APPLICATIONS

- Streaming
- Property Testing
- game theory
- TIME/SPACE Turing Machine LBs
- Circuit complexity
- Proof complexity
- Extension Complexity
- Clique/Coclique, Graph Theory, Learning Partial Functions
  Partition vs CC

Partition Number vs Deterministic CC

Let $F : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$.

<u>Defn</u> (Partition Number of F).

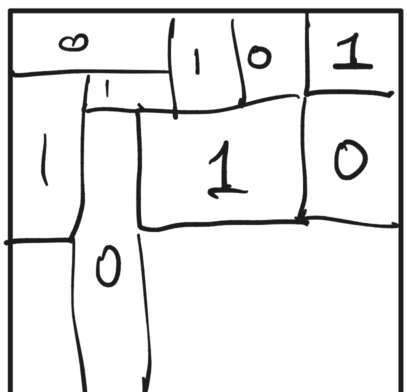
The Partition number of F, $\chi(F) \overset{d}{=} \chi_1(F) + \chi_0(F)$

where $\chi_i(F)$ = min number of i-monochromatic rectangles needed to partition $F^{-1}(i)$

M_F :

all Bob's inputs

rows with all inputs to Alice



$$cc(F) \leq \log(\chi(F)^2)$$

Partition Number vs Deterministic CC

Open Question (Partition vs Det CC) $P^{cc}(F) \overset{?}{=} \Theta(\log(\chi(F)))$

Known $\forall F \quad P^{cc}(F) \leq O(\log^2 \chi(F))$

$\tilde{\Omega}(\log^2 \chi(F))$

Theorem 1 [GPW'15, K'15] $\exists f$ st. $P^{cc}(F) = \tilde{\Omega}(\log^2 \chi_1(F))$ (Q1 false)

Corollary: $\exists F$ st. $P^{cc}(F) = \tilde{\Omega}(\log^2 \operatorname{rank}(F))$ since $\chi_1(F) \geq \operatorname{rank}(F)$

Pf (theorem 1)

Prove query separation + deterministic lifting (last class)

Clique vs Independent Set & Applications

CIS_g :

Clique $\alpha \subseteq [n]$

Independent set $\beta \subseteq [n]$

$CIS_g(\alpha, \beta) = 1$ iff $\alpha \cap \beta \neq \emptyset$

* Note $|\alpha \cap \beta|$ is 0 or 1, so $UP^{cc}(CIS_g) = O(\log n)$

Nondet protocol where every 1 input has exactly one accepting path

* $P^{cc}(CIS_g) = O(\log^2 n)$

Clique vs Independent Set & Applications

CIS_g: Alice given clique α in g
Bob given indep. set β in g
Output 1 iff $\alpha \cap \beta \neq \phi$

Open Question (clique vs Ind. set) $P^{cc}(CIS_g) \overset{?}{=} O(\log n)$
$coNP^{cc}(CIS_g) \overset{?}{=} O(\log n)$

Theorem 2 (GPW) $\exists g$ $P^{cc}(CIS_g) = \Omega(\log^2 n)$ ← deterministic

Theorem 3 (Göös, BBBJK) $\exists g$ $coNP^{cc}(CIS_g) = \Omega(\log^2 n)$ ← conondet lifts

Corollaries of thm 3: ASS conjecture, learning partial functions

Proofs via Lifting

Theorem 1 (GPW) [Kothari] $\exists f$ st. $P^{cc}(F) = \tilde{\Omega}(\log^2 \chi(F))$

Theorem 2 (GPW) $\exists g$ $P^{cc}(CIS_g) = \Omega(\log^2 n)$

Theorem 3 (Göös, BBBJK) $\exists g$ $coNP^{cc}(CIS_g) = \Omega(\log^2 n)$

$P^{cc}(F) \Rightarrow P^{dt}(f)$ = decision tree complexity

$NP^{cc}(F) \Rightarrow P^{Ndt}(f)$ = min width DNF for f

$\log \chi_1(F) \Rightarrow UP^{dt}(f)$ = min width of an unambiguous DNF for f ($\forall \alpha$ at most one term satisfied by α)

$\log \chi_0(F) \Rightarrow coUP^{dt}(f) = UP^{dt}(\neg f)$

ALON SAKS SEYMOUR CONJ

Theorem [Graham-Pollak '72]
If G is an edge-disjoint union of K complete bipartite graphs then max-clique size is $K+1$

Q: What is max chromatic number of a graph with bipartition number K?

Easy: maximum $\leq K^{O(\lg K)}$

Counterexample: [Huang, Sudakov] $\exists$ graph w/ chromatic # $K^{6/5}$

ASS conjecture: Max chrom # is poly(K)

CIS CC EQUIVALENT TO ALON-SAKS-SEMOUR

Theorem

$$\underbrace{\boxed{\exists g \ \mathrm{coNP}^{cc}(\mathrm{CIS}_g) = \Omega(\log n)^2}}_{\text{Theorem 3}} \iff \underbrace{\boxed{\begin{array}{l}\exists H \text{ a union of } n \text{ disjoint bipartite cliques, and} \\ \log[\chi(H)] = \Omega(\log n)^2\end{array}}}_{\text{ASS conjecture false}}$$

s.t.

ALON SAKS SEYMOUR CONJECTURE

Claim $\boxed{\exists \mathcal{G} \quad coNP^{cc}(CIS_{\mathcal{G}}) = \Omega(\log^2 n)}$ ← Theorem 3

$\Longleftrightarrow$ $\forall k \, \exists H$ st. H has bipartition number k, and chromatic number $k^{\log k}$

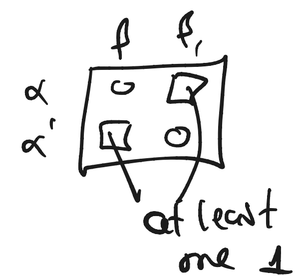
Proof Let $\mathcal{G} = (V,E)$, $|V| = n$ witness Theorem 3.

So $coNP^{cc}(CIS_{\mathcal{G}}) = \Omega(\log^2 n)$.

Constructing H:

$$V(H) = \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{l} \alpha \text{ a clique in } \mathcal{G} \\ \beta \text{ an IS in } \mathcal{G} \end{array}, \quad \alpha \cap \beta = \phi \right\}$$

$$E(H) = \left\{ ((\alpha,\beta),(\alpha',\beta')) \;\middle|\; \begin{array}{l} \text{either } \alpha \cap \beta' \neq \phi \\ \text{or } \quad \alpha' \cap \beta \neq \phi \end{array} \right\}$$



ALON SAKS SEYMOUR CONJECTURE

<u>Claim</u> $\boxed{\exists g \quad coNP^{cc}(CIS_g) = \Omega(\log^2 n)}$ ← Theorem 3

$\Longleftrightarrow$ $\boxed{\forall k \, \exists H \text{ st. } H \text{ has bipartition number } k, \text{ and chromatic number } k^{\log k}}$

$$V(H) = \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{l} \alpha \text{ a clique in } g \\ \beta \text{ an IS in } g \end{array}, \quad \alpha \cap \beta = \phi \right\}$$

$$E(H) = \left\{ ((\alpha, \beta), (\alpha', \beta')) \;\middle|\; \begin{array}{l} \text{either } \alpha \cap \beta' = \phi \\ \text{or } \alpha' \cap \beta = \phi \end{array} \right\}$$

Let
$$\left.\begin{array}{l} A_i = \{ (\alpha, \beta) \in V \mid i \in \alpha \} \\ B_i = \{ (\alpha, \beta) \in V \mid i \in \beta \} \end{array}\right\}$$
① $A_i \times B_i$ is a complete bipartite subgraph in H

② every edge in H is covered by 1 or 2 edges of $\bigcup_i (A_i, B_i)$
(an edge can't appear in both (A_i, B_i) + (A_j, B_j))

α

$A_i = \{(\alpha, \beta) \mid i \in \alpha\}$

$B_i = \{(\alpha, \beta \mid i \in \beta\}$

$\underbrace{(\alpha, \beta)}_{\text{in } A_i}, \underbrace{(\alpha', \beta')}_{\text{in } B_i}$ $\quad i \in \alpha$ and $i \in \beta'$

$(\alpha, \beta), (\alpha', \beta')$ is an edge so

$\exists i$ s.t. $\alpha \cap \beta' = i \longleftarrow$

or $\alpha' \cap \beta = i$

$$\begin{array}{c|cc} & \beta & \beta' \\ \hline \alpha & 0 & \\ \alpha' & 1 & 0 \end{array}$$

ALON SAKS SEYMOUR CONJECTURE

Claim $\boxed{\exists g \quad coNP^{cc}(CIS_g) = \Omega(\log^2 n)}$ ← Theorem 3

$\Longleftrightarrow$ $\boxed{\begin{array}{c}\forall k\ \exists H \text{ st. } H \text{ has bipartition number } k, \text{ and} \\ \text{chromatic number } k^{\log k}\end{array}}$

$V(H) = \{ (\alpha, \beta) \mid \alpha \text{ a clique in } g,\ \beta \text{ an IS in } g,\ \alpha \cap \beta = \phi \}$

$E(H) = \{ ((\alpha, \beta), (\alpha', \beta')) \mid \text{either } \alpha \cap \beta' = \phi \text{ or } \alpha' \cap \beta = \phi \}$

Let $A_i = \{ (\alpha, \beta) \in V \mid i \in \alpha \}$
$B_i = \{ (\alpha, \beta) \in V \mid i \in \beta \}$
} ① $A_i \times B_i$ is a complete bipartite subgraph

② every edge in H is covered by 1 or 2 edges of $\bigcup_i (A_i, B_i)$

∴ H has bp_2 number n

↖ covering by n complete bipartite subgraphs, each edge appears twice

ALON SAKS SEYMOUR CONJECTURE

Claim $\boxed{\exists g \quad coNP^{cc}(CIS_g) = \Omega(\log^2 n)}$ ← Theorm 3

$\Longleftrightarrow$ $\boxed{\forall k \, \exists H \text{ st. } H \text{ has bipartition number } k, \text{ and chromatic number } k^{\log k}}$

$$V(H) = \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{l} \alpha \text{ a clique in } g \\ \beta \text{ an IS in } g \end{array}, \quad \alpha \cap \beta = \phi \right\}$$

$$E(H) = \left\{ ((\alpha,\beta),(\alpha',\beta')) \;\middle|\; \begin{array}{l} \text{either } \alpha \cap \beta' = \phi \\ \text{or } \alpha' \cap \beta = \phi \end{array} \right.$$

Want to show: For any proper coloring of H, the colors correspond a separator family $\mathcal{F}$ for g.

Lemma H has chromatic number $\Omega(n^{\log n})$

Defn Let $\mathcal{J} = \{V^1, V^2, \dots, V^t\}$, $V^i \subseteq V(g)$ $\forall i$

$\mathcal{J}$ is a separator for g if $\forall (\alpha, \beta)$ s.t. $\alpha \cap \beta = \phi$, $\exists i$ such that V^i separates α from β

Yannakakis showed: $coNP^{cc}(CIS_g) = \log(\text{min size of separator for } g)$

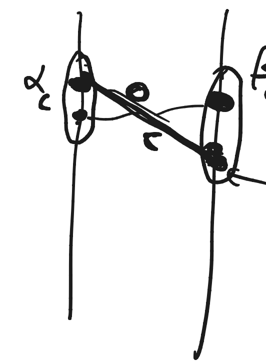
$\therefore$ any separator for g has size $\Omega(n^{\log n})$.

Claim A proper coloring of H with C colors implies a separator $\mathcal{J}$ for g of size C.

Let $\mathcal{C}$ be a coloring of H

For each color c, Let α_c = all α's s.t. $\exists \beta$ st $\mathcal{C}(\alpha, \beta) = c$

β_c similar



* there can be No edges between α_c & β_c in H so c separates these α's from β's

<u>Claim</u> Let $\alpha_c = \{\alpha \mid \exists \beta \text{ s.t. } (\alpha, \beta) \text{ colored } c\}$

$\beta_c = \{\beta \mid \exists \alpha \text{ st } (\alpha, \beta) \text{ " } c\}$

Then $\forall$ pairs (α, β) such that $\alpha \in \alpha_c$, $\beta \in \beta_c$, α & β are disjoint

Let $(\alpha, \beta), (\alpha', \beta')$ both be colored by c.

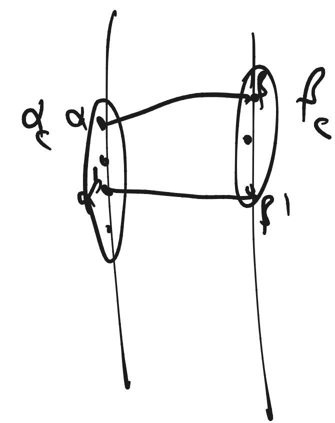
Then there is NO edge in H between (α, β) and (α', β').

$\therefore$ (by defn of when an edge is in H)

this means $\alpha \cap \beta = \phi$, $\alpha \cap \beta' = \phi$, $\alpha' \cap \beta = \phi$, $\alpha' \cap \beta' = \phi$

$\therefore$ $\alpha \cup \alpha'$ is disjoint from $\beta \cup \beta'$

$\therefore$ $\bigcup_{\alpha \in \alpha_c} \alpha$ separates α_c from β_c



A separator $\mathcal{F}$ for g of size m implies a coNP^{cc} protocol for CIS_g of cost $\log m$:

Let $\mathcal{F}$ be a separator for g $\mathcal{F} = V^1, V^2, \ldots V^t$

Claim ① if $CIS_g(\alpha,\beta) = 0$ then $\exists i$ that separates α from β

② If $CIS_g(\alpha,\beta) = 1$ then $\forall i$ V^i does not separate α from β.

Given $\mathcal{F}$ ~~NP~~ NP^{cc} protocol for $\overline{CIS_g}$ on input (α,β):

guess some $i \in [t]$ (~~assume $\alpha \in V_i$~~

Alice + Bob check if V^i separates α from β
(constant # of bits)

If V^i separates α, β output 1
else 0

PAC LEARNING + VC DIMENSION [Alon, Hanneke, Holzman, Moran]

<u>Concept class</u> $\mathcal{H}$ = set of functions $f: \mathcal{X} \to R$

typical: $\mathcal{X} = \{0,1\}^n$, $R = \mathbb{R}$ or $R = \{0,1\}$

$\mathcal{H}$ is (ε, δ)-PAC learnable with sample complexity m if:

there is a learning alg A s.t $\forall \mathcal{D}$ over $\mathcal{X}$, $\forall h \in \mathcal{H}$

A gets as input a set S of random labelled pairs, $|S| = m$

$(x, h(x))$, $x \sim \mathcal{D}$ and outputs some function $f: \mathcal{X} \to R$ st. wp $\geq 1-\delta$

$$\mathrm{Error}(\mathcal{D}, f) = \Pr_{x \sim \mathcal{D}}[f(x) \neq h(x)] \leq \varepsilon$$

$\mathcal{H}$ has poly sample complexity if $m = \mathrm{poly}(n, \frac{1}{\varepsilon}, \frac{1}{\delta})$

$\mathcal{H}$ is polytime (ε, δ)-PAC learnable if further, A runs in time $\mathrm{poly}(n, \frac{1}{\varepsilon}, \frac{1}{\delta})$

PAC LEARNING + VC DIMENSION

<u>Concept class</u> $\mathcal{H}$ = set of functions $f : X \to R$

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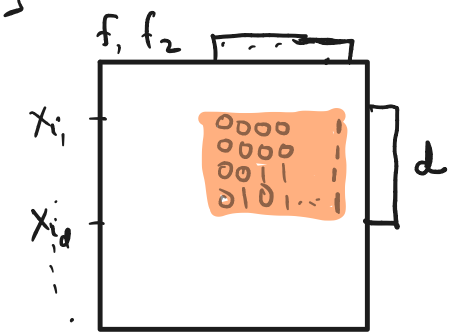


<u>VC-Dim ($\mathcal{H}$)</u> : Max d st. $\exists x_{i_1} \dots x_{i_d}$ st

$\forall \alpha \in \{0,1\}^d$ $\exists f_i \in \mathcal{H}$ st $f_i\big|_{x_{i_1} \dots x_{i_d}} = \alpha$

<u>Thm</u> A (total) concept class $\mathcal{H}$ is PAC-learnable iff VC-dim ($\mathcal{C}$) is finite

further, any A that outputs concept f that is <u>consistent</u> with samples suffices.

Partial concept Classes

$f_i : \mathcal{X} \to \{0, 1, *\}$

don't care

	f_1	f_2	f_3	f_4	f_5	. . .		
x_1	1	*	0	0	*	1	0	1
x_2	0	0	1	1	1	*	*	1
x_3	0	0	*	*	*	0	0	1
x_4	1	0	*	0	0	*	1	1

Theorem $\exists$ a partial concept class $\mathcal{H}$ of VC Dim 1 but such that any total class extending $\mathcal{H}$ has infinite VC dimension

PAC LEARNING + VC DIMENSION

<u>Concept class</u> $\mathcal{H}$ = set of functions $f : X \to R$

typical: $X = \{0,1\}^n$, $R = \mathbb{R}$ or $R = \{0,1\}$

all concepts in $\mathcal{H}$

$$
\begin{array}{c|ccccc}
 & f_1 & f_2 & \cdots & & \\
\hline
x_1 & 1 & 0 & 0 & 1 & 1 \\
x_2 & 0 & 0 & 1 & 1 & \\
x_3 & 1 & 1 & 1 & 0 & \\
 & 0 & 1 & 1 & 1 & \\
\vdots & & & & \cdots &
\end{array}
$$

all inputs $x \in X$

Theorem $\exists$ a partial concept class $\mathcal{H}$ of VC Dim 1, but any total concept class extending $\mathcal{H}$ has infinite VC dimension

Proof Let $H=(V,E)$ be disj union of k complete bipartite graphs $H_1, \dots, H_k$, $\text{chrom}\#(H) = \Omega(k^{\log k})$

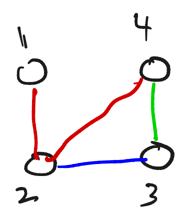
$|V| = N$

N rows/fxns for all $v \in H$

k columns

$$M(v,i) = \begin{cases} 1 & \text{if } v \text{ on LHS of } H_i \\ 0 & \text{if } v \text{ on RHS } \text{" "} \\ * & \text{ow} \end{cases}$$

Ex



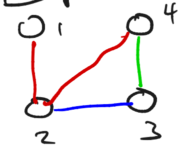
$$M: \begin{array}{c|ccc|} & H_1 & H_2 & H_3 \\ \hline 1 & 0 & * & * \\ 2 & 1 & 0 & * \\ 3 & * & 1 & 0 \\ 4 & 0 & * & 1 \\ \hline \end{array}$$

Theorem $\exists$ a partial concept class $\mathcal{H}$ of VC Dim 1, but any total concept class extending $\mathcal{H}$ has infinite VC dimension

Proof Let $G = (V, E)$ be disj union of K complete bipartite

$$M(v,i) = \begin{cases} 1 & \text{if } v \text{ on LHS of } H_i \\ 0 & \text{if } v \text{ on RHS " "} \\ * & \text{ow} \end{cases}$$

Example



$$A: \begin{array}{c|ccc} 1 & 0 & * & * \\ 2 & 1 & 0 & * \\ 3 & * & 1 & 0 \\ 4 & 0 & * & 1 \end{array}$$

- VC Dim of A is 1 (for every 2 bipartite graphs we have at most 1 vertex in common)
- For any extension of A to total matrix M':
  view distinct row vectors as colors. It will give a proper coloring of G
  (any edge (u,v) in H is coverd by some bipartite clique, H_j
  So entries $(u,j), (v,j)$ either 0/1 or 1/0 so different colors

  $\therefore$ $k^{\log k}$ distinct rows in A'

  $\therefore$ by Sauers Lemma VC Dim $\geq \log k$ (n^d distinct cols $\Rightarrow$ vc dim $\geq d$)