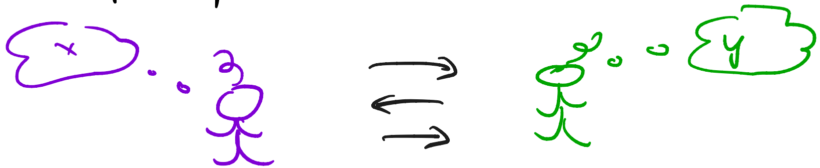
Last Class

1. 2-party basic model (deterministic)



$$P^{cc}(f) = \min_{\pi \text{ for } f} \ \max_{\substack{(x,y) \\ |x|=|y|=n}} \ \#\text{ bits sent on input } (x,y)$$

2. Randomized CC : Public vs Private coin model

BPP^{cc} : two-sided error

RP^{cc} : one-sided error

ZPP^{cc} : zero sided error.

cost = max # bits sent + $|r|$

3. Nondet cc / coNondet cc

$f(x,y)=1$ $\exists r$ $\pi(x,y,r)=1$
$f(x,y)=0$ $\forall r$ $\pi(x,y,r)=0$

$f(x,y)=0$ $\exists r$ $\pi(\ \ )=0$
$(\ \ )=1$ $\forall r$ $\pi(\ \ )=1$

BPP^{cc} : public coin

$\Pi(x, y, r)$: $\forall r$ $\Pi(x,y,r)$ is a deterministic protocol (outputs some value)

Π compute f in BPP^{cc} iff:

$\forall (x,y)$, $|x| = |y| = n$

$$\Pr_r \left[\Pi(x,y,r) = f(x,y)\right] \geq \underbrace{\frac{2}{3}}\quad \left(1 - \frac{1}{2^c}\right)$$

If we want bad prob. ε to be $\frac{1}{2^c}$

repeat $\sim c$ times

NP^{cc}: $\Pi(x,y,r)$

Π computes f NP^{cc} iff

$\forall (x,y)$ If $f(x,y) = 1$ then ~~$\forall$~~ $\exists r$ st $\Pi(x,y,r) = 1$

ow $f(x,y) = 0$ then $\forall r$ $\Pi(x,y,r) = 0$

Last class cont'd

✓ (1.) Protocols can be balanced

✓ (2.) Error ε can be amplified with little cost

✓ (3.) Can assume $|r|$ is $O(\log n)$ for randomized protocols

$\therefore$ Public coin & private coin randomized comm. nearly the same

Newman's Thm

(4.) $P^{cc}(f) \leq NP^{cc}(f) \cdot coNP^{cc}(f)$ [Yannakakis]

(5.) Log Rank conjecture

TODAY

RANK & DETERMINISTIC CC

$P^{cc}(f)$ = min cost of deterministic protocol for f

We know : $P^{cc}(f) \geq \log rk(M_f)$

RANK LOWER BOUND METHOD (for deterministic CC)

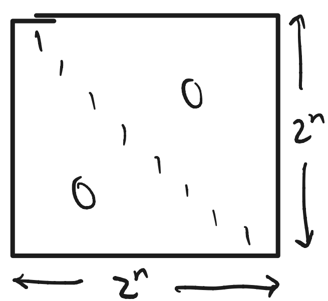
$rk(M_f) \geq r \implies P^{cc}(f) \geq \log r$

Example

$EQ(x,y) = 1 \Leftrightarrow x = y$

$|x| = |y| = n$

M_{EQ}



$rk(M_{EQ}) = 2^n$

$\implies P^{cc}(f) = \Omega(n)$

LOG RANK CONJECTURE (LRC)

$P^{cc}(f)$ = min cost of deterministic protocol for f

We know: $P^{cc}(f) \geq \log rk(M_f)$

$\boxed{\text{LRC [Lovász-Saks '88]: } P^{cc}(f) \leq O\left(\log rk(M_f)\right)^c}$

Says if rank small, deterministic cc is small

Log Rank Conjecture (LRC)

$P^{cc}(f)$ = min cost of deterministic protocol for f

We know: $P^{cc}(f) \geq \log rk(M_f)$

$$\boxed{\underline{\text{LRC}}: \quad P^{cc}(f) \leq O\left(\log rk(M_f)\right)^c}$$

State of the art:

$$\Omega(\log rk(M_f)^2) \leq P^{cc}(f) \leq O\left(\sqrt{rk(M_f)} \cdot \log rk(M_f)\right)$$

Göös, P, Watson

Lovett

LRC: a weaker (but equivalent) formulation

$$\boxed{\underline{\text{Weak LRC}}: \quad BPP^{cc}(f) \leq O(\log rk(M_f))^c}$$

Says low rank implies low randomized communication

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Says low rank implies low randomized communication

$$\boxed{\underline{\text{Weaker LRC}}: \quad POSTBPP^{cc}(f) \le O(\log rk(M_f))^c}$$

LRC: a weaker (but equivalent) formulation

$\boxed{\underline{\text{Weak LRC}}: \quad BPP^{cc}(f) \leq O(\log rk(M_f))^c}$

Says low rank implies low randomized communication

$\boxed{\underline{\text{Weaker LRC}}: \quad POSTBPP^{cc}(f) \leq O(\log rk(M_f))^c}$

Theorem 1 The above weak versions of LRC imply LRC

LRC: Another equivalent formulation

Let $\text{mono}(M_f) = \max\limits_{\substack{\text{monochrom.} \\ \text{rectangle } R \subseteq M_f}} \dfrac{|R|}{|M_f|}$

$$P^{cc}(F) \geq \log rk(M_f)$$

$$P^{cc}(F) \geq \log\left(\frac{1}{\text{mono}(M_f)}\right)$$

$$\boxed{\underline{\text{Conjecture}} \quad \log\left(\frac{1}{\text{mono}(M_f)}\right) \leq O\left(\log rk(M_f)\right)^c}$$

Says small rank implies large mono. rectangle

<u>Theorem 2</u> The above conjecture implies the LRC

Thm 3

Let $r = rk(M_f)$, $\delta(r)$ be arbitrary ~~nondecreasing~~ function

If $mono(M_f) \geq \delta(r)$ then $P^{cc}(f) \leq O(\log^2 r + \log r \cdot \log(\frac{1}{\delta(r)}))$

Thm 3 $\Rightarrow$ Thm 2:

* If conjecture is true, $mono(M_f) \geq 2^{-O(\log r)^c}$ for some c

so by Thm3 (with $\delta(r) = 2^{-O(\log r)^c}$),

$P^{cc}(f) \leq O(\log^2 r + \log r + O(\log r)^c) = O(\log r)^{c'}$ so LRC true.

Thm 3

Let $r = rk(M_f)$, $\delta(r)$ be arbitrary nondecreasing function

If $\mathrm{mono}(M_f) \geq \delta(r)$ then $P^{cc}(f) \leq O\left(\log^2 r + \log r \cdot \log\left(\frac{1}{\delta(r)}\right)\right)$

Proof idea

Let M_f have small rank, and let R be large mono. subrectangle.

Write M_f as: $\begin{array}{|c|c|} \hline R & A \\ \hline B & C \\ \hline \end{array}$

Since R rank 1, one of A, B has $rk \leq rk(M_f)/2$

Spose A has $rk \leq \frac{r}{2}$

Players determine if $x \in \boxed{R \mid A}$

If yes $\rightarrow$ recurse on submatrix of $rk \sim \frac{r}{2}$

If NO $\rightarrow$ recurse on submatrix $\boxed{B \mid C}$ of much smaller size

Thm 3 assume conj & $r = rk(M_f)$ then

Let $r = rk(M_f)$, $\delta(r)$ be arbitrary nonincreasing function

Assume for any rectangle $R_1 \subseteq M_f$ $\exists$ subrectangle $R_2 \subseteq R_1$ such that R_2 is monochromatic and $|R_2| \geq \delta(r) \cdot |R_1|$

then $P^{cc}(f) \leq O\left(\log^2 r + \log r \cdot \log\left(\frac{1}{\delta(r)}\right)\right)$

Proof (of thm)

Let $R \subseteq M_f$, $|R| \geq \delta(r) |M_f|$

Write $M_f = \begin{array}{|c|c|} \hline R & A \\ \hline B & C \\ \hline \end{array}$

Then by subadditivity of rank,

$$rk(A) + rk(B) \leq rk\left(\begin{array}{|c|c|} \hline R & A \\ \hline B & 0 \\ \hline \end{array}\right) - rk\left(\begin{array}{|c|c|} \hline R & 0 \\ \hline 0 & 0 \\ \hline \end{array}\right) \leq rk(M_f) + 1$$

wlog assume $rk(A) \leq rk(B) \leq \frac{r}{2} + 1$

so $rk\left(\boxed{R \mid A}\right) \leq \frac{r}{2} + 2$

$M_f = \begin{array}{|c|c|} \hline R & A \\ \hline B & C \\ \hline \end{array}$ $\qquad rk\left(\boxed{R \mid A}\right) \le \frac{r}{2} + 1$

Protocol

Row player (Alice) sends 1 bit specifying if $x \in \boxed{R \mid A}$ or
$x \in \boxed{B \mid C}$

(if $rk(B) < rk(A)$ then Bob specifies if $y \in \begin{bmatrix} R \\ \hline B \end{bmatrix}$ or $y \in \begin{bmatrix} A \\ \hline C \end{bmatrix}$)

Recurse.

$$M_f = \begin{array}{|c|c|}\hline R & A \\ \hline B & C \\ \hline\end{array} \qquad rk\left(\boxed{R \mid A}\right) \le \frac{r}{2} + 1$$

<u>Protocol</u>

Row player (Alice) sends 1 bit specifying if $x \in \boxed{R \mid A}$ or $x \in \boxed{B \mid C}$

(if $rk(B) < rk(A)$ then Bob specifies if $y \in \boxed{\begin{array}{c} R \\ \hline B \end{array}}$ or $y \in \boxed{\begin{array}{c} A \\ \hline C \end{array}}$)

Recurse.

Analysis:

Let $L(m,r)$ = # leaves in above protocol where $m = |M|$, $r = rk(M)$

Then $L(m,r) \le L(m, \frac{r}{2} + 2) + L(m(1 - \delta(r)), r)$

<u>Claim</u> $L(m,r) \le \exp\left(\log^2 r + \log r \cdot \log\left(\frac{1}{\delta(r)}\right)\right)$

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Then $L(m,r) \leq L(m, \frac{r}{2}) + L(m(1-\delta(r)), r)$ (same asymptotic behavior)

<u>Claim</u> $L(m,r) \leq \exp\left(\log^2 r + \log r \cdot \log\left(\frac{1}{\delta(r)}\right)\right)$

$$L(m,r) \leq L(m, \tfrac{r}{2}) + \underbrace{L(m(1-\delta), r)}$$

$$\leq L(m, \tfrac{r}{2}) + \underbrace{L(m(1-\delta), \tfrac{r}{2}) + \underbrace{L(m(1-\delta)^2, r)}}$$

$$\leq L(m, \tfrac{r}{2}) + L(m(1-\delta), \tfrac{r}{2}) + L(m(1-\delta)^2, \tfrac{r}{2}) + L(m(1-\delta)^3, r)$$

$$(1-\delta)^{\frac{1}{\delta}} \sim \frac{1}{e}$$

$$\leq \frac{\log m}{\delta} L(m, \tfrac{r}{2})$$

$$\leq \frac{\log m}{\delta}^{\log r}$$

Analysis:

Let $L(m,r)$ = # leaves in above protocol where $m = |M|$, $r = rk(M)$

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$$
\begin{aligned}
L(m,r) &\le L(m, \tfrac{r}{2}) + L(m(1-\delta), r) \\
&\le L(m, \tfrac{r}{2}) + L(m(1-\delta), \tfrac{r}{2}) + L(m(1-\delta)^2, r) \\
&\le L(m, \tfrac{r}{2}) + L(m(1-\delta), \tfrac{r}{2}) + L(m(1-\delta)^2, \tfrac{r}{2}) + L(m(1-\delta)^3, r) \\
&\quad \vdots \\
&\le \frac{\log m}{\delta} L(m, \tfrac{r}{2}) \\
&\le \left[\frac{\log m}{\delta}\right]^{\log r}
\end{aligned}
$$

$(1-\delta)^{\frac{1}{\delta}} \sim \frac{1}{e}$

Since protocols can be balanced (last lecture)

$$
\begin{aligned}
P^{cc}(f) \le \log L(m,r) &\lessapprox \log r \left[\log\log m + \log \tfrac{1}{\delta}\right] \\
&\le (\log r)^2 + \log r \cdot \log\left(\tfrac{1}{\delta}\right)
\end{aligned}
$$

∎

Theorem 1 If $BPP^{cc}(f) \leq (\log rk(M_f))^c$ for some $c > 0$ then LRC true.

(so to prove LRC, just need to show that any M_f of rank r has a BPP^{cc} protocol of cost $(\log r)^{O(1)}$)

Theorem 1

$$P^{cc}(f) \leq O\left(BPP^{cc}(f) \cdot \log^2 rk(M_f)\right)$$

Lemma: Let $r = rk(M_f)$, $\varepsilon = \frac{1}{8r}$, Let $BPP^{cc}_{\varepsilon}(f) = c$

then $\exists$ mono R with $|R| \geq \frac{2^{-c}}{16} |M_f|$

Lemma $\rightarrow$ Theorem 1:

Let $\delta(r) \geq 2^{-c}/16$

$$P^{cc}(f) \leq O(\log^2 r + \log r \cdot c)$$

$$c \leq O\Big(BPP^{cc}(f) \cdot \underbrace{\log\left(\tfrac{1}{\varepsilon}\right)}_{\log r}\Big)$$

$$P^{cc}(f) \leq O(\log^2 r + \log^2 r \cdot BPP^{cc}(f))$$

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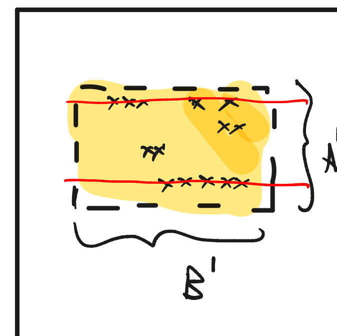
Proof sketch

Let π be randomized BPP^{cc} protocol for f with cost c

- By averaging $\exists\ (R', b)$ s.t. $|R'| \geq \frac{1}{2} \frac{|M_f|}{2^c}$ s.t. fraction of $(x,y) \in R'$ not labelled by b is $\leq 2\varepsilon$

$R' = A' \times B'$



- Remove from R' any bad rows with $\geq 4\varepsilon$ fraction of errors
  By Markov, # remaining rows is $\geq \frac{\#\text{rows}(R')}{2}$
  Let A'' = good rows of A'

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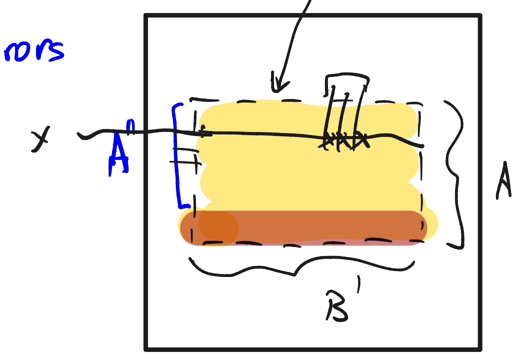
- Remove from R' any bad rows with $\geq 4\varepsilon$ fraction of errors
  Let A'' = good rows of R' $(|A''| \geq |A'|/2)$

- Since M_f has rank $\leq r$, $Q = A'' \times B'$ has rank $\leq r$
  Let $x_1 \ldots x_r$ be a basis for Q
  Let $B_i = \{ y \in cols(R') \mid f(x_i, y) \neq b \}$. $|B_i| \leq 4\varepsilon \cdot [$



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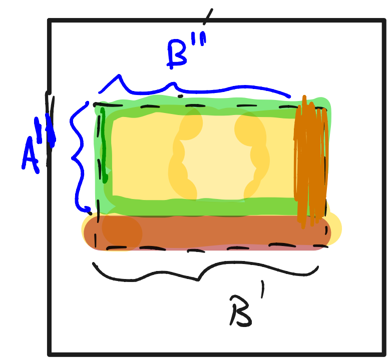
  Let $B'' = B' - \bigcup_i B_i$. $|B''| \geq (1 - 4\varepsilon r)|B'|$



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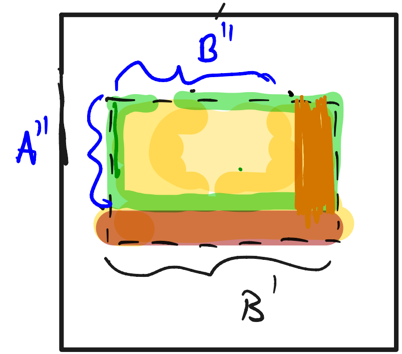
then $\exists$ mono R with $|R| \geq \frac{2^{-c}}{16} |M_f|$

Proof sketch

Let π be randomized BPP^{cc} protocol for f with cost c

- By averaging $\exists (R', b)$ st. $|R'| \geq \frac{1}{2} \frac{|M_f|}{2^c}$ st. fraction of $(x,y) \in R'$ not labelled by b is $\leq 2\varepsilon$
- Remove from R' any bad rows with $\geq 4\varepsilon$ fraction of errors
  Let A' = good rows of R'. $|A| \geq |A'|/2$
- Since M_f has rank $\geq r$, Q has $rk \geq r$
  Let $x_1 \ldots x_r$ be a basis for Q
  Let $B_i = \{ y \in cols(R') \mid f(x_i, y) \neq b \}$. $|B_i| \leq 4\varepsilon |B'|$
  Let $B'' = B' - \bigcup_i B_i$. $|B''| \geq (1 - 4\varepsilon r) |B'|$



- The matrix $A'' \times B''$ is spanned by rows all equal to b
  So all rows of $A'' \times B''$ are constant.
  Let R_2 = rows of $A'' \times B''$ taking most popular value
  $|R_2| \geq \frac{1}{2} |A''| \times |B''| \geq \frac{1}{8} |A'| |B'| = \frac{1}{16} 2^{-c} |R'|$

Note Theorem holds for even more powerful POSTBPP^{cc} model of cc.

$\boxed{\text{POSTBPP}^{cc}(f):}$ zero communication randomized protocol

on (x, y, r) protocol output is in $\{0, 1, \perp\}$ ← don't know

Correctness:

$$\Pr_r\left[\Pi(x,y) = f(x,y) \;\middle|\; \Pi \text{ doesn't output } \perp\right] \geq 2/3$$

Cost: k, where $\forall (x,y)$ $\Pr_r\left[\Pi(x,y) \neq \perp\right] \geq 2^{-k}$

Thm

$$P^{cc}(f) \leq O\left(\text{PostBPP}^{cc}(f) \cdot \log^2 rk(M_f)\right)$$

Next : Two Theorems of Yannakakis

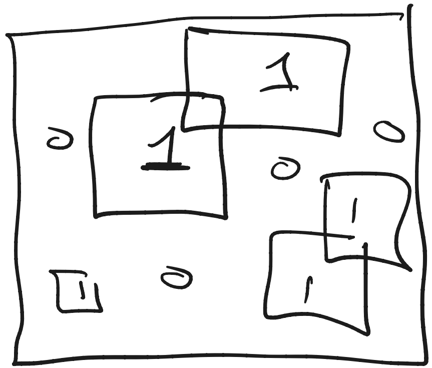
① $P^{cc}(f) \leq NP^{cc}(f) \cdot coNP^{cc}(f)$

② Log of Partition Number of M_f $= \Theta\left(P^{cc}(f)\right)^2$

Nondet. protocol Π for f:

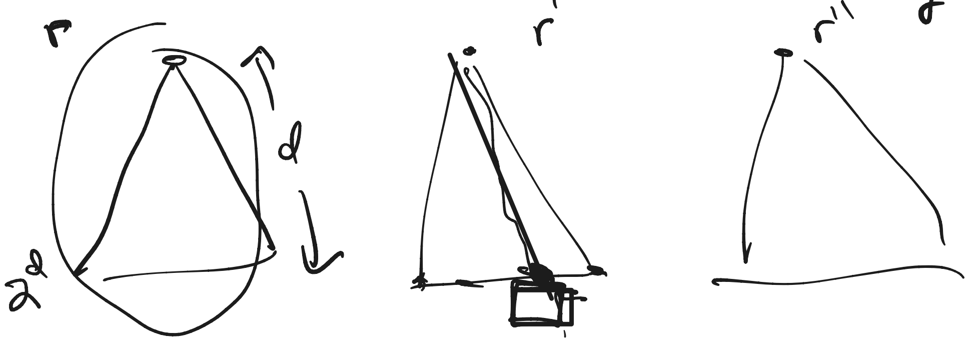
gives rise to a covering of 1's of f by monochrom. subrectangles

M_f:



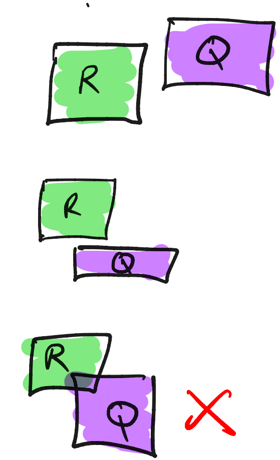
If cost of protocol is $c = d + d'$ then #rectangle in cover is $\le 2^c$

conondet protocol: gives rise to covering of 0's of f by 0-mono. subrectangles



cost $\Pi = d + \lceil r \rceil d'$

Theorem $P^{cc}(f) \le NP^{cc}(f) \cdot coNP^{cc}(f)$.



Pf

- Observation: Let R be a 0-mono rectangle of M_f
  Q " " 1-mono " " "

  Then either $cols(R)$, $cols(Q)$ are disjoint
  or $rows(R)$, $rows(Q)$ " "

- Spose $(x,y) \in f^{-1}(1)$, $(x,y) \in R$

  Then either R row-intersects $\le \frac{1}{2}$ 0-rect's
  or R column-intersects $\le \frac{1}{2}$ 0-rect's

Protocol π:

(0) $\mathcal{R}$ = all 1-mono rectangles of NP^{cc} protocol for f
$\mathcal{Q}$ = " " " " " $coNP^{cc}$ " " "

Repeat until no 0-rectangles in $\mathcal{Q}$:

(1A) Alice looks for a 1-mono R such that $x \in rows(R)$ and s.t. R row-intersects with $\le \frac{1}{2}$ 0-rect's

If she finds such an R, she sends name of R to Bob. + they can prune # possible 0-rect's by $\frac{1}{2}$

(1B) OW (Alice cant find such an R), Bob looks for a 1-mono R st $y \in R$ and R col-intersects with $\le \frac{1}{2}$ 0-rect's

If Bob finds R, he sends name of R to Alice + they can prune # possible 0-rect's by $\frac{1}{2}$

(2) If (1A), (1B) fail $\longrightarrow$ $\pi(x,y)$ outputs 0

$\boxed{\text{Protocol } \Pi}$:

0. $\mathcal{R}$ = all 1-mono rectangles of NP^{cc} protocol for f
   Q = " " " " " $coNP^{cc}$ " " "

Repeat until no 0-rect's in Q:

1A. Alice looks for a 1-mono R such that $x \in rows(R)$ and st. R row-intersects with $\leq \frac{1}{2}$ 0-rect's

   If she finds such an R, she sends name of R to Bob. + they can prune # possible 0-rect's by $\frac{1}{2}$

1B. OW (Alice cant find such an R), Bob looks for a 1-mono R st $y \in R$ and R col-intersects with $\leq \frac{1}{2}$ 0-rect's

   If Bob finds R, he sends name of R to Alice + they can prune # possible 0-rect's by $\frac{1}{2}$

2. If 1A, 1B fail $\longrightarrow$ $\Pi(x,y)$ outputs 0

cost of Π: # iterations = $\log(coNP^{cc}(f))$

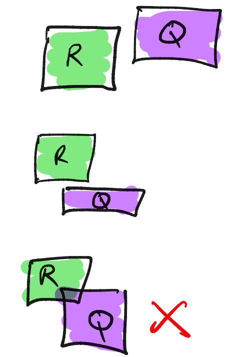
each iteration has cost $\sim \log(NP^{cc}(f))$

Next : Two Theorems of Yannakakis

✓ ① $P^{cc}(f) \leq NP^{cc}(f) \cdot coNP^{cc}(f)$

② Log of Partition Number of M_f $= O\left(P^{cc}(f)\right)^2$

Theorem $\log(\text{Partition Number of } M_f) = O(P^{cc}(f))^2$

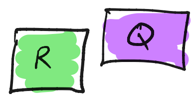


Recall

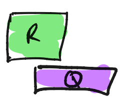
observation: Let R be a 1-mono rectangle of M_f
Q " " 0-mono " " "

Then either cols(R), cols(Q) are disjoint
OR rows(R), rows(Q) " "

Protocol:



} R, Q row-intersect



} R Q column-intersect

Let $\mathcal{P}$ be a partition of M_f into mono rectangles.

$\forall (x,y)$ either $\exists R \in \mathcal{P}$ such that $x \in R$ and R row-intersects $\leq$ half the rectangles in $\mathcal{P}$

OR $\exists R \in \mathcal{P}$ such that $y \in R$ and R column-intersects $\leq$ half the rectangles in $\mathcal{P}$

Players find such an R, which prunes $|\mathcal{P}|$ by half & recurse.