CS 2429
Communication Complexity and Applications
HOMEWORK ASSIGNMENT
Due: Nov 25, 2014

1. Let f be a boolean function on $X \times Y$. prove that if all of the rows of M_f are distinct, then $D(f) \geq \log\log|X|$. Prove that $D(f) \leq rank(f) + 1$.

2. $MED(x, y)$ is defined to be the median of the multiset $x \cup y$. (Here we are viewing x and y as n-bit binary strings each representing subsets of $[n]$.) Using binary search, one can show that $D(MED) = O(\log^2 n)$. Give an $O(\log n)$-bit protocol for MED.

3. $GT(x, y)$, the greater-than function, is 1 if and only if $x > y$ (viewing x and y as numbers expressed in binary, each as n-bit numbers). What is the communication comlexity of the greater-than function?

   (a) Prove a lower bound of n on the deterministic communication complexity of GT.

   (b) Given an upper bound of $O(\log^2 n)$ on the randomized complexity.

4. Consider the following communication complexity problem. Alice holds an n bit binary string x and Bob holds two strings y', y where $y' \neq y$ and either (1) $y = x$ or (2) $y' = x$. Decide which case (1) or (2) holds.

   (a) Given an $O(1)$ public-coin randomized communication protocol for this problem.

   (b) Prove that $\Omega(\log n)$ bits of communication is necessary for deterministic protocols. (In fact, $\Omega(\log n)$ bits of communication are even required for nondeterministic protocols. You will receive extra credit for proving this stronger result.)

5. Recall that for a boolean function $f : \{0,1\}^n \to \{0,1\}$, the block sensitivity of f at α, $bs(f, \alpha)$ is the maximum number of blocks bs such that there exist disjoint sets $B_1, \ldots, B_{bs}$, $B_i \subset [n]$ such that for all $i \leq bs$, $f(\alpha) \neq f(\alpha^{B_i})$, where α^{B_i} is just like α but where the values indexed by B_i are flipped. The block sensitivity of f is the maximum over all α of $bs(f, \alpha)$.

   - Prove that the block sensitivity of f is greater than or equal to the decision tree complexity of f. That is, show $bs(f) \leq D(f)$.
   - Given an example of a boolean function f such that $bs(f) = \sqrt{n}$ but $D(f) = n$.
   - (Extra Credit) Prove $D(f) \leq bs(f)^3$.
   - (Open) It is open to improve this relationship. For example it is not known whether $D(f) \leq bs(f)^2$.