, B are $\mathcal{L}$-formulas, so are $\neg A$, $(A \wedge B)$, $(A \vee B)$, $\forall x A$, $\exists x A$

Example : FO Formulas in $\mathcal{L}_A$

③ Fermat's Last Theorem (actually Andrew Wiles' theorem)

$$\forall n \geq 3 \; (\forall a, b, c \quad a^n + b^n \neq c^n)$$

Problem: How to say a^n ?

(we'll see later how to do this!)

FREE / BOUND VARIABLES

- An occurrence of x in A is bound if x is in a subformula of A of the form $\forall x B$, or $\exists x B$ (otherwise x is free in A)

Example

$$\exists y (x = y + y)$$

$$Px \wedge \forall x (\neg (x + sx = x))$$

- A formula/term is closed if it contains no free variables
- A closed formula is called a sentence

SEMANTICS OF FO LOGIC

An $\mathcal{L}$-structure $\mathfrak{M}$ (or model) consists of:

1. A nonempty set M called the universe (variables range over M)

2. For every n-ary function symbol f in $\mathcal{L}$, an associated function $f^{\mathfrak{M}} : M^n \to M$

3. For each n-ary relation symbol P in $\mathcal{L}$, an associated relation $P^{\mathfrak{M}} \subseteq M^n$

* Equality predicate $=$ is always true equality relation on M.

Example

$$\mathcal{L}_A = \{0, S, +, \cdot, =\}$$

(1) $\mathbb{N}$: standard model of $\mathcal{L}_A$

$$M = \mathbb{N}$$

$$0 = 0 \in \mathbb{N}$$

$+$, $\cdot$, S are usual plus, times, successor functions

Jumping ahead a bit: Evaluation of a formula in $\mathbb{N}$

$$\forall x \, \forall z \, \left(\exists z' \, (\neg(z' = 0) \wedge z + z' = x) \rightarrow \exists z'' \, (Sz + z'' = x) \right)$$

Definition: Evaluation of terms/formulas over $\mathfrak{M}, \sigma$

Let $\mathfrak{M}$ be an $\mathcal{L}$-structure,

σ an object assignment for $\mathfrak{M}$

Evaluation of terms over $\mathfrak{M}, \sigma$

(a) $x^{\mathfrak{M}}[\sigma]$ is $\sigma(x)$ for all variables x

(b) $(f t_1 \ldots t_n)^{\mathfrak{M}}[\sigma] = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}[\sigma], \ldots, t_n^{\mathfrak{M}}[\sigma])$

Evaluation of formulas over $\mathfrak{M}, \sigma$

Let A be an $\mathcal{L}$-formula. $\mathfrak{M} \models A[\sigma]$ ($\mathfrak{M}$ satisfies A under σ) iff

(a) $\mathfrak{M} \models Pt_1 \ldots t_n[\sigma]$ iff $\langle t_1^{\mathfrak{M}}[\sigma], \ldots, t_n^{\mathfrak{M}}[\sigma] \rangle \in P^{\mathfrak{M}}$

(b) $\mathfrak{M} \models (s = t)[\sigma]$ iff $s^{\mathfrak{M}}[\sigma] = t^{\mathfrak{M}}[\sigma]$

(c) $\mathfrak{M} \models \neg A[\sigma]$ iff not $\mathfrak{M} \models A[\sigma]$

(d) $\mathfrak{M} \models (A \vee B)[\sigma]$ iff $\mathfrak{M} \models A[\sigma]$ or $\mathfrak{M} \models B[\sigma]$

(e) $\mathfrak{M} \models (A \wedge B)[\sigma]$ iff $\mathfrak{M} \models A[\sigma]$ and $\mathfrak{M} \models B(\sigma)$

(f) $\mathfrak{M} \models \forall x A[\sigma]$ iff $\forall m \in M$ $\mathfrak{M} \models A[\sigma(m/x)]$

(g) $\mathfrak{M} \models \exists x A[\sigma]$ iff $\exists m \in M$ $\mathfrak{M} \models A[\sigma(m/x)]$

Example

$\mathcal{L} = \{ ; R, = \}$

$\mathfrak{M} = (\mathbb{N}; \leq, =)$

$R^{\mathfrak{M}}(m,n)$ iff $m \leq n$

Then

$\mathfrak{M} \overset{?}{\models} \forall x \exists y R(x,y)$ yes

satisfiable by $\mathfrak{M}$

$\mathfrak{M} \overset{?}{\models} \exists y \forall x R(x,y)$ No

but $\exists y \forall x R(x,y)$ is also satisfiable

IMPORTANT DEFINITIONS

① A is satisfiable iff there exists a model $\mathfrak{M}$ and an object assignment σ such that $\mathfrak{M} \vDash A[\sigma]$

② A set of formulas Φ is satisfiable iff $\exists \mathfrak{M}, \sigma$ such that $\mathfrak{M} \vDash \Phi[\sigma]$ [$\mathfrak{M} \vDash A[\sigma]$ for all $A \in \Phi$]

③ $\Phi \vDash A$ (A is a logical consequence of Φ) iff $\forall \mathfrak{M} \, \forall \sigma$ if $\mathfrak{M} \vDash \Phi[\sigma]$ then $\mathfrak{M} \vDash A[\sigma]$

$\vDash A$ (A is valid) iff $\forall \mathfrak{M}, \sigma$ $\mathfrak{M} \vDash A[\sigma]$

FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

$$\left. A_1, \ldots, A_k \longrightarrow B_1, \ldots, B_\ell \right\} S$$

where each A_i, B_j is a proper $\mathcal{L}$-formula

$$A_S : \quad A_1 \wedge A_2 \wedge \ldots \wedge A_k \supset B_1 \vee \ldots \vee B_\ell$$

FIRST ORDER SEQUENT CALCULUS LK

Lines are again sequents

$$A_1, \ldots, A_k \rightarrow B_1, \ldots, B_\ell$$

where each A_i, B_j is a proper $\mathcal{L}$-formula

RULES

OLD RULES OF PK

PLUS NEW RULES FOR $\forall$, $\exists$

like a large AND

Large OR

New Logical Rules for $\forall$, $\exists$

$\forall$-left
$$\frac{A(t), \Gamma \to \Delta}{\forall x\, A(x), \Gamma \to \Delta}$$

$\forall$-Right
$$\frac{\Gamma \to \Delta, A(b)}{\Gamma \to \Delta, \forall x\, A(x)}$$

$\exists$-left
$$\frac{A(b), \Gamma \to \Delta}{\exists x\, A(x), \Gamma \to \Delta}$$

$\exists$-right
$$\frac{\Gamma \to \Delta, A(t)}{\Gamma \to \Delta, \exists x\, A(x)}$$

* A, t are proper
* b is a free variable Not appearing in lower sequent of rule

Example of an LK proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Pa \to Pa}{Pa, Qa \to Pa}}{Pa \wedge Qa \to Pa}}{Pa \wedge Qa \to \exists x Px}\;\exists\text{-rt}}{\exists x(Px \wedge Qx) \to \exists x Px}\;\exists\text{-left} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Qa \to Qa}{Pa, Qa \to Qa}}{Pa \wedge Qa \to Qa}}{Pa \wedge Qa \to \exists x Qx}\;\exists\text{-rt}}{\exists x(Px \wedge Qx) \to \exists x Qx}\;\exists\text{-left}}{\exists x(Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}\;\wedge\text{-rt}$$

Theorem (LK Soundness)

Every sequent provable in LK is valid

Pf by induction on the number of sequents in proof.

Axiom $A \to A$ is valid

Induction step: use previous soundness lemma

Soundness (Proof): By induction on the number of sequents in proof

Example: $\exists$ Left

Assume: $A(b), \Gamma \Rightarrow \Delta$ has an LK proof and is valid

show: $\exists x A(x), \Gamma \Rightarrow \Delta$ also Valid

By defn $\overline{A(b)} \vee \overline{\Gamma}_1 \vee .. \vee \overline{\Gamma}_k \vee \Delta_1 \vee .. \vee \Delta_k$ is valid

Let $\mathfrak{M}$ be any structure, σ any object assignment.

Show: $\mathfrak{M} \models \neg \exists x A(x) \vee \overline{\Gamma}_1 \vee .. \vee \overline{\Gamma}_k \vee \Delta_1 \vee .. \vee \Delta_k [\sigma]$ (*)

<u>case 1</u> $\mathfrak{M} \models \overline{\Gamma}_1 \vee .. \vee \overline{\Gamma}_k \vee \Delta_1 \vee ... \vee \Delta_k [\sigma]$. Then (*) holds

<u>Case 2</u> Case 1 does not hold.

Soundness (Proof): By induction on the number of sequents in proof

Example: $\exists$ Left

Assume: $A(b), \Gamma \Rightarrow \Delta$ has an LK proof and is valid

Show: $\exists x\, A(x), \Gamma \Rightarrow \Delta$ also valid

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Let $\mathfrak{M}$ be any structure, σ any object assignment.

Show: $\mathfrak{M} \models \neg \exists x\, A(x) \vee \overline{\Gamma}_1 \vee \dots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \dots \vee \Delta_k\ [\sigma]$ $(*)$

<u>case 1</u> $\mathfrak{M} \models \overline{\Gamma}_1 \vee \dots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \dots \vee \Delta_k\ [\sigma]$. Then $(*)$ holds

<u>case 2</u> Case 1 does not hold.

Since b does not occur in Γ or Δ,

$$\mathfrak{M} \not\models \overline{\Gamma}_1 \vee \dots \vee \overline{\Gamma}_k \vee \Delta_1 \vee \dots \vee \Delta_k\ [\sigma(m/b)] \text{ for all } m \in M$$

Since $A(b), \Gamma \Rightarrow \Delta$ is valid, $\mathfrak{M} \models \overline{A(b)}\ [\sigma[m/b]]\ \forall m \in M$

Thus $M \models \neg \exists x\, A(x)\ [\sigma]$, & thus $\exists x\, A(x), \Gamma \Rightarrow \Delta$ is valid.

TODAY: gödels COMPLETENESS THEOREM

Defn An LK-Φ proof is an LK-proof, but leaves are either axioms $(A \Rightarrow A)$ or of the form $\Rightarrow A$ for $A \in \Phi$

goal prove that if $\Gamma \Rightarrow \Delta$ is a logical consequence of Φ, then there is an LK-Φ proof of $\Gamma \Rightarrow \Delta$ (called Derivational completeness)

Defn Let $A(a_1 .. a_n)$ be a formula with free variables $a_1 .. a_n$. Then $\forall A$ is $\forall x_1 \forall x_2 .. \forall x_n A(x_1 .. x_n)$ (called universal closure of A)

TODAY: LK COMPLETENESS

(MAIN LEMMA) Completeness Lemma

If $\Gamma \to \Delta$ is a logical consequence of a set of (possibly infinite) formulas $\forall \Phi$ then there exists a finite subset $\{C_1, \ldots, C_n\}$ of Φ such that

$$\forall C_1, \ldots, \forall C_n, \Gamma \to \Delta \quad \text{has a (cut-free) PK proof}$$

* We will assume = Not in Language for Now

Derivational Completeness Theorem

Let Φ be a set of sequents or formulas such that the sequent $\Gamma \to \Delta$ is a logical consequence of $\forall\Phi$.

Then there is an LK-Φ proof of $\Gamma \to \Delta$.

$\Uparrow$

Proof follows from Completeness Lemma (similar to derivational completeness of PK from completeness)

Proof of LK Completeness Lemma

High Level idea (assume Φ is empty for now)

- As in PK completeness, we want to construct an LK proof in reverse.
- Start with $\Gamma \Rightarrow \Delta$ at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones)
- Tricky rules are $\exists$right & $\forall$left. When applying one of these in reverse, Need to "guess" a term

New Logical Rules for $\forall$, $\exists$

$\forall$-left $\quad \dfrac{A(t), \Gamma \rightarrow \Delta}{\forall x A(x), \Gamma \rightarrow \Delta}$ $\qquad$ $\forall$-Right $\quad \dfrac{\Gamma \rightarrow \Delta, A(b)}{\Gamma \rightarrow \Delta, \forall x A(x)}$

$\exists$-left $\quad \dfrac{A(b), \Gamma \rightarrow \Delta}{\exists x A(x), \Gamma \rightarrow \Delta}$ $\qquad$ $\exists$-right $\quad \dfrac{\Gamma \rightarrow \Delta, A(t)}{\Gamma \rightarrow \Delta, \exists x A(x)}$

* A, t are proper
* b is a free variable Not appearing in lower sequent of rule

Proof of LK Completeness Lemma

High Level idea (assume Φ is empty for now)

- As in PK completeness, we want to construct an LK proof in reverse.
- Start with $\Gamma \Rightarrow \Delta$ at root, and apply rules in reverse (to break up a formula into one or 2 smaller ones)
- Tricky rules are $\exists$right + $\forall$left.
  When applying one of these in reverse, Need to "guess" a term
- Key is to systematically try all possible terms — without going down a rabbit hole.

Example of an LK proof

$$
\dfrac{
\dfrac{\dfrac{?\quad \dfrac{\dfrac{Pa \to Pa}{Pa, Qa \to Pa}}{Pa \wedge Qa \to Pa}}{Pa \wedge Qa \to \exists x Px}}{\exists x (Px \wedge Qx) \to \exists x Px}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{Qa \to Qa}{Pa, Qa \to Qa}}{Pa \wedge Qa \to Qa}\quad ?}{Pa \wedge Qa \to \exists x Qx}}{\exists x (Px \wedge Qx) \to \exists x Qx}
}{
\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx
}
$$

Example of an LK proof

$$Pa, Qa \rightarrow Pb$$

$$\dfrac{Pa \wedge Qa \rightarrow Pb}{Pa \wedge Qa \rightarrow \exists x Px}$$

$$\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x(Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x(Px \wedge Qx) \rightarrow \exists x Qx}$$

$$\dfrac{\exists x(Px \wedge Qx) \rightarrow \exists x Px \qquad \exists x(Px \wedge Qx) \rightarrow \exists x Qx}{\exists x(Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Instead:

$Pa, Qa \to Pb, \exists x Px$

$Pa \wedge Qa \to Pb, \exists x Px$

$$\dfrac{Pa \wedge Qa \to \exists x Px}{\exists x (Px \wedge Qx) \to \exists x Px} \qquad \dfrac{Pa \wedge Qa \to \exists x Qx}{\exists x (Px \wedge Qx) \to \exists x Qx}$$

$$\overline{\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx}$$

Instead

Try again

$$Pa, Qa \to Pb, Pfa, \exists x Px$$

$$Pa, Qa \to Pb, \exists x Px$$

$$Pa \wedge Qa \to Pb, \exists x Px$$

$$\dfrac{Pa \wedge Qa \to \exists x Px}{\exists x (Px \wedge Qx) \to \exists x Px} \qquad \dfrac{Pa \wedge Qa \to \exists x Qx}{\exists x (Px \wedge Qx) \to \exists x Qx}$$

$$\exists x (Px \wedge Qx) \to \exists x Px \wedge \exists x Qx$$

Instead

and again

and again

$Pa, Qa \to Pb, Pfa, Pfb, \exists x Px$

Try again

$Pa, Qa \to Pb, Pfa, \exists x Px$

$Pa, Qa \to Pb, \exists x Px$

$Pa \land Qa \to Pb, \exists x Px$

$$\dfrac{Pa \land Qa \to \exists x Px}{\exists x(Px \land Qx) \to \exists x Px} \qquad \dfrac{Pa \land Qa \to \exists x Qx}{\exists x(Px \land Qx) \to \exists x Qx}$$

$$\exists x(Px \land Qx) \to \exists x Px \land \exists x Qx$$

Instead

and again

and again

$Pa, Qa \rightarrow Pb, Pfa, Pfb, \exists x Px$

Try again

$Pa, Qa \rightarrow Pb, Pfa, \exists x Px$

$Pa, Qa \rightarrow Pb, \exists x Px$

$Pa \wedge Qa \rightarrow Pb, \exists x Px$

$$\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}$$

$$\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx$$

There are infinitely many choices! Need a systematic way to try all

Instead

There are infinitely many choices! Need a systematic way to try all and for all frontier sequents in current proof!

and again

$\vdots$

and again

$$Pa, Qa \rightarrow Pb, Pfa, Pfb, \exists x Px$$

Try again

$$Pa, Qa \rightarrow Pb, Pfa, \exists x Px$$

$$Pa, Qa \rightarrow Pb, \exists x Px$$

$$Pa \wedge Qa \rightarrow Pb, \exists x Px$$

$$\dfrac{\dfrac{Pa \wedge Qa \rightarrow \exists x Px}{\exists x (Px \wedge Qx) \rightarrow \exists x Px} \qquad \dfrac{Pa \wedge Qa \rightarrow \exists x Qx}{\exists x (Px \wedge Qx) \rightarrow \exists x Qx}}{\exists x (Px \wedge Qx) \rightarrow \exists x Px \wedge \exists x Qx}$$

Completeness: Proof Search Algorithm

Enumeration of formulas + terms:

Since the number of underlying symbols of $\mathcal{L}$ is finite, there is an <u>enumeration</u> of pairs $\langle A_1, t_1\rangle, \langle A_2, t_2\rangle, \langle A_3, t_3\rangle, \ldots$ such that <u>every</u> term and <u>every</u> formula in $\mathcal{L}$ occur infinitely often in the enumeration

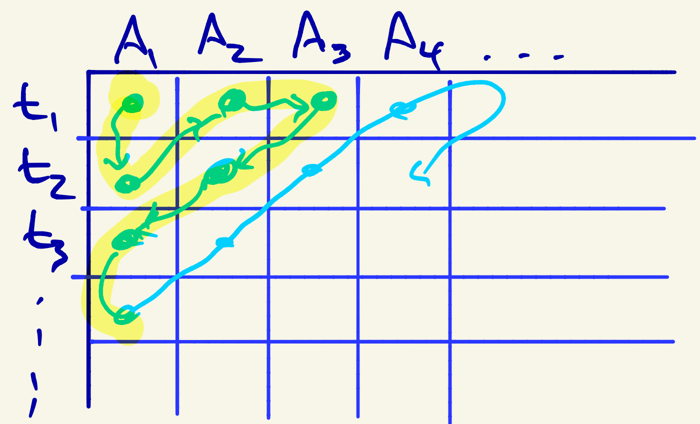
More details of enumeration ($\mathcal{L}$ finite)

Enumerate all $\mathcal{L}$-formulas $A_1, A_2, \ldots$

Enumerate " $\mathcal{L}$-terms $t_1, \ldots$

such that every formula/term occurs infinitely often

Enumerate all pairs to have same property



Completeness: Proof Search Algorithm

- Initially Π is the sequent $\Gamma \to \Delta$
- At each stage, modify Π by adding some $A_i \in \Phi$ to antecedent of all sequents in Π, and adding onto the "frontier" or "active" sequents in Π
- Active sequent: a leaf sequent in Π, not a weakening of $A \to A$
- at stage k: we will use the k^{th} pair $\langle A_k, t_k \rangle$ in the enumeration

Completeness: Proof Search Algorithm

Stage k: $\langle A, t \rangle_k$

(1) If $A_k \in \Phi$, replace $\Gamma' \Rightarrow \Delta'$ in Π by $\Gamma', A_k \Rightarrow \Delta'$

(2) If A_k atomic, skip this step. Otherwise for all leaf sequents containing A_k, break up outermost connective of A_k using the appropriate logical rule, and t_k if necessary.

Completeness: Proof Search Algorithm

<u>Stage k:</u>

(1) If $A_k \in \Phi$, replace $\Gamma' \to \Delta'$ in Π by $\Gamma', A_k \to \Delta'$

(2) If A_k atomic, skip this step. Otherwise for all leaf sequents containing A_k, break up outermost connective of A_k using the appropriate logical rule, and t_k if necessary.

Examples:

- $A_k = \exists x B x$

$$\frac{\Gamma, B(c) \to \Delta}{\Gamma, \exists x B(x) \to \Delta}$$

c is a new variable

$$\frac{\Gamma \to \Delta, \exists x B(x), B(t_k)}{\Gamma \to \Delta, \exists x B(x)}$$

keep <u>both</u> here

Completeness: Proof Search Algorithm

Stage k:

(1) If $A_k \in \Phi$, replace $\Gamma' \Rightarrow \Delta'$ in Π by $\Gamma', A_k \Rightarrow \Delta'$

(2) If A_k atomic, skip this step. Otherwise for all leaf sequents containing A_k, break up outermost connective of A_k using the appropriate logical rule, and t_k if necessary.

Examples:

- $A_k = \forall x\, B(x)$

$$\frac{\Gamma \Rightarrow \Delta, B(c)}{\Gamma \Rightarrow \Delta, \forall x\, B(x)}$$

c a new variable

$$\frac{B(t_k), \forall x\, B(x), \Gamma \Rightarrow \Delta}{\Gamma, \forall x\, B(x) \Rightarrow \Delta}$$

Keep both here

Exit when no more active sequents

Proof of Correctness

We want to show:

- If Algorithm halts, Π is an LK-Φ proof of $\forall c_1, \ldots, \forall c_n \; \Gamma \rightarrow \Delta$ ✓
- If Algorithm never halts, then $\forall \Phi \not\vdash \Gamma \rightarrow \Delta$

Proof of Correctness

We want to show: If Algorithm never halts, then $\forall \Phi \not\models \Gamma \to \Delta$

Suppose Algorithm doesn't halt and let Π be the (typically infinite) tree that results

Leaf "sequents" of Π look like $\Gamma', \underbrace{C_1, C_2, \ldots}_{\text{infinite sequence containing all of } \Phi \text{ each infinitely often}} \to \Delta'$

Find a bad path β in the tree:

If Π finite, $\exists$ some active leaf node containing only atomic formulas. Choose β to be path from root to this leaf

Proof of Correctness

We want to show: If Algorithm never halts, then $\forall \Phi \not\models \Gamma \to \Delta$

Find a bad path β in the tree:

If Π finite, $\exists$ some active leaf node containing only atomic formulas. Choose β to be path from root to this leaf

If Π infinite by König's Lemma, $\exists$ an infinite path. Let β be this path

Proof of Correctness

Properties of β

(1) β is a path starting at root
(2) all sequents in β were once active
(3) for all sequents in β, no formula occurs on both the Left and right side of sequent
(4) all atomic formulas $A \in \Phi$ in root sequent of β on LEFT, and thus occur on LEFT of all sequents in β

By (3)+(4), we know that NO atomic $A \in \Phi$ occurs on the Right of any sequent in β

Proof of Correctness (cont'd)

We will construct a "term" model $\mathfrak{M}$, & object assignment σ from β such that $\mathfrak{M} \models \Phi[\sigma]$ but $\mathfrak{M} \not\models \Gamma \Rightarrow \Delta$ (and thus our algorithm fails to halt & produce a proof only when $\Gamma \Rightarrow \Delta$ is not a logical consequence of Φ.)

Proof of correctness (cont'd)

We will construct a "term" model $\mathcal{M}$, + object assignment σ from β such that $\mathcal{M} \models \Phi[\sigma]$ but $\mathcal{M} \not\models \Gamma \to \Delta$

Universe M: all $\mathcal{L}$-terms t (containing only free vars)

σ : map variable a to itself

$$f^{\mathcal{M}}(r_1 \ldots r_k) \stackrel{d}{=} f r_1 \ldots r_k$$

$$P^{\mathcal{M}}(\underline{r_1 \ldots r_k}) \stackrel{d}{=} \text{true if and only if } P r_1 \ldots r_k$$

is on the LEFT of some sequent in β

$$f^{\mathcal{M}}(\underline{r_1 \ldots r_k}) \stackrel{d}{=} \underline{f r_1 \ldots r_k}$$

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}_\sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}_\sigma$ falsifies A iff A is on the RIGHT of some sequent in β

Proof (induction on A)

A atomic: A cannot occur on LEFT of some sequent in β and on RIGHT of some sequent in β (since A persists up β)

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

Proof (induction on A)

Induction Step Example $A = \exists x\, B(x)$ on RIGHT

high level: if A occurs in some sequent in β, then A persists upward until it becomes the active formula (at stage k, $A_k = A$) then use inductive hypothesis

Proof of correctness (cont'd)

Claim: For every formula A,

$\mathcal{M}, \sigma$ satisfies A iff A is on the LEFT of some sequent in β, and

$\mathcal{M}, \sigma$ falsifies A iff A is on the RIGHT of some sequent in β

Proof (induction on A)

Induction Step $A = \exists x B(x)$ on RIGHT

By Ind hyp, $\mathcal{M}, \sigma$ falsify $B(t_j)$

Since $\exists x B(x)$ persists, we have $\forall t$ $B(t)$ on RIGHT of some sequent in β

Thus $\mathcal{M}, \sigma$ falsify $B(t)$ for all terms t

β: $\vdots$

$\cdots \rightarrow B(t_k), \exists x B(x), \ldots$

$\cdots \rightarrow \exists x B(x), \ldots$