CSC 438/2404 Lecture 5 (plus tutorial)

- HW2 DUE OCT 18

* LATE ASSIGNMENTS WILL NOT BE ACCEPTED!

- Midterm in class OCT 21 3 - 5
- Extra office hours Wed OCT 16 and FRI OCT 18
- Study Problems - see course website

TODAY:

- Corollaries of completeness
- Dealing with Equality
- Theories of Arithmetic

Corollaries of Completeness

① **Lowenheim-Skolem Theorem**. Let $\mathcal{L}$ be countable, Φ a set of sentences over $\mathcal{L}$.

Φ satisfiable $\Rightarrow$ Φ is satisfiable in a countable universe.

Proof Follows from completeness proof. Let Φ be satisfiable

Let $A : \;\longrightarrow$ (empty sequent is unsatisfiable)

Then $\Phi \not\models A$, so proof of completeness constructs a countable model where Φ is satisfiable. $\boxtimes$

Corollaries of Completeness

(2) First Order Compactness Theorem.

An infinite set of first order sentences Φ is unsatisfiable if and only if some finite subset of Φ is unsatisfiable

Proof Let A be the empty sequent (or any unsatisfiable formula)
Φ unsatisfiable means $\Phi \models A$.
Thus (by completeness) there is a Φ-LK proof of A
proof. Thus there is a finite subset Φ' of Φ such that there is a Φ'-LK proof of A
$\therefore$ Φ' is unsatisfiable.

(other direction is easy)

Dealing with Equality

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Definition A *weak $\mathcal{L}$-structure* is an $\mathcal{L}$-structure where $=$ can be any binary predicate

Question: Can we define a finite set of sentences $\mathcal{E}$ that defines equality? (That is, a proper structure satisfies $\mathcal{E}$ and any weak structure satisfying $\mathcal{E}$ must have $=$ be true equality?)

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No! Let $M' = M \cup \{m'\}$ New element

Fix some $m \in M$, and let $m \overset{\mathfrak{M}'}{=} m'$

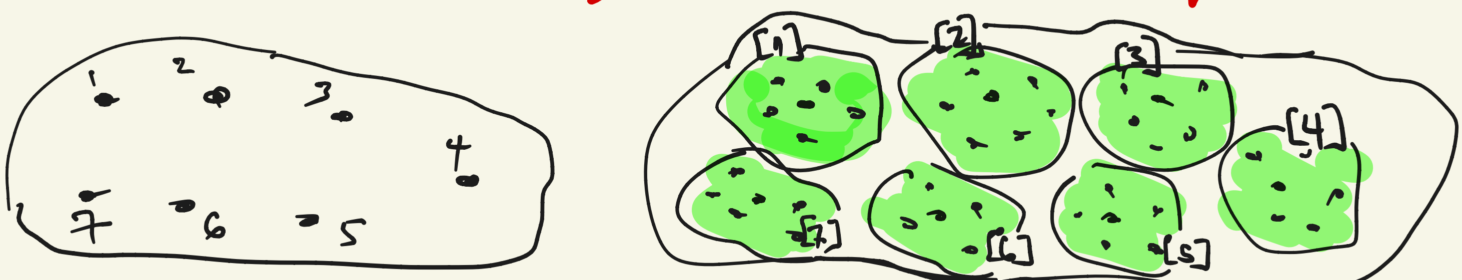
and otherwise $\mathfrak{M}'$ on m' behaves like $\mathfrak{M}$ on m

Dealing with Equality

Question: Can we define a finite set of sentences $\mathcal{E}$ that defines equality? (That is, a proper structure satisfies $\mathcal{E}$ and any weak structure satisfying $\mathcal{E}$ must have $=$ be true equality?)

But this is the only counterexample.
There is a natural, finite set of axioms that characterizes true equality (up to isomorphism)

M



M'

Dealing with Equality

Equality Axioms for $\mathcal{L}$ ($\mathcal{E}_{\mathcal{L}}$)

= is an equiv reln

E1. $\forall x (x = x)$

E2. $\forall x \forall y (x = y \supset y = x)$

E3. $\forall x \forall y \forall z ((x = y \wedge y = z) \supset x = z)$

E4. $\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n (x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset f x_1 \ldots x_n = f y_1 \ldots y_n$

for all n-ary function symbols, and for all $n \geq 1$

E5. $\forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_n ((x_1 = y_1 \wedge \ldots \wedge x_n = y_n) \supset (P x_1 \ldots x_n \supset P y_1 \ldots y_n))$

equivalence relation preserved by functions and predicates

Equality Theorem

Theorem Let Φ be a set of $\mathcal{L}$-sentences. Φ is satisfiable iff $\Phi \cup \mathcal{E}_{\mathcal{L}}$ is satisfied by some weak $\mathcal{L}$-structure.

Proof straightforward (see Lecture Notes)

LK with Equality

Add these axioms for all terms $u, t, u_1, \ldots, t_1 \ldots$

L1 $\quad \longrightarrow t = t$

L2 $\quad t = u \longrightarrow u = t$

L3 $\quad t = u, u = v \longrightarrow t = v$

L4 $\quad t_1 = u_1, \ldots, t_n = u_n \longrightarrow f t_1 \ldots t_n = f u_1 \ldots u_n$

L5 $\quad t_1 = u_1, \ldots, t_n = u_n, P t_1 \ldots t_n \longrightarrow P u_1 \ldots u_n$

Now an LK-Φ proof of $\longrightarrow A$ means an LK proof of A from Φ and from above axioms

Models of $\mathcal{L}_A$

Recall $\mathcal{L}_A = \{0, s, +, \cdot\ ;\ =\}$ Language of arithmetic

the **standard model** for $\mathcal{L}_A$: $\underline{\mathbb{N}}$

$M = \mathbb{N}$, $0, s, +, \cdot$ have usual meanings

Th(A) or TA: the set of all sentences of $\mathcal{L}_A$ that are true in $\mathbb{N}$

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Defn A set Φ of sentences is decidable if there is an algorithm (that always halts) that given a sentence B, outputs 1 if B is in Φ and otherwise outputs 0

We will soon see that TA is **not** decidable.

On the other hand, restricted systems of TA <u>are</u> decidable ($\mathcal{L}_s$, $\mathcal{L}_+$)

Theories

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Definition A theory (over $\mathcal{L}$) is a set Σ of sentences closed under logical consequence. ($\Sigma \models A$ then $A \in \Sigma$)

We can specify a theory by a finite or countable set of sentences Ψ -- the theory corresponding to Ψ is $\{A \mid \Psi \models A\}$

Notation Σ a theory $\Sigma \vdash A$ means $A \in \Sigma$

Definition For a language $\mathcal{L}$, $\Phi_0^{\mathcal{L}}$ is the set of all sentences over $\mathcal{L}$

Theories

Definition Σ is consistent if and only if $\Sigma \neq \Phi_0$

(If $\Sigma = \Phi_0$ then Σ contains A & $\neg A$

Conversely if Σ contains A & $\neg A$ then Σ contains all of Φ_0)

Theories

Definición Σ is consistent if and only if $\Sigma \nvdash \Phi_0$

Σ is complete iff Σ is consistent and for all sentences A, either $\Sigma \vdash A$ or $\Sigma \vdash \neg A$

Theories

Definición Σ is **consistent** if and only if $\Sigma \nvdash \Phi_0$

Σ is **complete** iff Σ is consistent and for all sentences A, either $\Sigma \vdash A$ or $\Sigma \vdash \neg A$

Example $\mathcal{L}_A = \{0, s, +, \cdot\,; =\}$

TA = all sentences over $\mathcal{L}_A$ that are true in $\mathbb{N}$

is consistent and complete

Theories

Definición Σ is **consistent** if and only if $\Sigma \nvdash \Phi_0$

Σ is **complete** iff Σ is consistent and for all sentences A, either $\Sigma \vdash A$ or $\Sigma \vdash \neg A$

Definición A theory Σ over $\mathcal{L}_A$ is **sound** iff $\Sigma \subseteq TA$

Subsystems of True Arithmetic

- Theory of Successor $(0, s; =)$
- Presburger Arithmetic $(0, s, +; =)$

Defn $\mathcal{L}_s = \{0, s; =\}$ Language of successor

The standard model for $\mathcal{L}_s$, $\mathbb{N}_s$:

$M = \mathbb{N}$, 0 and s have usual meaning $(s(x) = x+1)$

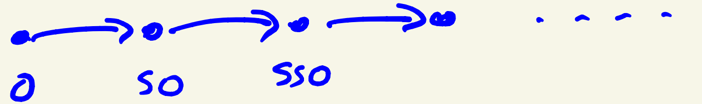
Let $Th(s)$ (theory of successor) be the set of all sentences of $\mathcal{L}_s$ that are true in $\mathbb{N}_s$

Th(S): There is a simple (infinite but countable) complete set of axioms for Th(S), Ψ_S

Ψ_S:
(S1) $\forall x\,(Sx \neq 0)$
(S2) $\forall x \forall y\,(Sx = Sy \supset x = y)$
(S3) $\forall x\,(x = 0 \vee \exists y\,(x = Sy))$
(S4) $\forall x\,(Sx \neq x)$
(S5) $\forall x\,(SSx \neq x)$
(S6) $\forall x\,(SSSx \neq x)$
(S7)
⋮

Models for Ψ_s: A model for Ψ_s is a model/structure over $\mathcal{L}_s$ that satisfies all formulas in Ψ_s

(1)



← isomorphic to $\mathbb{N}$

up to renaming

(2)

← $\mathbb{N}$ plus a copy of integers

plus

* In class I said that $\underline{\mathbb{N}}$ was the only model for Ψ_s

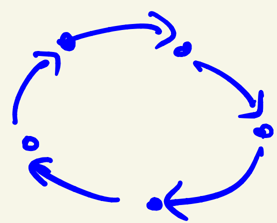
this is incorrect

③ generalizing ②, models contain one copy of $\mathbb{N}$, plus any number of copies (isomorphic to) the integers

Note without all axioms S4, S5, S6, ...
we could have additional models with loops

$$0 \to S0 \to SS0 \to \cdot \quad \cdots$$

plus



any number of cycles

Theorem Ψ_S is complete and consistent

(proof omitted)

Therefore although Ψ_S has both the standard model $\mathbb{N}$ as well as nonstandard models, all models $\mathcal{M}$ of Ψ_S have the same set of true sentences.

Theorem Ψ_s is complete and consistent

(proof omitted)

Therefore although Ψ_s has both the standard model $\mathbb{N}$ as well as nonstandard models, all models $\mathcal{M}$ of Ψ_s have the same set of true sentences.

We'll see later that when a set of sentences (such as $Th(S)$) has a nice (enumerable) axiomatization, then $Th(S)$ is decidable.

Defn $\mathcal{L}_+ = \{0, s, + \,;\, =\}$ Language of Presburger arithmetic

the **standard model** for $\mathcal{L}_+$, $\underline{\mathbb{N}_+}$:

$M = \mathbb{N}$, $0, s, +$ have usual meaning

Th(+): (theory of Presburger arithmetic, or standard model for $\mathcal{L}_+$): all sentences of $\mathcal{L}_+$ that are true in $\underline{\mathbb{N}_+}$

Presburger (1928) showed that Th(+) is also characterized by a countable set of axioms (like the theory of successor)

so it is also consistent and complete

BACK TO TA (TRUE ARITHMETIC)

the standard model for $\mathcal{L}_A$, $\underline{\mathbb{N}}$:

$M = \mathbb{N}$, $0, S, +, \cdot$ have usual meaning

$\underline{Th(A) \text{ or } TA}$: (Theory of True Arithmetic): set of all $\mathcal{L}_A$ sentences that are true in standard model $\underline{\mathbb{N}}$

$\underline{\text{Theorem}}$ TA has a nonstandard model

Theorem TA has a nonstandard model

Proof Let c be a constant symbol (not in $\mathcal{L}_A$)

$$\Psi = \{ c \neq 0,\ c \neq S0,\ c \neq SS0,\ c \neq SSS0,\ \ldots \}$$

- every finite subset of Ψ is satisfiable
- so by compactness, $TA \cup \Psi$ has a model $\mathcal{M}$
- $\mathcal{M}$ is not isomorphic to $\underline{\mathbb{N}}$ (standard model) since c cannot be any element of $\mathbb{N}$

MIDTERM REVIEW

Material covered:

1. Propositional Calculus (pp 1-17 of Notes and Notes on Resolution)
2. Predicate Calculus (pp 18-30 of Notes)
3. Completeness (pp. 31-38 of Notes)
4. Equality Axioms (pp. 43-47)
   Corollaries of Completeness (48-53)

MIDTERM REVIEW

Study Tips

- Read Lecture Notes and Course Notes carefully first
- Then do/review solutions to homework questions and tutorial problems
- Then do practice questions (see handout "Midterm Study Problems")

Office hours

Wed Oct 16 3-4
Fri Oct 18 2-3

MIDTERM REVIEW

Study Tips

- given a propositional or first order formula/sequent, produce a (RES, PK, LK) proof
- Run completeness Algorithm(s)
- Compactness: what is it? how to use it? why is it true?
- give a model for Φ; does $\Phi \models A$? is Φ valid? satisfiable? invalid/unsat.?