A Reconciliation of Logical Representations of Space: from Multidimensional Mereotopology to Geometry

by

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Abstract

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Reasoning about spatial knowledge is an important aspect of computational intelligence. Humans easily switch between high-level and low-level spatial knowledge, while computers have traditionally relied only on low-level spatial information. Qualitative spatial representation and reasoning is concerned with devising high-level, qualitative, representations of certain aspects of space using small sets of intuitive spatial relations that lend themselves to efficient reasoning. Many such representations have been developed over the years, but their use in practical applications seems to be inhibited.

One reason preventing more widespread adoption of qualitative spatial representations may be the gap between simple but inexpressive qualitative representations at one end and geometric or quantitative representations with the expressivity of Euclidean geometry at the other end. Another factor may be the lack of semantic integration between the various spatial representations ranging from qualitative to geometric ontologies. We will address both issues in this thesis with a focus on spatial ontologies that involve some kind of mereotopological relations such as contact and parthood.

We design a family of spatial ontologies with varying restrictiveness and increasingly more expressive nonlogical languages, organized into hierarchies of ontologies of equal expressivity. As the most foundational spatial ontology we propose a multidimensional mereotopology based only on 'containment' and 'relative dimension' as undefined concepts. By adding either 'boundary containment' or 'betweenness' as new concepts, we further extend the expressivity without impairing the qualitative character.

Tools from mathematical logic, such as interpretability and definability, are used to semantically integrate other spatial ontologies into our hierarchies. Moreover, we show how mereotopological theories, incidence geometries, and ordered incidence geometries are formally related to our theories. We thereby better understand differences in expressivity, restrictiveness, and ontological assumptions between a broad range of spatial ontologies. Throughout the thesis, we utilize automated theorem provers to assist with the verification of all ontologies by constructing nontrivial models and by proving key properties about the axiomatized relations and functions. Theorem provers are also utilized to obtain some of the integration results.

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a'	the complement (or unique complement) of the element a in a complemented lattice	53
$a^{\perp}$	the orthocomplement of the element a in an orthocomplemented lattice	54
a^{*}	the pseudocomplement of the element a in a pseudocomplemented lattice	54
a^{+}	the quasicomplement of the element a in a quasicomplemented lattice	54
T^{0}	logical theory $T \cup$ Z-A1	99
$T^{\neg 0}$	logical theory $T \cup$ NZ-A1	99
$\overrightarrow{v}$	a finite set of parameters of a logical formula	16
$\ominus x$	nonlogical function symbol denoting the unique complement of the variable x in an equidimensional mereotopology	50
$\prod \mathbf{X}$	the intersection of a possibly infinite set of entities	58
$\bigodot \mathbf{X}$	the intersection of a possible infinite set of entities	51
$\sum \mathbf{X}$	the sum of a possibly infinite set of entities	58
$\bigoplus \mathbf{X}$	the sum of a possible infinite set of entities	51
$\lvert\mathbf{X}\rvert$	the cardinality of the set $\mathbf{X}$	86
$\overline{X}$	manifold closure: smallest manifold with boundary with point set X as subset	82
$\mathbf{X}^{\circ}$	interior of the point set $\mathbf{X}$ that is a subset of a topological space or interior of a manifold with boundary	82
$\mathbf{X}^{-}$	exterior of the point-set $\mathbf{X}$ that is a subset of a topological space with respect to the topological space or exterior of an atomic or composite manifold with respect to a complex manifold	90
$\overleftrightarrow{xy}$	line segment with the endpoints x and y	249
$\mathcal{M} \vDash \sigma$	the $\mathcal{L}$-structure $\mathcal{M}$ satisfies the sentence $\sigma \in \mathcal{L}$	15
$T \vDash \sigma$	the sentence $\sigma \in \mathcal{L}$ is entailed by the $\mathcal{L}$-theory T (is a logical consequence of the theory T)	15
$x \odot y$	nonlogical function symbol denoting the unique intersection of the variables x, y in an equidimensional mereotopology	50
$x \oplus y$	nonlogical function symbol denoting the unique sum of the variables x, y in an equidimensional mereotopology	50

$\mathbf{A} \subseteq \mathbf{B}$	the set $\mathbf{A}$ is a subset of the set $\mathbf{B}$	35
$\mathbf{A} \subset \mathbf{B}$	the set $\mathbf{A}$ is a proper subset of the set $\mathbf{B}$	35
$\mathbf{A} \setminus \mathbf{B}$	the point-set difference between the point sets $\mathbf{A}$ and $\mathbf{B}$; if $\mathbf{A}$ and $\mathbf{B}$ are sets of axioms describing theories, the difference between their axioms	35
$\mathbf{A} \cap \mathbf{B}$	the point-set intersection of point sets $\mathbf{A}$ and $\mathbf{B}$; if $\mathbf{A}$ and $\mathbf{B}$ are sets of axioms the union of their axioms	35
$\mathbf{A} \cup \mathbf{B}$	point-set union of point sets $\mathbf{A}$ and $\mathbf{B}$; if $\mathbf{A}$ and $\mathbf{B}$ are sets of axioms or sentences, $\mathbf{A} \cup \mathbf{B}$ denotes the set of shared axioms or sentences	35
$\mathbf{A} \sqcap \mathbf{B}$	a customized intersection operation for point sets $\mathbf{A}$ and $\mathbf{B}$	60
$\mathbf{A} \sqcup \mathbf{B}$	a customized union operation for point sets $\mathbf{A}$ and $\mathbf{B}$	60
α	first-order formula	16
β	first-order formula	16
γ	a first-order sentence that is an explicit definitions	18
Γ	set of first-order sentences that are explicit definitions	19
∂	boundary of a subset of a topological space or of a manifold with boundary	82
Δ	(outer) boundary of a composite manifold	86
Δ_i	interior boundary of a composite manifold	86
Θ	interior of a composite manifold	86
ϑ	some mapping	25
λ	a signature, i.e., a set of nonlogical symbols	15
$\lambda(T)$	the signature, i.e., the nonlogical symbols of the theory T	15
$\Lambda(T)$	the primitives, i.e., the undefinable nonlogical symbols of the theory T	18
$\mu(d)$	a bijection from $\mathrm{Dom}(\mathfrak{M})$ (the composite manifolds of a complex manifold in the class $\mathbb{M}$) into the domain $\mathbf{M}$ of a model of $CODI_{\downarrow}$ or $CODIB_{\downarrow}$	131
$\pi(\alpha)$	the interpretation of formulas of a nonlogical language in another nonlogical language; it translates formula from the source language into the target language	25
$\varrho(d)$	a bijection from the the domain $\mathbf{M}$ of a model of $CODI_{\downarrow}$ or $CODIB_{\downarrow}$ into the composite manifolds in $Dom(\mathfrak{M})$ of a complex manifold in the class $\mathbb{M}$	134
σ	some first order sentence	16
Σ	the point-set area of a composite manifold	86
τ	the topology of a topological space	35
Υ	some function or relation	28
Φ	the mapping of function symbols, including constant symbols, to functions in an interpretation	16
φ	some mapping	19
Ψ	the mapping of relation symbols to relations in an interpretation	16

Chapter 1

Introduction

Knowledge and information are central to all of computer science. Spatial knowledge is among the most ubiquitous knowledge we have, playing a key role in many areas of computer science and many interdisciplinary fields: Artificial intelligence in the broadest sense including knowledge representation, planning, autonomous robots, natural language processing, and computer vision; computer graphics, computational geometry and computational topology; geographic information systems (GIS) as used for cartography, surveying, or geology; computed-aided design and manufacturing software (CAD and CAM) as used in architecture, civil engineering, manufacturing, or product design; medicine (image processing, diagnosis); and cognitive science.

For centuries, spatial knowledge has been collected and curated manually for cadastres, military purposes, and infrastructure planning. This involved laborious and expensive processes. With technological advances in remote sensing, in particular through aerial and satellite-based sensing, and advances in computer vision and object recognition, collecting and maintaining spatial information is nowadays much cheaper, faster, and more accurate. This has spurred a growth in location-aware applications for a multitude of everyday applications. For example, the availability of cheap, accurate, and up-to-date spatial maps has been a major driver for the widespread use of navigation systems. For the same reason, more and more companies, government agencies, and even individuals nowadays collect or produce and publish spatial information for a variety of purposes. This has resulted in ever-growing amounts of spatial information in formats as varied as 2D maps, aerial and satellite images, GPS coordinates, routes, 2D and 3D plans, height profiles, sketches, and textual descriptions.

Coping with unprecedented amounts of information is not limited to spatial information, it has received increasingly prominent attention throughout computer science. But dealing with the variety of underlying spatial representation formats that differ in language, in expressivity, and in their implicit semantic assumptions is equally important. This challenge has not been addressed sufficiently and is currently a main hindrance for better integration of spatial knowledge from diverse sources. Exchanging spatial knowledge between sources and combining it from multiple sources opens many opportunities to address some of today's societal challenges, such as urban and transportation planning, vehicle routing, emergency response, or environmental and climate monitoring. Likewise, integration of spatial knowledge may enable novel kinds of applications, similar to the recent rise in location-aware services.

Much available spatial knowledge as expressed by humans is often very high-level and imprecise. One approach to emulate the *spatial intelligence* of humans replaces or supplements low-level spatial calculations with more high-level spatial reasoning reliant on small sets of "commonsensical" spatial relations frequently found in human language [BIC97a]. Such high-level representations of space are known as *qualitative theories of space*. Finding qualitative representations of space suitable for human-computer-interaction and for efficient spatial reasoning is the main objective of the research area of qualitative spatial reasoning (QSR), with contributions from disciplines as diverse as Ontology, Formal Logic, Artificial Intelligence, Cognitive Science, Cartography, Geographic Information Systems, Computer Vision, and Computational Geometry. Extensive introductions and overviews of QSR can be found, for example, in [CH01; CR08; Vie97]. Despite much progress throughout the last 25 years, work on qualitative representations of space has been primarily of theoretical nature. While an abundance of scenarios that could put qualitative spatial representations to good use have been identified, the more widespread use of qualitative representations seems to be inhibited by other factors. Many of the simple but well-understood theories are rather limited in their expressivity, while the few more expressive theories are either very similar to traditional geometry or are overly complicated in that humans cannot easily and intuitively work with the proposed relations and concepts.

The development of qualitative representations of space has not been accompanied by a concentrated effort to formally relate the qualitative spatial representations to geometric representations and to one another. If we want to exchange spatial knowledge across systems, we must ensure that the exchanged knowledge is interpreted equally by all systems, in other words, the meaning of the exchanged knowledge must be preserved. We can achieve such a level of interoperability by semantically integrating the systems' implicit or explicit spatial ontologies, which capture their assumptions about space. Generally, in computer science an ontology is an artifact designed with the purpose of expressing the intended meaning—the semantics—of a vocabulary (consisting a set of concepts, relations, and functions) in terms of the nature and structure of the entities it refers to. This is usually done in an *ontological language* that may range from very informal languages to more rigidly formalized ontological languages, such as logics. The logics used vary widely in their expressivity, from rather inexpressive description logics (the OWL family [OWL04; OWL11] of ontology languages containing the most prominent ones) to very expressive first- or second-order logic. In this thesis we use first-order logic as the ontological language of choice; a spatial ontology is nothing but a first-order theory.

The general issue of semantic interoperability has been discussed in detail in, e.g., [Har+99].

Semantic integration of spatial ontologies can help translate spatial knowledge between diverse *spatial information systems*, encompassing all software systems that deal with spatial information in some way, such as mapping software, GIS software, special-purpose spatial reasoners, spatial databases, or CAM and CAD software. Moreover, it can facilitate the use of different qualitative, geometric, or quantitative representations of space within highly-optimized and specialized software systems, which can be integrated into larger and more powerful spatial information systems that could—ideally—mimic spatial inferences that humans draw. Thereby, we would leverage the various spatial information systems that can reason efficiently about certain aspects of space in a more general spatial reasoning framework.

1.1 Research challenges

There are two research challenges pertaining to spatial ontologies that we address in this thesis. The first one concerns the semantic integration of ontologies of space ranging from qualitative representations to geometric and quantitative representations. The second one is the need to find more expressive qualitative representations of space. We will describe both challenges in more detail now.

1.1.1 Semantic integration of ontologies of space

The advance of the World Wide Web and the multiplication of network capacities have laid the technical foundation for large-scale publishing and sharing of digital information amongst government, companies, and individuals. The Semantic Web has promised seamless integration of information systems through shared formal specification of the semantics of their information models in a formal ontology—a description of the concepts and relations between the concepts in the domain of interest using a logical language. However, these prospects are still far from becoming reality, in particular because a wealth of different ontologies have been developed for individual applications without regards for reusability. Essentially, proprietary data formats have been replaced by proprietary ontologies, which do not permit semantic integration per se. Though the reasons for this development may be manifold, a contributing factor is the lack of understanding of ontologies and the ensuing lack of trust in the ontologies developed by others. Attempts to overcome this dilemma by standardizing ontologies for particular domains or aspects have had limited success because of the immense difficulties for all stakeholders to agree on a standardized ontology. However, semantic integration may overcome this problem by formally relating different ontologies to one another. Then the promise of the Semantic Web can be fulfilled without the need for a single shared ontology.

Semantic integration of two ontologies means to understand whether and how their concepts and relations vary in meaning. Often, two ontologies have concepts or relations of the same name or that are in other ways superficially equivalent, but that satisfy different constraints and thus differ in meaning. We can use logical languages to formally specify those commonalities and differences in meaning between two ontologies, resulting in a sub-ontology that captures the set of concepts and relations shared by the two ontologies. The sub-ontology is likely less precise and more narrow than either of two ontologies we want to integrate, but captures the portion of the two ontologies' knowledge whose semantics are preserved when translating knowledge between the ontologies' languages. In other words, models of the sub-ontology can be exchanged without loss or change in meaning between information systems that implement either of the ontologies.

In this thesis, we are concerned with semantic integration only of spatial ontologies. While the essential differences between various geometric representations of space—such as the difference between a raster-based and a vector-based representation—are well-understood, we have only started to understand their more subtle differences and the implications of these differences. This applies even more so to many qualitative ontologies of space that have been proposed to address cognitive inadequacies of traditional geometry-based ontologies and to deal with vague or imprecise spatial knowledge. For many of these qualitative theories, we lack a full understanding of the differences between them and of their relationships to geometry-based ontologies.

1.1.2 Expressive qualitative representations of space

Qualitative descriptions of space are pervasive in human language: many of our everyday descriptions of space are of qualitative nature. For example, we rely a lot on geographic directions (or "turns"), connectivity ("turn left at the next light", "follow the road until you cross Main Street"), relative positions of features or landmarks ("in front of the church", "across from the park") and order among spatial objects ("just after the post office across the bridge", "between the gas station and the supermarket") when giving driving or walking directions. While names (most frequently names of streets or towns) and categories of spatial features (rivers, lakes, islands, hills, forests) or landmarks ('church', 'gas station', 'post office') are also commonly used in human descriptions of space, quantitative descriptions are less common and mostly approximate ("after 100 meters", "walk 10 min along the lake", "at the fifth traffic light"). Humans have become accustomed to highly quantitative directions only with the advance of navigation systems.

Despite recent adoption of small sets of simple qualitative spatial relations for uses in spatial information systems, the full potential of qualitative representations of space has not been realized for various reasons. We believe more widespread use of qualitative representations is inhibited because many of the simple but well-understood theories are in fact too limited in their expressiveness, an argument supported by our analysis of equidimensional mereotopologies in Chapter 4. On the other side, the few more expressive theories that have been proposed either have expressive powers similar to those of classical geometries and are thus no longer purely qualitative; are overly complicated; or are not well-understood so that humans cannot easily and intuitively work with their sets of proposed relations and concepts in interaction with spatial information systems. In other words, there is a gap between very basic qualitative ontologies of space such as equidimensional mereotopologies and extremely expressive geometric theories of space.

Therefore, there is a clear need for a commonsensical and expressive, but qualitative theory of space: a theory that is qualitative but still expressive enough to describe key aspects of space and that is intuitive enough to be used in interaction with humans. By overcoming the limited expressiveness of equidimensional mereotopologies, it may lead to more widespread adoption of qualitative representations of space and, ultimately, it may help to realize the full potential of qualitative spatial reasoning.

1.1.3 Research scope

Before we describe our contributions towards the two mentioned research challenges in more detail, remarks about two fundamental distinctions that are key in order to understand our approach, its basic assumptions, and its limitations, are in order.

Qualitative vs. geometric and quantitative representations of space

Before we proceed we need to clarify what we mean by qualitative representations of space and how they differ from quantitative or geometric representations. Quantitative space is equipped with a metric, usually a distance metric, giving us precise information about how long a certain line segment is, or how far one point is from another point. Any coordinate system allows to precisely specify such information. In a quantitative treatise of space, we can apply metric calculations as a form of reasoning. But often, such information is either not available, imprecise, or simply unnecessary to answer certain questions. In those cases, we can apply qualitative reasoning, in which we only have a small set of values (or relation symbols) available to specify a certain conceptual relationship [FNF91; KB85]. A classic example includes cardinal directions: we may have information about how objects (or regions for that matter) are located relative to one another using only the four cardinal directions North, East, South, West, defining a cardinal direction calculus [Fra96]. For example, from the information that Calgary is West of Toronto, and Edmonton is North of Calgary, we can infer that Edmonton is either North or West of Toronto.

The same applies to topological relations: instead of expressing how far two objects are away from each other or how large their overlap is, we only express whether one regions is completely contained in the other, whether they partially overlap, only 'touch', or are not in contact at all.

Of course, the boundary between qualitative and quantitative representations is fluid. There is no limit on the "small" number of values or relations one can use for a specific conceptual relationship in a qualitative representation. This is well demonstrated by the general star calculus presented in [RM04]: we can take n lines to divide space into $2n$ sectors, resulting in $4n + 1$ base relations that express the direction of an object relative to a fixed point. For example, for $n = 1$ we have the five relations 'equivalent position', 'north' (N), 'east' (E), 'south' (S), 'west' (W). For $n = 2$ these five relations are supplemented by the four relations 'north-east' (NE), 'south-east' (SE), 'south-west' (SW), and 'north-west' (NW). For $n = 3$ each of the four relations 'north-east', 'south-east', 'south-west', and 'north-west' is split into two. For $n = 4$, we can use the 9 relations from $n = 2$ and add another eight relations that include, for example, 'north-north-east' (NNE), 'east-north-east' (ENE), and 'east-south-east'. If we choose a fairly large n, the resulting set of base relations can hardly be called a qualitative ontology. Therefore, it makes little sense to categorize spatial representations into qualitative and quantitative ones. Instead, it is a continuous scale with two extremes: a minimum of two base relations (or a single relation with Boolean values) on the one extreme and an infinite number of base relations (or a relation with an infinite number of values) on the other extreme. We call it the *quantitative-qualitative-continuum*, illustrated by the diagonal in Figure 1.1. Euclidean geometry is at the very extreme end due to its continuous nature and its ability to specify congruence.

Finding good qualitative representations depends on the given context: humans can only deal with very few relations while machines may be capable of dealing with many more. Equally, for some application, more fine-grained distinctions may be necessary than for another application. For example, in many domains it is not sufficient to only state whether two regions overlap at all, but it is necessary to know whether they overlap in only their boundaries or in their interiors as well. The precision with which we measure and record spatial knowledge may also influence how many relations we distinguish between a pair of regions. It makes no sense to work with a qualitative representation that is more fine-grained than the accuracy of the input data. For all those reasons, it is reasonable not to focus on a single ontology with a fixed set of relations; instead it is better to have a family of ontologies with the flexibility to expand expressivity as necessary.

Representations of abstract vs. physical space

Our work presupposes that we are comfortable with distinguishing abstract space regions from material objects, the later just happening to occupy abstract space regions (as a property). Space regions are purely mathematical-geometric abstractions that have no material properties, while objects are grounded in some physical reality, in which the materiality of objects is crucial. This separation into two levels of space, which we call *abstract space* and *physical space*, gives us the freedom to talk about arbitrary abstract regions without having to worry about whether they correspond to physically meaningful objects. We can talk about the intersection and sums of regions, about overlapping regions, regions' boundaries, and regions of arbitrary dimensions without assuming that they exist in some physical reality.

While our ultimate goal is to model physical space, most of the work in this thesis focuses on the underlying theories that capture abstract space. Only in Chapter 11 we will show how the theories of abstract space can be utilized to model arrangements of physical entities, especially material bodies, and their features. In that sense, *physical space* as a term refers to the collection of the physical entities of a domain of interest—it is a conceptual "space" populated by physical objects and their features. To ground physical space in abstract space, we will reuse a portion of the DOLCE ontology [Mas+03] for physical entities. A function assigns each physical entity the abstract region of space it occupies. Of course, this assignment is not surjective in that there are many abstract regions of abstract space that are not occupied by a concrete physical endurant.

The distinction between physical and abstract space allows us to formalize multidimensional mereotopology without the looming question of its philosophical adequacy. In the theories of abstract space we can talk about many abstract concepts of space that are pervasive in how humans treat space and talk about it. Moreover, it allows us to formalize space without a specific bias to a particular metaphysical stance such as nominalism, conceptualism, or realism. Our work is not a study in Ontology (in the philosophical sense of the word): We do not try to convince the reader that the abstract spatial entities we talk about do exist in the real world, in our perceived image of the world, or in our mind.

1.2 Contributions

In this thesis, we study the continuum of spatial ontologies—logical theories of space—depicted in Figure 1.1 with mereotopologies as the most basic qualitative theories on the one end and classical geometries as extremely expressive, essentially quantitative theories on the other end. Our exploration into this continuum of spatial ontologies offers two overarching contributions towards the outlined research challenges. The contributions can be summarized as follows.

1. We partially fill the gap in between the two extremes on the quantitative-qualitative-continuum by proposing a multidimensional mereotopology that is more expressive and more general than currently available mereotopologies and by showing how this multidimensional mereotopology can be extended to increase its expressivity without obtaining the full expressive power of classical geometries. The extensions are motivated by how humans describe space qualitatively and by the spatial relations necessary to define Euclidean geometry. The result is a *family of spatial ontologies*, grouped into hierarchies of equally expressive but differently constrained theories. With increasing expressivity the theories in this family define successively more fine-grained qualitative relations.

2. We formally relate spatial ontologies within this family to one another, but also relate the family's theories to previously proposed mereotopological and geometric ontologies. Thereby, we semantically integrate a wide range of spatial ontologies, not only with our theories but indirectly also with one another.

Next, we will give an overview of the most important individual results as the pieces that accrue to the two overarching contributions. First, we discuss the results that help us fill the mereotopology-geometry gap; subsequently, we elaborate on the major integration results. Finally, we highlight how the semantic integration of qualitative spatial theories with geometries allows as by-product the construction of a qualitative analogue of geometry. This result does not support one of the two overarching contributions, but is an interesting consequence of our work.

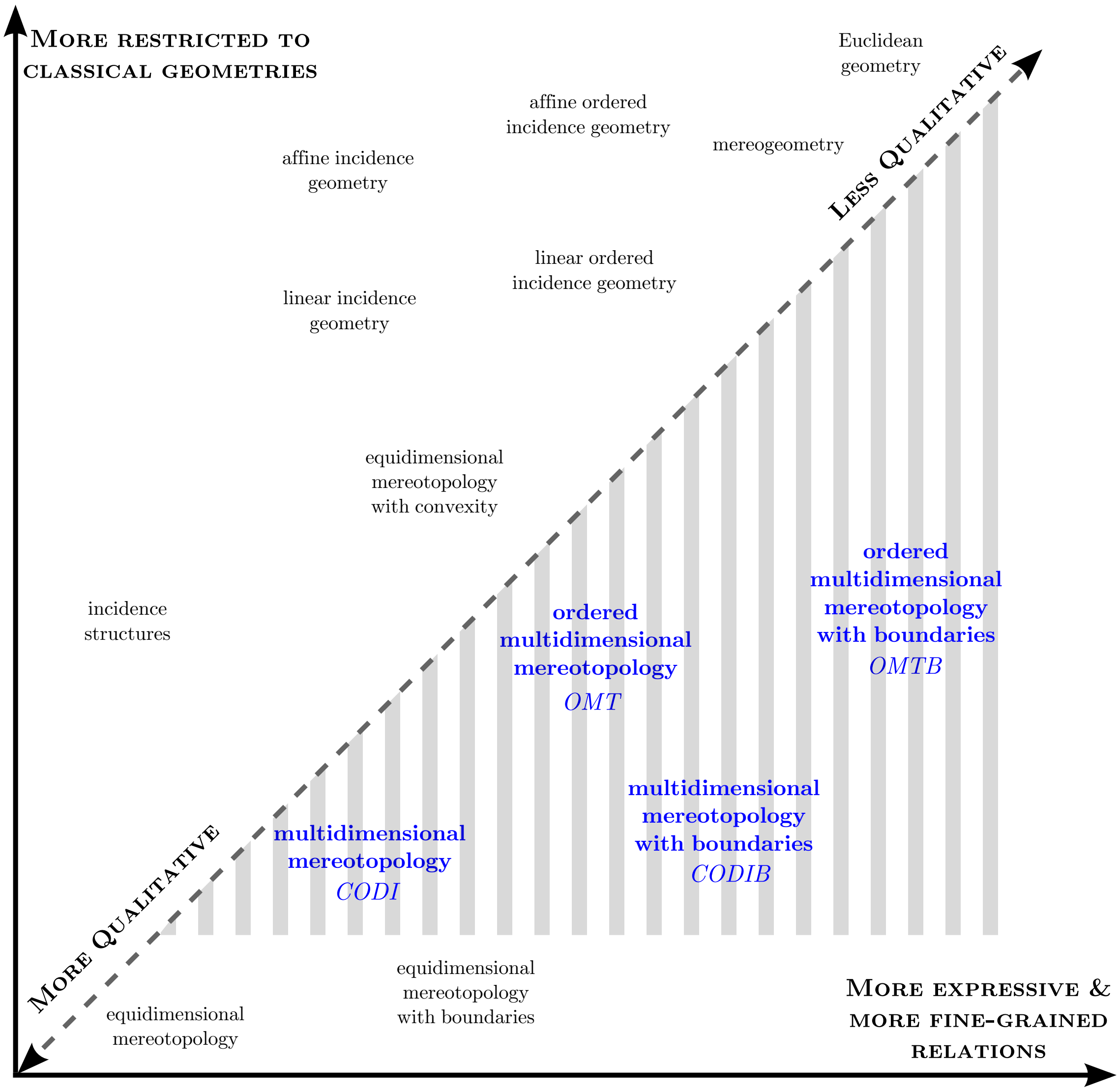


Figure 1.1: A graphic illustration of the essence of the thesis showing how the spatial theories developed in this thesis (shown in blue in the striped area) relate to other spatial theories according to two criteria: the restrictiveness in comparison to geometric spatial theories (y-axis) and the number of independent spatial relations we can distinguish (x-axis), which tells us how fine-grained distinctions we can make. By proposing spatial theories located in the striped area—which was previously unexplored—we bridge the gap between chiefly geometric theories that are less qualitative in nature and the purely qualitative theories of equidimensional mereotopologies with puny expressivity. The placement in the continuum is purely illustrative with the distance to the origin approximating the "qualitative" nature of a theory without implying any scale.

1.2.1 Multidimensional mereotopology for expressive qualitative spatial ontologies

Due to the foundational role topological (of *contact*) and mereological (of *parthood*) relations play in space, mereotopology is an indispensable part of any comprehensive ontology of qualitative space. In Chapter 4 we show that there is an narrow range of traditional, so-called equidimensional, mereotopologies for modeling space. The restriction of equidimensional mereotopology to models with domains of only regions of equal dimension makes it very difficult to extend equidimensional mereotopologies by other intuitive qualitative relations without defining a spatial theory with the expressive power of Euclidean geometry. As Borgo and Masolo [BM10] showed, most extensions of equidimensional mereotopology are indeed as expressive as Euclidean geometry, the extension by a convexity relation or a convex hull operation being the sole exception [CR08].

One way to build more expressive non-geometric qualitative theories is to relax the restriction on the dimensions of the objects in the domain. This results in so-called *multidimensional mereotopologies*. We will develop an eminently general axiomatic theory of multidimensional mereotopology. But unlike previous GIS-focused studies of the mereotopological relations between spatial entities of different dimensions, we provide an axiomatization whose spatial relations apply to entities independent of their concrete numeric dimensions or codimensions. That is, our theory is not restricted to entities of maximal two or three dimensions such as in the analyses of mereotopological relations in [CDF98; EH91; ME94; McK+05] or in the General Formal Ontology (GFO) [BH11]. In our work in Chapters 6, 7, 9, and 10, we will give a more general account of multidimensional mereotopological relations that is designed to work for any finite number of dimensions, not just for two, three, or four dimensions. In that way, our theory is as general as possible in the spirit of Belot's argument [Bel11, p. 15]:

> "At one time, of course, philosophers trafficked in arguments purporting to show that space was necessarily three-dimensional. But few today are likely to deny that space could have had two, or four, or twenty-six dimensions. And if these are allowed, it seems parochial to exclude Euclidean spaces of any finite dimension."

As a result, we do not need separate relations for each combination of dimensions of a pair (or set) of entities. For example, in our theories a single relation may apply to pairs of entities with the dimensions (3D,2D), (3D,1D), (4D,3D, (4D, 2D), (4D 1D) in a four-dimensional space. This helps keep the number of distinct relations down, defining a fairly small set of intuitive relations. To accommodate practical applications, in which spaces of more than four spatio-temporal dimensions may not be frequently used[1],

[1] Though spaces of more than four spatio-temporal dimensions many not be of great practical importance, we can use them for a multitude of purposes. For example, we can use them to model space-dependent properties, such as temperature, lighting, or signal strength for cell phone reception throughout a building. Such properties can be treated spatially in terms of "region": the range of comfortable working temperature or ranges of signal strength, for example strong, medium, and weak reception. We can also use more than four dimensions to model standard three- or four-dimensional space in a rather unusual but interesting way by introducing new dimensions. For example, we can treat a "highway" or any other road with more than a single lane in each direction as having a higher dimension than roads with only one lane in each direction. Or, we could treat a path and any road with a single lane (such as some mountain road or a small one-way street) as having a lower dimension than roads with multiple lanes. This captures the idea that, a road with multiple lanes has an extra dimension: the number of lanes. Compare the examples of how roads are treated in different spatial representations from [Har+99]. Equally, we could express that any path has a higher dimension than an abstract, truly linear boundary of no width, but is considered of a lower dimension than an area, which can have arbitrary width. An intersection or a square may also be treated differently than a point specified solely in terms of GPS coordinates. The benefit of modeling space with more than four dimensions needs further careful investigation outside the scope of this thesis. We consider the elegance associated with a dimension-independent approach and its benefits in terms of extracting a small set of dimension-independent spatial relations as reasons sufficient to warrant a dimension-independent treatise of qualitative space.

we can voluntarily restrict the theory to a specific number of dimensions as necessary— as we will do in our mappings to two- and three-dimensional incidence geometries in Chapter 10. Typically, such a restriction is not even necessary; this is demonstrated by the way we model hydrogeological space in Chapter 11 using exclusively the entities of maximal dimension in a model.

The only other proposals for multidimensional mereotopologies that are not restricted to specific combinations of dimensions are those of Galton and Gotts [Gal96; Gal04; Got96], which inspired our work here. In particular, Galton [Gal04] gave the only mathematical model for multidimensional mereotopology. However, it rests on the premise that lower-dimensional entities occur only as boundaries of higher-dimensional entities. For example, to model a piece of string as a one-dimensional object, the string must be part of the boundary of some two-dimensional area or plane. Equally, to model a membrane as a two-dimensional object, it must be part of the boundary of some three-dimensional physical body[2]. Often, such higher-dimensional objects simply do not exist. Our theory overcomes this limitation: it does not assume that lower-dimensional entities only arise as boundaries of entities of the next higher dimension. Our approach also avoids the rather awkward and unintuitive primitive relation of 'x includes a chunk of y', $INCH(x, y)$, used in Gott's INCH Calculus [Got96][3]. Altogether, we have had the following five design criteria for our multidimensional theory of qualitative space in mind:

Multidimensional: Admits models in which entities of multiple dimensions can coexist;

Commonsensical: Defines an intuitive set of spatial relations;

Dimension-independent: All relations are as general as possible, in particular not dependent upon specific dimensions or codimensions;

Atomicity-neutral: Admits discrete and continuous models;

Geometry-consistent: Generalizes classical geometries.

Our most basic multidimensional mereotopology *CODI*, introduced in Chapter 6, is based on two primitive relations: *spatial containment* and *relative dimension*. In Chapter 7 we extend *CODI* by binary mereological closure operations that assign a spatial entity to the intersection, difference, and sum of any pair of spatial entities. In contrast to the closure operations defined in [Gal96; Gal04; Got96], our closure operations are *total functions* (see Theorems 7.1, 7.2, and 7.5), that is, they are not only defined for pairs of entities of equal dimension, but also for pairs of entities of different dimensions.

While this approach increases the expressivity of mereotopology by choosing a multidimensional approach, it is still fairly limited. In order to use qualitative spatial relations to represent the space that surrounds us without losing too much essential spatial information, we need even more expressive qualitative theories. But because many extensions of mereotopologies have turned out to have expressive powers comparable to those of Euclidean geometry [BM10], we want to extend our multidimensional mereotopology in ways that increases its expressivity without becoming as expressive as Euclidean geometry. We consider two extensions: first, by a relation of boundary-containment and, second, by an order relation.

The first extension in Chapter 9 with a primitive relation of *boundary-containment* allows us to identify, for example, whether a room is accessible from the outside of a building or not; whether according to a construction drawing two pieces of copper pipe simply need to be coupled at their ends

[2] The string and membrane examples have been discussed as limitations of the proposed model in [Gal04].

[3] We will later relate our theory to Gott's, in the process of which we identify some problems with the INCH Calculus that are probably caused by the difficulty of operating with the relation *INCH*.

or whether a tee-coupling needs to be inserted into one of them to connect it to the other; or whether a city is landlocked or accessible by sea. We show how the such extended multidimensional theories, which form the hierarchy *CODIB*, are expressive enough to define natural spatial distinctions such as between *interior* and *tangential containment* or *parthood*. More generally, we define all nine intersection relations from Egenhofer's 9-intersection model [Ege91; EF91; EH91] without restrictions on the involved entities' dimensions or codimensions. We can also define bodiless ('thin') as well as bulky ('think') boundaries in this theory. Those two kinds of boundaries are sufficient to model most, if not all, abstract and physical boundaries.

In Chapter 10, a second extension by a betweenness relation as a notion of order allows us to qualitatively capture another fundamental spatial relation without constructing full mereogeometry. Order relations play a key role in how humans navigate space: they are one of the few relations, next to topological relations, that are most frequently preserved in human sketch maps [WL12; WS09]. Order among street intersections often allows humans to complete easy navigation tasks in a city without knowledge about exact distances or cardinal directions. However, order as defined in classical geometries is narrow in scope. We propose a more general relation of *relativized betweenness* that is applicable to multidimensional space without requiring common geometrical restrictions. This constructs the hierarchy *OMT* of *ordered multidimensional mereotopologies*.

1.2.2 Semantic integration of spatial ontologies that contain some mereotopological relations

We integrate first-order spatial ontologies with one another using methods from mathematical logic. More precisely, we utilize (a) interpretations between theories and (b) relationships between classes of structures. As reference for all of our integration results, we use the family of multidimensional theories that we concurrently develop. Specifically, in Chapter 8 we relate other mereotopologies, namely the equidimensional Region Connection Calculus (RCC) [Coh+97b; RCC92] and the multidimensional INCH Calculus [Got96], to extensions of our multidimensional mereotopology. Equally, we construct axiomatic extensions of our multidimensional mereotopology to relate them to incidence structures and incidence geometries (Section 10.2).

For the relationships to other mereotopologies and incidence geometries no new primitives in addition to those available in our basic multidimensional mereotopology are necessary. To integrate spatial theories that distinguish contact to the boundary of another entity from contact to the interior of that entity (based on Egenhofer's 9-intersection relations [Ege89; Ege91; EF91; EH91]) into our family, we use in Section 9.5 the theories from the hierarchy *CODIB*. The extension with betweenness is used in Section 10.3.3 to formally integrate ordered incidence geometries into our hierarchy *OMT* as restrictions of ordered multidimensional mereotopology. This establishes a direct relationship between the *OMT* theories and classical geometries. it further shows that the *OMT* theories approach the expressivity of geometry without explicitly or implicitly defining congruence. Thus the *OMT* theories are strictly less expressive than geometries and they still exhibit a qualitative conceptualization of space.

These results show how, in principle, other spatial theories can be seen as extensions of our multidimensional mereotopology. It makes clear how the various integrated spatial ontologies differ in the expressivity of their nonlogical languages and in the kind of models they admit: because they are formally related to theories in our family, we can compare them based on their sets of primitive relations and their sets of axioms, which are formally captured by partial orders within and between hierarchies.

All theories formally related to theories in our family are thus implicitly integrated with one another: we can compare them by extracting from our family the most restrictive, most expressive sub-theory that they share. In that sense, the ontologies proposed in this thesis can serve as a family of *reference ontologies* for the integration of many other spatial ontologies. The results further suggest that our most general multidimensional mereotopology can serve as a key piece of an upper ontology of space: it is general enough to accommodate any other spatial ontology that preserves topological and mereological relations.

1.2.3 Qualitative generalization of classical geometry

As part of our effort to relate various spatial ontologies to one another, we relate intrinsically qualitative theories to geometries. Inevitably, the question arises to what constitutes a geometry and how does it differ from a qualitative theory of space? This is closely related to understanding the continuum of spatial theories: Where can we draw the line that separates geometric from qualitative theories of space? Posed differently, we ask when a spatial theory becomes inherently geometrical. We suggest the following answer to that question in Chapter 10: It is not necessarily the existence of a metric as the name implies, but the presence of two "geometric" assumptions. First, the assumption that lines are straight in that any two points uniquely define a line and, second, the assumption that lines are dense total orders of points. With this explanation, we are equipped to generalize geometries to qualitative theories of space by omitting those two assumptions. As specific contribution, we propose the theory $CODI_{\mathrm{plp-g}}$ as mereotopological abstraction of three-dimensional incidence geometry. Ordered incidence geometry could be generalized similarly by finding a suitable extension of $CODI_{\mathrm{plp-g}}$ in the hierarchy of ordered multidimensional mereotopologies, OMT.

1.3 Outline of the thesis

The thesis is structured as illustrated in Figure 1.2. In Chapter 2 we review the logical methods used to compare and to semantically integrate spatial theories with one another by grouping them into hierarchies of theories with equal expressivity. In Chapter 3 we give the necessary background on spatial ontologies; we review the basics of equidimensional mereotopology, previous work on multidimensional mereotopologies, and the treatment of boundaries and geometric relations in mereotopology. We also give some examples of how mereotopology has been put to practical use. Chapter 4 studies equidimensional mereotopologies from the perspective of their *spatial representability*, addressing the question of what constitutes an equidimensional mereotopology that adequately captures space wherein complements must exist. This chapter stands by itself, the results primarily motivate a closer examination of multidimensional mereotopology.

In Chapter 5 we characterize the intended multidimensional structures using the mathematical notion of *manifolds with boundaries* and developing more complex spatial entities analogue to the constructive definition of simplicial complexes. We will later use this characterization to show that any intended structure satisfies the axioms of the two key spatial theories developed in Chapter 7 and Chapter 9, respectively.

In Chapter 6 we develop a general axiomatization of multidimensional space that will form the basis for the remainder of the thesis. Chapter 7 extends that axiomatization by defining so-called mereological closure operations: binary functions that denote the intersection, difference, and sum for any pair of spatial entities, as well as a constant that denotes the universal entity. In Chapter 8 we use the theories from Chapters 6 and 7 to establish relationships to two other mereotopologies, namely to the equidimensional RCC and to the multidimensional INCH Calculus.

Chapters 9 and 10 each propose an extension of the language of the basic multidimensional mereotopology. Chapter 9 considers an extension by the notion of *boundary-containment*. This extension of the language suffices to draw the interior-boundary distinctions that are necessary to define the nine

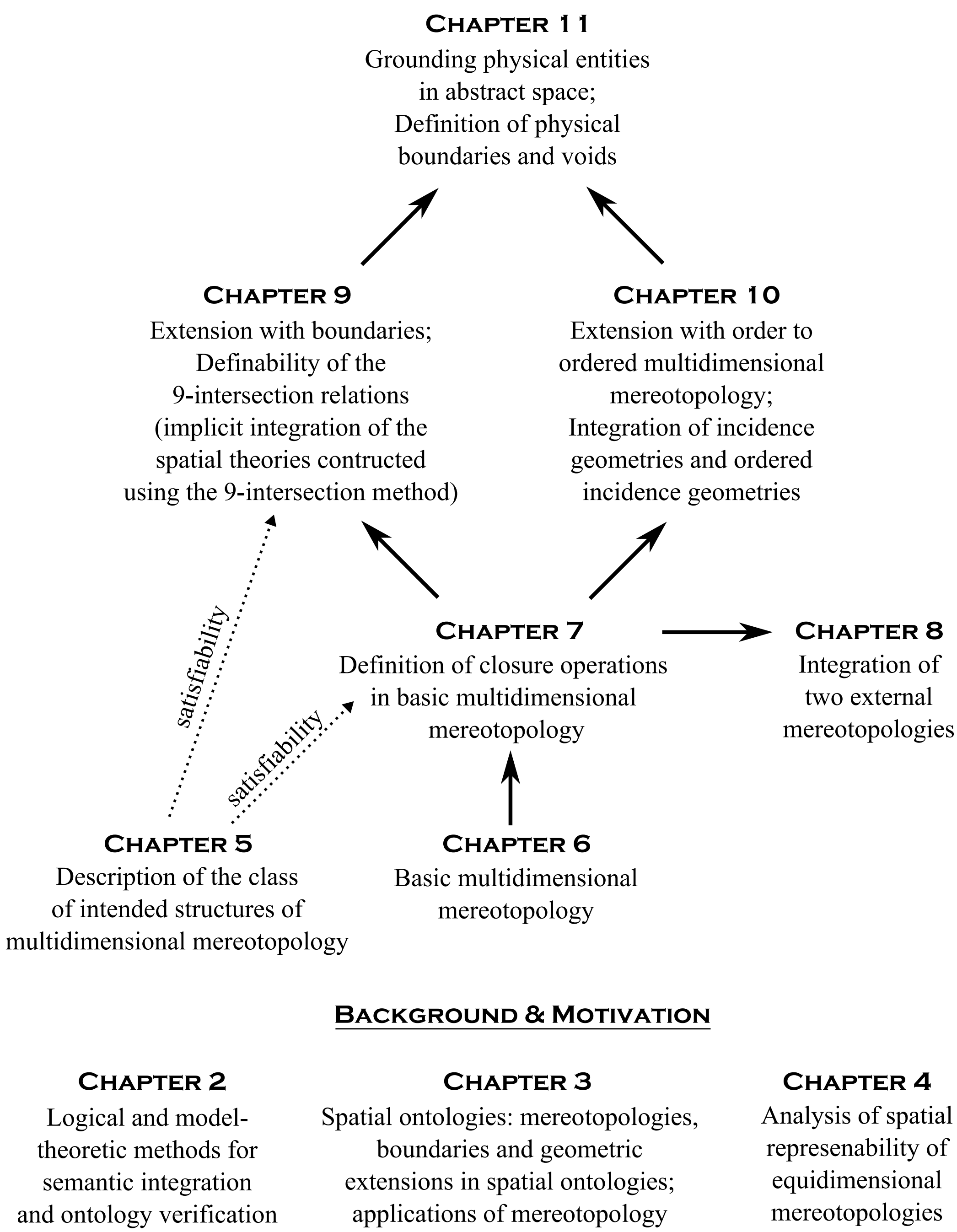


Figure 1.2: The synopsis of each thesis chapter and the dependencies between the chapters.

possible intersection relations arising from the intersection of the interior, boundary, or exterior of one entity with the interior, boundary, or exterior of a second entity. Chapter 10 examines an extension by a multidimensional version of betweenness as a notion of spatial order. The chapter relates the basic multidimensional mereotopology to incidence geometries and its extension by the betweenness relation to ordered incidence geometries.

Our final technical chapter, Chapter 11, utilizes the abstract spatial theories that we developed in the Chapters 6 to 10 to model physical space. In particular, we show how, in principle, the various conceptions of physical boundaries and surfaces can be defined using the abstract spatial theories and we give a spatial account of physical voids. The chapter finishes with a use case scenario that demonstrates interesting hydrological distinctions that we can make using our multidimensional spatial theories as underlying theory of abstract space.

While each chapter contains a summary of its results, we summarize the key findings and insights of the entire thesis in Chapter 12.

Chapter 2

Methodology: theory extensions and interpretations[1]

In our work, the ontologies of space that we are interested in are all axiomatized or axiomatizable as theories in first-order logic. In fact, for the purpose of the thesis we consider spatial ontologies as nothing but first-order axiomatizations of spatial relations.

First-order ontologies are vastly expressive but at the same time very difficult to integrate compared to less expressive languages such as description logics or the OWL ontology languages. We now give an overview of the formal methods we use to verify the spatial ontologies we develop and to integrate spatial ontologies with one another. Most of these methods are well-known in mathematical logic or in the field of model theory. Through joint effort in our lab [Grü09; GHK11; Grü+10; Grü+12; KG10], those general logical methods have been customized for the verification of ontologies and the semantic integration of ontologies. Some methods are further refined here.

On a more general note, the work in this thesis is conducted in the spirit of the axiomatic method, which some consider as one of the oldest methods of science. In its modern mathematical and geometrical form it dates back to Moritz Pasch [Pas88], who revolutionized the study of 'physical space' (which is what used to be meant by 'geometry' until the end of the 19th century) by providing a set of axioms in a formal language and proving theorems about geometry *from the axioms alone*. Hilbert's work "Grundlagen der Geometrie" [Hil71], first published in 1899, is probably the most famous piece of work applying the axiomatic method to geometry, even though his presentation is sometimes far from clear. Indeed, much axiomatic work in geometry, for example, uses a formal language that is still less rigorous than what we aim for. For the purpose of automated reasoning with spatial ontologies, the degree of formality provided by most geometry textbooks is still insufficient. In particular, we want to avoid set-theoretic definitions of lines or planes, because those do not lend themselves to easily implementable first-order axiomatizations. We differ in one point from most work that applies the axiomatic method; we are not only interested in abstract mathematical theories, but rather study them in relationship to particular classes of intended structures that we want to capture. Therefore, we must show that the axioms of a particular theory are actually satisfied in the class of intended structures, given a particular meaning of all nonlogical symbols in the theory's language. Nevertheless, we can work purely axiomatically when we compare two theories to another, for example, we can show that the axioms of one theory are entailed

[1] The work in this chapter extends joint work previously published as [GHK11; Grü+10; Grü+12].

by the axioms of another theory. But we also want to compare theories with different languages; to do so, we discuss the ideas of *expressiveness* and *definability*.

This chapter is structured as follows. In Section 2.1 we review the logical terminology and notation used in the thesis as well as two notions of definability: one regarding theories and the other regarding structures. We also show how those two notions are related to each other. In the subsequent Sections 2.2 and 2.3, we specifically discuss the logical and model-theoretic methods we utilize to compare, relate, verify, and integrate spatial ontologies. Throughout, we point to thesis results that have been obtained using the discussed methods.

2.1 Tools from mathematical logic and model theory

We now review the logical terminology and notation used throughout the thesis; most of it can be found in standard references on mathematical logic, such as [End72; Hod93; Kle67; Mar02], though the presentations differ significantly in terminology, notation, and organization.

2.1.1 First-order theories

Throughout, we assume standard first-order logic with equality, using the standard symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftarrow$, and $\leftrightarrow$[2] as logical connectives and $\exists$ and $\forall$ as quantifiers. The symbol $=$ denotes the distinguished binary relation of equality. We assign meaning to terms (denoting individuals of the domain), well-formed formulas, and sentences (having truth values) in the standard inductive way, as explicated, for example, in [End72].

We call a set T of first-order sentences an axiomatization. The sentences of an axiomatization are usually called its *axioms* and *definitions*, we will formalize the difference between axioms and definitions later on. We write $T \models \sigma$ to express that σ is a logical consequence—or theorem—of T, we also say T *entails* σ. A first-order *theory* is an axiomatization closed under *logical consequences*. Because all our theories are assumed to be first-order, we use the term *theory* throughout to refer to a first-order theory.

Definition 2.1. *A theory is a set of first-order sentences closed under logical consequence.*

Strictly speaking, a theory T as a set of sentences closed under logical consequence is different from its *axiomatization* as a set of sentences whose closure under logical consequences is the theory. We will not make this distinction unless absolutely necessary; for convenience both are from now on referred to as a theory. Two theories or two axiomatizations in the same language are *logically equivalent* if and only if they have identical closures under entailments.

Every theory is implicitly associated with a *signature* and *language* specified by the nonlogical symbols that appear in the sentences of the theory.

Definition 2.2. *The signature of a theory T, denoted by $\lambda(T)$, is the set of all nonlogical symbols, i.e., all constant symbols, function symbols, and relation symbols, that appear in T.*
The language of a theory T, denoted by $\mathcal{L}(T)$, is the set of all well-formed first-order formulas that only use the nonlogical symbols in the signature $\lambda(T)$.
The language of a signature λ, denoted by $\mathcal{L}_\lambda$, is the set of all well-formed first-order formulas that only use the nonlogical symbols in the signature λ.

[2] The symbols $\Rightarrow$, $\Leftarrow$, and $\Leftrightarrow$ do not denote logical connectives, but rather abbreviate the meta-mathematical expressions 'implies', 'is implied by', and 'if and only if'.

Note that a theory's language is equivalent to the language of the theory's signature. But the definition of the *language of a signature* also allows us to talk about languages not associated with a concrete theory.

We say a theory T is a $\mathcal{L}$-theory if and only if $\mathcal{L}$ is the language of T. Equally, we call a formula α or a sentence σ an *$\mathcal{L}$-formula* or an *$\mathcal{L}$-sentence* if and only if $\sigma \in \mathcal{L}$ or $\alpha \in \mathcal{L}$. Generally, we will use α and β to denote formulas and σ to denote a sentence. We use lower-case letters to denote function symbols and upper-case letters to denote relation symbols, except in Chapter 11 where some relations are denoted by lower-case symbols to preserve the relation names from previous work. ω and Ω denote an arbitrary function or relation symbol, respectively. Appendix A contains a list of all nonlogical symbols used in this thesis.

2.1.2 Interpretations and models of first-order theories

A theory is a purely abstract mathematical object, only when we interpret the nonlogical symbols of the theory, we can use the theory to capture knowledge about the world (or about a hypothetical world).

An *interpretation* of an $\mathcal{L}$-theory T is a tuple $\mathcal{I} = \langle D, \Phi, \Psi \rangle$[3] that assigns a meaning to all symbols in $\lambda(T)$. An interpretation of a theory of the language $\mathcal{L}$ is also called an *$\mathcal{L}$-structure* [Ebb94; End72; Mar02] with $\lambda(\mathcal{I})$ denoting the signature of the structure. D denotes a nonempty domain, Φ a mapping of all n-place function symbols $\omega \in \lambda(T)$ (including constants) to functions $\Phi(\omega) : D^n \rightarrow D$, Ψ is a mapping of all n-place relation symbols (predicates) $\Omega \in \lambda(T)$ to relations $\Psi(\Omega) : D^n \rightarrow \{\mathsf{True} \mid \mathsf{False}\}$, where True means the relation holds and False means the relation does not hold. An $\mathcal{L}$-structure $\mathcal{I}$ *satisfies* an $\mathcal{L}$-sentence σ if and only if the structure assign σ the truth value True; we then write $\mathcal{I} \models \sigma$. Two $\mathcal{L}$-structures $\mathcal{I}_1$ and $\mathcal{I}_2$ are *elementarily equivalent* if for all $\mathcal{L}$-sentences σ, $\mathcal{I}_1 \vDash \sigma$ if and only if $\mathcal{I}_2 \vDash \sigma$.

We call an interpretation $\mathcal{I}$ a *model* of a theory T if and only if all axioms (or equivalently, all logical consequences) of T are satisfied, that is, all axioms are evaluated to statements with the truth value $True$; we can write $\mathcal{I} \models T$. A theory is *consistent* (or *satisfiable*) if it has some model. T entails the sentence σ if and only if the sentence is satisfied in all models of T.

Models of a logical theory are denoted by calligraphic upper case Latin letters such as $\mathcal{M}$. If necessary, a subscript indicates the theory of which it is a model of, such as $\mathcal{M}_{DI_{linear}}$. If the theory is clear from the context, we omit the subscript. The domain of a model is denoted by a bold upright version of the letter used to denote the model, e.g., $\mathbf{M}$ is the domain of the model $\mathcal{M}$. Mod(T) denotes the set of all models of the theory T.

If Ω is an n-place relation symbol in the signature of a theory T, every model $\mathcal{M}$ of T specifies a set $\boldsymbol{\Omega}_{\mathcal{M}} \subseteq \mathbf{M}^n$ such that $\langle \overrightarrow{a} \rangle \in \boldsymbol{\Omega}_{\mathcal{M}}$ if and only if $\Psi(\Omega)(\overrightarrow{a}) = \mathsf{True}$. Equally, any n-place function symbol $\omega \in \lambda(T)$ specifies a set $\boldsymbol{\omega}_{\mathcal{M}} \subseteq \mathbf{M}^{n+1}$ such that $\langle \overrightarrow{a}, b \rangle \in \boldsymbol{\omega}_{\mathcal{M}}$ if and only if $\Phi(\omega)(\overrightarrow{a}) = b$. We call the sets $\boldsymbol{\Omega}_{\mathcal{M}}$ and $\boldsymbol{\omega}_{\mathcal{M}}$ the *extensions* of the symbols Ω and ω, respectively. Notice that any n-place functions has an extension that could also be considered as the extension of an $(n+1)$-place relations. In that sense, functions are nothing more than special kinds of relations.

Extensions of particular functions and relations in a model are always denoted by the bold relation or function symbol with the model name as subscript, such as $\boldsymbol{\Omega}_{\mathcal{M}}$ or $\boldsymbol{\omega}_{\mathcal{M}}$. For example, $\mathbf{ZEX}_{\mathcal{M}}$, $\mathbf{Cont}_{\mathcal{M}}$, and $(\leq_{\mathbf{dim}})_{\mathcal{M}}$ denote the extension of the zero region ZEX, the containment relation $Cont$, and the

[3] Technically, a variable assignment function is also needed to assign each variable to an element in D; we take the variable assignment for granted.

relative dimension relation $\leq_{\text{dim}}$ of a model $\mathcal{M}$. $\mathbf{boundary}_{\mathcal{M}}$ denotes the extension of the boundary function of a model $\mathcal{M}$. Extensions of unary relations are sets, we write $a \in \mathbf{ZEX}_{\mathcal{M}}$ to say that the domain element $a \in \mathbf{M}$ is in the extensions of ZEX in the model $\mathcal{M}$. Equally, we write $\langle a, b\rangle \in \mathbf{Cont}_{\mathcal{M}}$ or $\langle a, b\rangle \in (\boldsymbol{\leq}_{\mathbf{dim}})_{\mathcal{M}}$ to state that for the pair $a, b \in \mathbf{M}$ the relations $Cont$ or $\leq_{\text{dim}}$ evaluate to $True$ in the model $\mathcal{M}$. We often abuse the terser notation $\mathbf{Cont}(a, b)$ or $a \boldsymbol{\leq}_{\mathbf{dim}} b$ to denote the same fact within semantic proofs. In combination with this notation, we sometimes use the logical connectives but mean their semantics. For example, the expression

$$\mathbf{P}(z, x - y) \leftrightarrow \mathbf{P}(z, x - (x \cdot y))$$

says that within the model $\mathcal{M} = \langle \mathbf{M}, \Phi, \Psi\rangle$ of interest (which is obvious from the context) with $x, y, z \in \mathbf{M}$, there exist $u, v, w \in \mathbf{M}$ such that $u = \Phi(-)(x, y)$, $v = \Phi(\cdot)(x, y)$, and $w = \Phi(-)(x, v)$ and such that

$$\langle z, u\rangle \in \mathbf{P}_{\mathcal{M}} \quad \text{if and only if} \quad \langle z, w\rangle \in \mathbf{P}_{\mathcal{M}}.$$

The function symbols $\cdot$, $-$, and $+$ are not shown in bold, even when they are used semantically. The difference should always be clear from the context.

2.1.3 Theory extensions

Definition 2.3. *A theory T_2 is an* <u>*extension*</u> *of a theory T_1 (or T_2 extends T_1) iff for any sentence $\sigma \in \mathcal{L}(T_1)$,*

$$\textit{if} \quad T_1 \models \sigma \quad \textit{then} \quad T_2 \models \sigma.$$

T_2 is a <u>*conservative extension*</u> *of T_1 iff for any sentence $\sigma \in \mathcal{L}(T_1)$,*

$$T_1 \models \sigma \quad \textit{iff} \quad T_2 \models \sigma.$$

T_2 is a <u>*nonconservative extension*</u> *of T_1 iff T_2 is an extension of T_1 and some sentence $\sigma \in \mathcal{L}(T_1)$ exists such that*

$$T_1 \not\models \sigma \quad \textit{and} \quad T_2 \models \sigma.$$

Strictly speaking, this definition only applies to theories T_1 and T_2 with $\lambda(T_1) \subseteq \lambda(T_2)$.

If $\lambda(T_1) = \lambda(T_2)$ and T_2 is a nonconservative extension of T_1, then T_2 must contain at least one axiom that is *independent*, i.e., not provable from the axioms of T_1. Two theories with $\lambda(T_1) = \lambda(T_2)$ that extend each other must be logically equivalent. Conversely, two logically equivalent theories conservatively extend each other.

2.1.4 Definability of nonlogical symbols

Just as we can check a set of axioms for independence, we can also check whether the nonlogical symbols in the language of a theory are independent from another in the sense that all nonlogical symbols are *undefined* concepts. An undefined concept in the language of a theory is a nonlogical symbol that is not definable in terms of the other nonlogical symbol of the language. To formalize this notion of an undefined concept, we need to explicate when a nonlogical symbol is *explicitly definable*, an idea that reaches back to work by Padoa in 1900 (reprinted in [Hei67]) and that has been investigated by

Tarski [Tar56c] and Beth [Bet53]. Explicit definability essentially requires that it is possible to give an explicit definition.

Definition 2.4. *Let λ be a signature.*
An explicit definition of a function symbol $\omega \notin \lambda$ in terms of λ is a sentence $\gamma \in \mathcal{L}_{\lambda \cup \{\omega\}}$ such that

$$\gamma = \forall \overrightarrow{v} \left[\forall x \left[\omega(\overrightarrow{v}) = x \leftrightarrow \alpha(\overrightarrow{v}, x) \right] \wedge \exists x \left[\alpha(\overrightarrow{v}, x) \right] \right]$$

where $\alpha(\overrightarrow{v}, x)$ is a formula in $\mathcal{L}_\lambda$ in which only $\overrightarrow{v}$ and x occur free.
An explicit definition of a relation symbol $\Omega \notin \lambda$ in terms of λ is a sentence $\gamma \in \mathcal{L}_{\lambda \cup \{\Omega\}}$ such that

$$\gamma = \forall \overrightarrow{v} \left[\Omega(\overrightarrow{v}) \leftrightarrow \alpha(\overrightarrow{v}) \right]$$

where $\alpha(\overrightarrow{v})$ is a formula in $\mathcal{L}_\lambda$ in which only $\overrightarrow{v}$ occur free.

We use γ throughout to denote explicit definitions. A nonlogical symbol is *definable in a theory* if and only if an explicit definition for the symbol exists in the theory [Ben04; Kle67; Tar56c]. That captures the idea of dispensable symbols, i.e., we can substitute every occurrence of such a symbol with its definition without logically changing the theory.

Definition 2.5. *A nonlogical symbol $\Omega \in \lambda(T)$ is (explicitly) definable in a theory T iff there is a possible definition γ of Ω in terms of $\lambda(T) \setminus \Omega$ such that $T \models \gamma$.*

More generally, we can say that a nonlogical symbol is definable in a theory using only a particular subset of the other nonlogical symbols of the theory's signature.

Definition 2.6. *A nonlogical symbol $\Omega \in \lambda(T)$ is (explicitly) definable in a theory T in terms of $\lambda' \subseteq \lambda(T)$ iff there is a possible definition γ of Ω in terms of $\lambda' \setminus \Omega$ such that $T \models \gamma$.*

Definability allows us to separate a set of mutually undefinable nonlogical symbols from the definable nonlogical symbols. In that way, we reduce a theory's signature to a set of so-called *primitives*, of which none is definable in terms of the others.

Definition 2.7. *Let T be a theory with signature $\lambda(T)$. The symbols within a set of nonlogical symbols $\mathbf{\Lambda}(T) \subseteq \lambda(T)$ are called a set of primitives of T iff*

1. *no symbol $\Omega \in \mathbf{\Lambda}(T)$ is definable in T in terms of $\mathbf{\Lambda}(T) \setminus \Omega$, and*
2. *every symbol $\Omega \in \lambda(T)$ with $\Omega \notin \mathbf{\Lambda}(T)$ is definable in T in terms of $\mathbf{\Lambda}(T)$.*

There may be no unique set of primitives for a particular theory. But usually we are interested in sets of primitives that include some particular nonlogical symbols. We assume that every theory has a distinguished set of primitives (see Section 2.2.1), denoted by $\Lambda(T)$, such that $\Lambda(T) \subseteq \lambda(T)$. We call the set of sentences that only use the primitive nonlogical symbols of T the *primitive language* of T and refer to it as $\mathcal{L}_\Lambda(T)$.

Next, we formalize when an extension of a theory only introduces new definitions [Hod93, p. 60].

Definition 2.8. *A theory T_2 is a definitional extension of T_1 iff*

1. *T_2 is a conservative extension of T_1, and*

2. *any symbol $\Omega \in \lambda(T_2)$ is definable in T_2 in terms of $\lambda(T_1)$.*

The definition of a conservative extension already requires $\lambda(T_2) \supseteq \lambda(T_1)$. That means T_2 only introduces new definable symbols but does otherwise not differ logically from T_1. We can capture the definitions introduced by T_2 in a separate set of sentences Γ, called *definitions*, as follows [Hod93, p. 60].

Theorem 2.1. *T_2 is a definitional extension of T_1 iff there exists a set of sentences Γ and a bijection $\varphi : \lambda(T_2) \setminus \lambda(T_1) \rightarrow \Gamma$ such that*

1. *T_2 is logically equivalent to $T_1 \cup \Gamma$;*

2. *$\varphi(\Omega)$ is an explicit definition of Ω in terms of $\lambda(T_1)$;*

3. *$T_1 \models \Gamma$.*

Proof. $\Leftarrow$: This direction is trivial. T_2 is logically equivalent to $T_1 \cup \Gamma$ and thus a conservative extension of $T_1 \cup \Gamma$. Furthermore, $T_1 \models \Gamma$, hence T_2 is a conservative extension of T_1, satisfying condition (1) of Definition 2.8. Moreover, condition (2) of Definition 2.8 is satisfied, because Γ is a set of explicit definitions of the symbols in $\lambda(T_2) \setminus \lambda(T_1)$.
$\Rightarrow$: Assume that T_2 is a definitional extension of T_1.
Let us define Γ as the set of explicit definitions for all the symbols $\Omega \in \lambda(T_2) \setminus \lambda(T_1)$ in terms of $\lambda(T_1)$ that must exist because T_2 is a definitional extension of T_1 by Definitions 2.6 and 2.8. The set of sentences Γ thereby satisfies condition (2) and condition (3), the latter because $T_1 \models \gamma$ for every explicit definition $\gamma \in \Gamma$. It remains to prove that condition (1) is satisfied.

By our assumption, T_2 extends T_1 and by Definition 2.6 $T_1 \models \Gamma$, thus T_2 extends $T_1 \cup \Gamma$. If we can show that $T_1 \cup \Gamma$ also extends T_2, then $T_1 \cup \Gamma$ and T_2 are logically equivalent. It thereby suffices to show that every sentence $\sigma \in \mathcal{L}(T_2)$ entailed by T_2 is also entailed by $T_1 \cup \Gamma$.

Assume $\sigma \in \mathcal{L}(T_2)$ such that $T_2 \models \sigma$.
We can rewrite σ as a logically equivalent *unnested formula* σ' inductively following [Hod93, p. 58–59]. All atomic subformulas in σ' are unnested, that is, they are of the form $v = x$ or $\omega(\overrightarrow{v}) = x$, or $\Omega(\overrightarrow{v})$. Any symbol $\Omega \in \lambda(T_2) \setminus \lambda(T_1)$ that occurs in σ has an explicit definition $\gamma \in \Gamma$[4] such that

$$\gamma = \forall \overrightarrow{v}[\Omega(\overrightarrow{v}) \leftrightarrow \alpha(\overrightarrow{v})],$$

where $\alpha(\overrightarrow{v}) \in \mathcal{L}(T_1)$. Any occurrence of $\Omega(\overrightarrow{v})$ in σ', where $\Omega \in \lambda(T_2) \setminus \lambda(T_1)$, can thus be replaced by the formula $\alpha(\overrightarrow{v})$ to obtain a logically equivalent sentence $\sigma_{\lambda(T_1)}$ because $T_2 \models \gamma$. Hence, we have

$$T_2 \models \sigma \quad \Leftrightarrow \quad T_2 \models \sigma' \quad \Leftrightarrow \quad T_2 \models \sigma_{\lambda(T_1)}$$

$\sigma_{\lambda(T_1)}$ only uses symbols from $\lambda(T_1)$, thereby $\sigma_{\lambda(T_1)} \in \mathcal{L}(T_1)$. Hence $T_1 \models \sigma_{\lambda(T_1)}$ iff $T_2 \models \sigma_{\lambda(T_1)}$ because T_2 is a conservative extension of T_1 by Definition 2.8(1). Hence $T_1 \cup \Gamma \models \sigma_{\lambda(T_1)}$ iff $T_2 \models \sigma_{\lambda(T_1)}$. Consequently, every sentence σ entailed by T_2 is logically equivalent to a sentence $\sigma_{\lambda(T_1)}$ that is also entailed by $T_1 \cup \Gamma$. Therefore $T_1 \cup \Gamma$ extends T_2. □

[4] Because all n-place functions ω can be considered as $(n+1)$-place relations, we only deal with definitions γ of relations Ω; the functions being a special case thereof as long as $T_2 \models \gamma$.

2.1.5 Definability of relations in a structure

A related notion is concerned with the sets, functions, and relations that are definable in some structure. We maintain the definitions from [Grü09; Grü+10; Mar02].

Definition 2.9. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure.*
A set $\mathbf{X} \subseteq \mathbf{M}^n$ is a definable set in $\mathcal{M}$ iff there is a formula $\alpha(v_1, ..., v_n, w_1, ..., w_m)$ in $\mathcal{L}$ and $\overrightarrow{b} \in \mathbf{M}^m$ such that

$$\mathbf{X} = \{\overrightarrow{a} \in \mathbf{M}^n \ : \ \mathcal{M} \models \alpha(\overrightarrow{a}, \overrightarrow{b})\}.$$

A definable structure comprises a definable set together with some definable subsets, the definable relations. Again, all functions can be rewritten as relations and are thus not treated separately.

Definition 2.10. *Let $\mathcal{N}$ be an $\mathcal{L}_1$-structure and let $\mathcal{M}$ be an $\mathcal{L}_2$-structure with domain $\mathbf{M}$. $\mathcal{N}$ is definable in $\mathcal{M}$ (equivalently, $\mathcal{M}$ defines $\mathcal{N}$) iff we can find a definable subset $\mathbf{X} \subseteq \mathbf{M}^n$ and we can interpret the nonlogical symbols of $\mathcal{L}_1$ as definable subsets and functions on $\mathbf{X}$ so that the resulting $\mathcal{L}_1$-structure is elementarily equivalent to $\mathcal{N}$.*

Note that the definition here uses elementary equivalence instead of isomorphism as condition on the resulting structure because we are only interested in differences that can be expressed by some first-order sentence. In that sense, the notion of *definability* here is what has been called *weak definability* in [Grü+10].

If a structure $\mathcal{N}$ merely adds new relations and functions to a structure $\mathcal{M}$ without changing the domain or the existing relations or functions, then $\mathcal{N}$ is an *expansion* of a structure $\mathcal{M}$. More precisely, $\mathcal{N}$ is an *expansion* of a structure $\mathcal{M}$ if and only if $\lambda(\mathcal{N}) \supseteq \lambda(\mathcal{M})$ and we can obtain $\mathcal{N}$ from $\mathcal{M}$ by forgetting about all the relations and functions in $\lambda(\mathcal{M}) \setminus \lambda(\mathcal{N})$ [Hod93]. An expansion $\mathcal{N}$ of $\mathcal{M}$ is a *definitional expansion* if and only if all added relations and functions are definable in $\mathcal{M}$ [Hod93].

Definition 2.11. *A structure $\mathcal{N}$ is a definitional expansion of a structure $\mathcal{M}$ iff*

1. *$\mathcal{N}$ and $\mathcal{M}$ have equivalent domains, that is, $\mathbf{N} = \mathbf{M}$;*
2. *$\mathbf{\Omega}_{\mathcal{N}} = \mathbf{\Omega}_{\mathcal{M}}$ for all $\Omega \in \lambda(\mathcal{M})$;*
3. *$\mathbf{\Omega}_{\mathcal{N}}$ is a definable set in $\mathcal{M}$ for all $\Omega \in \lambda(\mathcal{N}) \setminus \lambda(\mathcal{M})$.*

We can then prove that proof-theoretic definability of symbols as captured by the notion that one theory is a definitional extension of another theory is directly related to the model-theoretic notion of definitional expansions between their models.

Theorem 2.2. *Let T_1 and T_2 be two theories. Then T_2 is a definitional extension of T_1 iff there is a bijection $\varphi : \text{Mod}(T_1) \to \text{Mod}(T_2)$ such that $\varphi(\mathcal{M})$ is the uniquely defined definitional expansion of $\mathcal{M}$.*

Proof. $\Rightarrow$: Assume T_2 is a definition extension of T_1. Then by Theorem 2.1 for all $\Omega \in \lambda(T_2) \setminus \lambda(T_1)$ there exists a explicit definition $\gamma \in \Gamma$ in T_2, which we can use to define $\mathbf{\Omega}_{\mathcal{M}}$ for any model $\mathcal{M}$ of T_1. Clearly, $\mathcal{M} \cup \{\mathbf{\Omega}_{\mathcal{M}} \ : \ \Omega \in \lambda(T_2) \setminus \lambda(T_1)\}$ is a uniquely defined definitional expansion of $\mathcal{M}$ and is also a model of T_2 because T_2 is logically equivalent to $T_1 \cup \Gamma$.
$\Leftarrow$: The symbols in $\lambda(T_2) \setminus \lambda(T_1)$ are implicitly defined in any model of T_1. Then by Beth's definability theorem [Bet53; Ebb94; Kle67] there is an explicit definition of them, which can serve as the set of sentences Γ in Theorem 2.1. □

2.1.6 Notation for theories and sentences

Logical theories are denoted by names with italic upper case Latin letters; logically distinct theories with the same set of primitives share a name, but have a different subscript, which indicates the restrictiveness of the axioms. For example, the theories DI_{linear} and $DI_{\text{linear-bounded}}$ share the language definable from the primitive relations *ZEX* and $<_{\text{dim}}$, but differ in the restrictiveness of their axioms. As another example, the abbreviation *CODI* is reserved for theories that only use the primitive nonlogical symbols *Cont* and $<_{\text{dim}}$. Appendix C contains a list of all named logical theories used in this thesis.

All logical sentences used for axiomatizing theories are labeled according to the schema '[theory]-[type][number]' as in EP-T1, where the first letter(s) indicate the theory (for example, D=dimension, C=containment, EP=parthood, CD=containment & dimension, INCH=INCH Calculus), while the type distinguishes axioms (A), definitions (D), theorems (T), extension axioms (E), and mapping axioms (M). Axioms and definitions are always included in the corresponding theories, while extension axioms are optional axioms that may be used to further extend the corresponding theories in a nonconservative way. Theorems are properties that can be proved from the axioms and definitions alone. Mapping axioms are only relevant when we prove definable interpretations or definable equivalence between two theories, see Section 2.2.4. All free variables that occur in sentences labeled according to this schema are assumed to be universally quantified. Appendix A contains a list of all named sentences used in the theories designed in this thesis.

2.2 Relationships between ontologies

In this section, we review meta-mathematical relationships between theories that will help us to organize theories into hierarchies, partially order them within hierarchies, and relate them across hierarchies. First, we review the sets of primitives used (Subsection 2.2.1) in the various chapters and how we can organize and relate theories with equivalent, related, or totally different sets of primitives using theory extensions and theory interpretations (Subsections 2.2.2 – 2.2.4). In those subsections we also show how the relationships between theories correspond to relationships between their classes of models because, ultimately, we want to exchange models of the ontologies between different spatial information systems. Thereby, Subsections 2.2.2 – 2.2.4 form the technical foundation on which we construct the family of hierarchies of spatial ontologies.

2.2.1 The primitives used in our multidimensional theories of space

In Chapter 4 we concentrate on equidimensional mereotopological theories that rely exclusively on the primitive concepts of *region* as a class, *connection* (or *contact*) as a topological relation, and *part* as a mereological relation. We investigate which of those equidimensional mereotopologies can be used to adequately axiomatize space such that models can be interpreted topologically and/or mereologically with a suitable complementation operation. In particular, we are interested in equidimensional mereotopologies that equally allow discrete and continuous models of space. This comprehensive study reveals that very few spatially representable equidimensional mereotopologies can exist, while all of them have an equivalent expressivity.

To overcome the limitation in expressivity, we subsequently focus on multidimensional mereotopology that can deal with spatial entities of varying dimension in a single model. We use the following set of primitive symbols in this thesis to design successively stronger multidimensional mereotopologies from Chapter 6 on:

- the zero region $ZEX(x)$ (Chapter 6, only for the theories of the *DI* hierarchy; *ZEX* becomes definable in any theory that extends a theory of the *CO* hierarchy),
- (relative) dimension $x <_{\dim} y$ (Chapter 6),
- spatial containment $Cont(x, y)$ (Chapter 6),
- boundary-containment $BCont(x, y)$ (specializes spatial containment; Chapter 9),
- betweenness $Btw(r, x, y, z)$ (Chapter 10), and
- convex hull function $\mathrm{ch}(x)$ (Chapter 11).

All symbols in the set $\mathbf{\Omega} = \{<_{\dim}, Cont, BCont, Btw, \mathrm{ch}\}$ are treated as primitives if they are in the signature of any multidimensional theory proposed in this thesis. More formally, let T be any of our theories. Then for any $\Omega \in \mathbf{\Omega}$,

$$\Omega \in \lambda(T) \Rightarrow \Omega \in \Lambda(T).$$

This does not apply to external spatial theories that we also discuss, formalize, and relate to our own theories.

In Chapter 7 we prove that the mereological closure functions are definable functions in the theory *CODI* (Theorems 7.1, 7.2, 7.5, and 7.7) and thus not needed as primitives. But instead of giving explicit definitions, we simply prove that the functions are implicitly defined. This is one example where definability plays a role.

The theory in Chapter 11 that grounds physical space in abstract space contains many more primitives, which we maintain from the DOLCE ontology [Mas+03]. In such upper ontologies it is generally difficult to discriminate primitives from defined symbols, since the primary objective of an upper ontology is to coarsely categorize various kinds of entities. Definitions for broad categories, such as that of a physical endurant, are difficult, if not impossible, to come up with.

2.2.2 Comparing theories with equivalent sets of primitives

Two theories with equivalent primitive languages, that is, with equivalent sets of primitives (up to symbol renaming), are said to be in the same hierarchy of ontologies. This considerably strengthens the definition of a hierarchy as containing only theories with the exact same languages from [Grü+12].

Definition 2.12. *A hierarchy* $\mathbb{H} = \langle \mathbf{H}, \leq \rangle$ *is a partially ordered, finite set of theories* $\mathbf{H} = T_1, ..., T_n$ *such that*

1. $\Lambda(T_i) = \Lambda(T_j)$*, for all* i, j*;*
2. $T_1 \leq T_2$ *iff* T_2 *is an extension of* T_1*;*
3. $T_1 < T_2$ *iff* T_2 *is a nonconservative extension of* T_1*.*

Out of two theories T_1 and T_2 in the same hierarchy, we say T_2 is (axiomatically) *more restricted* than T_1 if and only if T_2 is a nonconservative extension of T_1, i.e., $T_1 < T_2$. Because theories within a hierarchy use the same language apart from definitions, they are capable of expressing the same set of sentences. In other words, they have equivalent expressive power. However, an explicit definition of an intended relation given in terms of their common primitive language may be adequate in some highly restricted theory but inadequate in a less restricted theory. For example, in Chapter 10 we can give a definition of convexity in our ordered multidimensional mereotopology $OMT_{\downarrow}$, but the definition only captures the intended idea of a convex region for a subclass of the models of $OMT_{\downarrow}$, namely those that are ordered incidence geometries. Hence, the relation of being convex as defined in Euclidean geometry, is not definable for all models of $OMT_{\downarrow}$.

Generally, we have the following relationship between the models of two theories with the exact same language [Grü+12].

Theorem 2.3. *If T_1 and T_2 are theories in the same hierarchy with $\lambda(T_1) = \lambda(T_2)$, then*

$$T_1 < T_2 \iff \mathrm{Mod}(T_2) \subsetneq \mathrm{Mod}(T_1).$$

In other words, a theory that is a nonconservative extension of another theory in the same language and with the same primitives only restricts the set of models but does not change the structure of the models. We can then relate all theories in the same hierarchy by their models as follows.

Corollary 2.1. *Let T_1 and T_2 be theories in the same hierarchy.*
Then $T_1 \leq T_2$ iff there exists an injective mapping $\varphi : \mathrm{Mod}(T_2) \rightarrow \mathrm{Mod}(T_1)$ such that any model $\mathcal{M} \in \mathrm{Mod}(T_2)$ is a definitional expansion of $\varphi(\mathcal{M})$.

Proof. Because T_1 and T_2 are in the same hierarchy, the have the same set of primitives, that is, $\Lambda(T_1) = \Lambda(T_2)$.
$\Rightarrow$: Assume $T_1 \leq T_2$. Then the symbols in $\lambda(T_2) \setminus \lambda(T_1)$ are all definable in T_2 in terms of $\lambda(T_1)$. Hence by Theorem 2.1 there exists a theory T_3 in the language of T_1 that is logically equivalent to T_2 (compare also Theorem 2.4 which proves the existence of T_3 in the most general case). Then T_2 is a definitional extension of T_3. There also exists a set of definitions Γ such that T_2 is logically equivalent to $T_3 \cup \Gamma$. Because $T_1 \leq T_3$ and $\lambda(T_1) = \lambda(T_3)$, a trivial mapping $\vartheta : \mathrm{Mod}(T_3) \rightarrow \mathrm{Mod}(T_1)$ exists, namely the identity function where $\vartheta(\mathcal{M}) = \mathcal{M}$ for all $\mathcal{M} \in \mathrm{Mod}(T_3)$ that exists by Theorem 2.3.
Moreover, because T_2 is a definitional extension of T_3, by Theorem 2.2 there exists a bijection $\varphi : \mathrm{Mod}(T_2) \rightarrow \mathrm{Mod}(T_3)$ such that $\mathcal{M}$ is the uniquely defined definitional expansion of $\varphi(\mathcal{M})$.
Then the composition $\varphi \circ \vartheta : \mathrm{Mod}(T_2) \rightarrow \mathrm{Mod}(T_1)$ is an injective function because φ is injective and ϑ is the identity function defined for all models in $\mathrm{Mod}(T_3)$.
$\Leftarrow$: Assume there exists an injective mapping $\varphi : \mathrm{Mod}(T_2) \rightarrow \mathrm{Mod}(T_1)$ such that any model $\mathcal{M} \in \mathrm{Mod}(T_2)$ is a definitional expansion of $\varphi(\mathcal{M})$.
Further assume σ to be a sentence in $\mathcal{L}(T_1)$ such that $T_1 \models \sigma$. Then every model in $\mathrm{Mod}(T_1)$ satisfies σ. Because by our assumption every model in $\mathrm{Mod}(T_2)$ is a definitional expansion of a model in $\mathrm{Mod}(T_1)$, every model of T_2 satisfies σ. Therefore $T_1 \models \sigma$ implies $T_2 \models \sigma$, thus T_2 is an extension of T_1 and thus by Definition 2.12 we have $T_1 \leq T_2$. □

Throughout the thesis, we will study several hierarchies of spatial theories. Chapter 6 develops the hierarchies of relative dimension, *DI*, and of containment, *CO*, afterward combining theories from both hierarchies to basic theories of 'containment and dimension' in the hierarchy *CODI*. Chapter 7 focuses on the rather complex hierarchy *CODI*, finding a theory in *CODI* that explicitly closes the domain under standard mereological closure operations: intersection, difference, and sum. In Chapter 9, we add another primitive relation, *BCont* to the primitives of *CODI*, resulting in the hierarchy *CODIB*. Equally, in Chapter 10, we add *Btw* as primitive relation, which defines a hierarchy *BTW* of its own and lets us obtain another hierarchy called *OMT*, whose theories extend theories from *CODI* with the theory *BTW*.

Next we will show the tools we apply to compare theories that do not have equivalent sets of primitives and are thus in different hierarchies. First, we deal with the case where the primitives of one theory are a subset of the primitives of another theory. This enables us to formally relate the various hierarchies to one another.

2.2.3 Comparing theories with related sets of primitives

Two theories of which one has a primitive language that is a subset of the other can be compared with respect to their primitive language as follows.

Definition 2.13. *A theory T_2 expands the primitive language of a theory T_1 iff $\Lambda(T_2) \supseteq \Lambda(T_1)$.*

If $\Lambda(T_2) = \Lambda(T_1)$ then T_2 and T_1 are in the same hierarchy. In the more interesting case when $\Lambda(T_2) \supsetneq \Lambda(T_1)$, some primitive symbol $\Omega \in \Lambda(T_2) \setminus \Lambda(T_1)$ is not definable in T_1 in terms of $\Lambda(T_1)$. Then we say T_2 has a *more expressive language* (or simply is *more expressive than*) than T_1. Equally, we say the hierarchy $\mathbb{H}_2$ is *more expressive* than the hierarchy $\mathbb{H}_1$ if and only if any theory $T_2 \in \mathbb{H}_2$ is more expressive than any theory $T_1 \in \mathbb{H}_1$. This comparison is only based on the primitive languages of $\mathbb{H}_1$ and $\mathbb{H}_2$, which are equivalent for all theories within each hierarchy, thus it suffices to establish that one theory $T_2 \in \mathbb{H}_2$ is more expressive than one theory $T_1 \in \mathbb{H}_1$ in order for any theory in $\mathbb{H}_2$ to be more expressive than any theory in $\mathbb{H}_1$.

If T_2 is a nonconservative extension of T_1 that expands the primitive language of T_1, the following theorem from [Grü+12] guarantees the we can separate the language expansion from the nonconservative extension through another theory whose existence is guaranteed.

Theorem 2.4. *If theory T_2 is a nonconservative extension of theory T_1 with $\Lambda(T_2) \supsetneq \Lambda(T_1)$, then there exists a theory T_3 with $\Lambda(T_3) = \Lambda(T_1)$ such that T_2 is a conservative extension of T_3.*

Next we will relate the models of a theory that extends a second theory while also expanding the language of the second theory. Note that for two arbitrary theories T_1 and T_2 not in the same hierarchy with T_2 extending T_1, $\text{Mod}(T_2) \subseteq \text{Mod}(T_1)$ is not necessarily true. The models of T_2 may only define structures that are elementarily equivalent to models of T_1. For example, the models of the theory of bipartite ordered incidence structures in Chapter 10 are not models of the theory of bipartite incidence structures; rather they define structures by leaving out the betweenness relation that are bipartite incidence structures. We can state this more formally for the general case as follows.

Theorem 2.5. *If a theory T_2 extends a theory T_1 and expands the primitive language of T_1, then there is a function $\varphi : \text{Mod}(T_2) \to \text{Mod}(T_1)$ such that any $\mathcal{M} \in \text{Mod}(T_2)$ expands $\varphi(\mathcal{M})$.*

Proof. If T_2 is a nonconservative extension of T_1, then by Theorem 2.4 there exists a theory T_3 with $\Lambda(T_3) = \Lambda(T_1)$ such that T_2 is a conservative extension of T_3. If T_2 is a conservative extension of T_1, $T_3 = T_1$ satisfies the consequent of Theorem 2.4 as well (without having to use Theorem 2.4). In either case, T_1 and T_3 are in the same hierarchy with $T_1 \leq T_3$.

By Theorem 2.1 an injective function $\varphi : \mathrm{Mod}(T_3) \rightarrow \mathrm{Mod}(T_1)$ exists such that any model $\mathcal{M} \in \mathrm{Mod}(T_3)$ is a definitional expansion of $\varphi(\mathcal{M})$. Because T_2 is a conservative extension of T_3, there also exists a mapping $\vartheta : \mathrm{Mod}(T_2) \rightarrow \mathrm{Mod}(T_3)$ such that any model $\mathcal{M} \in \mathrm{Mod}(T_2)$ expands $\vartheta(\mathcal{M})$. The composition of ϑ and φ results in the desired function $\vartheta \circ \varphi : \mathrm{Mod}(T_2) \rightarrow \mathrm{Mod}(T_1)$ with $\mathcal{M} \in \mathrm{Mod}(T_2)$ expanding $\vartheta(\varphi(\mathcal{M})) \in \mathrm{Mod}(T_1)$. □

We will apply the technique of language expansions if the theories of one hierarchy are not expressive enough in that they cannot define certain relations of interest in the class of intended structures. For example, because the relation of a manifold being contained in another manifold's boundary is not definable in the primitive language of *CODI*, we expand it in Chapter 9 by *BCont* as a new primitive relation, resulting in the hierarchy *CODIB*, whose primitive language is a superset of the primitive language of *CODI*. Equally, in Chapter 10, we expand the language of *CODI* by an order relation, resulting in a new hierarchy *OMT*. In that way, language expansion is primarily a relation of interest for comparing related hierarchies by the expressive power of their primitive language.

Finally, we will use a tool from [Hod93, p. 66] that allows us to prove that the models of one theory can always be expanded to models of another theory in order to show that the latter theory is a conservative extension of the former theory. This method is particularly relevant in Chapter 10, it allows us to derive the relationship between the theories of ordered multidimensional mereotopology and ordered incidence geometry from the relationships between their models.

Theorem 2.6. *If T_2 extends T_1 and any model of T_1 can be expanded to a model of T_2, then T_2 conservatively extends T_1.*

2.2.4 Comparing theories with different sets of primitives

To compare theories in different primitive languages, we reuse the idea of interpretability from mathematical logic [End72; Tar68], in particular the notions of relative interpretations and faithful interpretations. We essentially maintain (with some corrections) the definition of interpretability from [Grü+12; SK08], which are based on [End72]. We assume that the languages of T_1 and T_2 use disjoint nonlogical symbols.

Definition 2.14. *An <u>interpretation</u> π of the theory T_1 into a theory T_2 is a function on $\lambda(T_1)$ and formulae in $\mathcal{L}(T_1)$ such that*

1. *π assigns to $\forall$ a formula $\pi_\forall \in \mathcal{L}(T_2)$ in which at most the variable v_1 occurs free, such that*

$$T_2 \models (\exists v_1)\, \pi_\forall$$

2. *π assigns to each n-place relation symbol $\Omega \in \lambda(T_1)$ a formula $\pi_\Omega \in \mathcal{L}(T_2)$ in which at most the variables $v_1, ..., v_n$ occur free.*

3. *π assigns to each n-place function symbol $\omega \in \lambda(T_1)$ a formula $\pi_\omega \in \mathcal{L}(T_2)$ in which at most the variables $v_1, ..., v_n, v_{n+1}$ occur free, such that*

$$T_2 \models \forall v_1, ..., v_n \Big[\pi_\forall(v_1) \wedge \cdots \wedge \pi_\forall(v_n) \rightarrow \exists x \big[\pi_\forall(x) \wedge \forall v_{n+1}[\pi_\omega(v_1, \ldots, v_{n+1}) \leftrightarrow (v_{n+1} = x)]\big]\Big]$$

4. *for any formula $\alpha \in \mathcal{L}(T_1)$,*

$$\pi(\neg\alpha) = \neg\pi(\alpha);$$

5. *for any formulae* $\alpha, \beta \in \mathcal{L}(T_1)$,
$$\pi(\alpha \to \beta) = \pi(\alpha) \to \pi(\beta);$$

6. *for any formula* $\alpha \in \mathcal{L}(T_1)$,
$$\pi(\forall x\, \alpha) = \forall x\, \pi_\forall \to \pi(\alpha);$$

7. *for any sentence* $\sigma \in \mathcal{L}(T_1)$,
$$T_1 \models \sigma \Rightarrow T_2 \models \pi(\sigma).$$

The first condition essential defines the restricted domain of T_2. The other conditions except for the last one allow us to translate arbitrary formulas and sentences in $\mathcal{L}(T_1)$ into sentences in the language of T_2. Only the last condition characterizes the relationship between the theories T_1 and T_2, stating that the mapping π is an interpretation of T_1 if it preserves the theorems of T_1. We say that T_2 *interprets* T_1, or, equivalently, that T_1 *is interpretable in* T_2. Notice that it suffices to map the primitive symbols in $\lambda(T_2)$, since all others symbols in $\lambda(T_2)$ are then implicitly mapped through their definition in terms of the primitives $\Lambda(T_2)$.

Interpretations are generalization of extensions to theories with different languages: if T_2 interprets T_1, then there exists a theory that is definably equivalent to T_2 and that extends T_1. As trivial case we have: if T_2 extends T_1, then T_2 also interprets T_1.

A stronger notion of interpretability is that of a *faithful interpretation* [End72; Grü+12].

Definition 2.15. *An interpretation π of a theory T_1 into a theory T_2 is* <u>*faithful*</u> *iff for any sentence $\sigma \in \mathcal{L}(T_1)$,*
$$T_1 \models \sigma \iff T_2 \models \pi(\sigma).$$

Equivalently, we can state that an interpretation π of a theory T_1 into T_2 is faithful if and only if it satisfies the condition
$$T_1 \not\models \sigma \Rightarrow T_2 \not\models \pi(\sigma).$$

Hence, faithful interpretations preserve not only theorems (as all interpretations do), but also satisfiability. Again, we can say that T_2 *faithfully interprets* T_1 or that T_1 *is faithfully interpretable in* T_2. Observe that any theory T_2 that conservatively extends T_1 automatically faithfully interprets T_1. If T_2 faithfully interprets T_1, then any model of T_2 is—once translated into the language of T_1—an expansion of some model of T_1. This is the analogue of Theorem 2.6 for theories with different primitive languages.

Finally, the strongest meta-mathematical relationship between two theories with different primitives is that of *definable equivalence*, which generalizes logical equivalence to theories in different languages.

Definition 2.16. *Two theories T_1 and T_2 are* <u>*definably equivalent*</u> *iff T_1 is faithfully interpretable in T_2 and T_2 is faithfully interpretable in T_1.*

Stated differently, T_1 and T_2 are *definably equivalent* if and only if there exists a theory T_3 with $\lambda(T_3) = \lambda(T_1) \cup \lambda(T_2)$ that is a definitional extension of both T_1 and T_2.

We use the relationships of faithful interpretations and definable equivalence to relate different spatial theories to one another. Most prominently, we use these relationships in Chapter 8 to construct two theories, one in our *CODI* hierarchy and one as a nonconservative extension of the INCH Calculus, that are definably equivalent. We also use theory interpretations in Chapter 10 to establish a relationship between ordered incidence geometries and a theory of our *OMT* hierarchy. There, the main theorems relate the models of the two theories, but the theory interpretations follow.

2.3 Verification of ontologies

In the previous section we have studied ways to relate hierarchies to one another. This is one way to verify hierarchy, we call this *relative verification* or *cross-verification* because it verifies one theory relative to another or two theories relative to each other. In this section we present other methods we use to verify ontologies. Subsections 2.3.1 and 2.3.2 present the methods we use to verify theories with respect to the class of structures they are intended to capture: satisfiability, axiomatizability, and expressivity. Other verification method is the classification of relations with a theory, which we discuss in Section 2.3.3. Finally, in Subsection 2.3.4 we discuss how we employ automated (mechanical) theorem proving to assist us in verifying individual spatial theories and in relating theories to one another.

2.3.1 Model characterization: satisfiability as definability in a structure

In mathematics, representation theorems are used to understand a class of structures by relating it to another class of structure: any structure in the first class is embedded, in the best case using an isomorphism, in a structure in the second class. That way, properties of the embedding structures can be directly transferred to the embedded structures—and vice versa in the case of an isomorphism. For example, the set of regular closed sets of a topological space always define a Boolean algebra [Hal63], and certain classes of lattices are embeddable in certain kinds of topological spaces, the most famous result (Stone's representation theorem) showing an isomorphism between Boolean algebras and Stone spaces [Sto36]. Equally, we try to understand and verify ontologies of some practical domain, such as space, time, or processes, by representing them in terms of well-understood mathematical structures such as algebraic structures, partial orders, or groups [GHK11; Grü+10]. The known properties of those mathematical structures help us to characterize the class of models of an ontology and to eliminate unintended models. For example, by representing the models of Asher and Vieu's mereotopology algebraically as Stonian p-ortholattices, which again can be represented topologically, we identified unintended models of Asher and Vieu's mereotopology, which do not have a topological representation in terms of the regular sets of a topological space [HWG09; WHG12].

Representation theorems for an ontology with respect to a class of intended structures are proven in two parts [GHK11; Grü+10]. In one direction, *satisfiability* shows that every structure in the class of intended structures is definably equivalent to a model of the theory in question. In the other direction, *axiomatizability* shows that every model of the theory is definably equivalent to a structure in the class of intended structures. Axiomatizability ensures that only structures in the class of intended structures are models of the theory in question.

To be able to talk about either property we must provide a precise description of the class of intended structures we aim to capture. For the multidimensional mereotopology developed in this thesis, we will present a constructive characterization of the class of intended structures based on the mathematical notion of a *manifold with boundary* in Chapter 5. Subsequently, we will prove satisfiability with respect to that class of intended structures for two key theories developed in the thesis: For the theory $CODI_{\downarrow}$ in Theorem 7.4 and for the theory $CODIB_{\downarrow}$ in Theorem 9.2. In the context of those proofs, we will explain why axiomatizability fails and we will discuss the difficulties involved in extending the theories to establish axiomatizability.

The two satisfiability proofs emphasize that theories of different expressiveness can be satisfiable with respect to the same class of structures: $CODI_{\downarrow}$ is less expressive than $CODIB_{\downarrow}$. The difference in their expressivity does, however, affect whether two intended structures can be discriminated by a sentence in the two theories. There are two intended structures that define distinct models of $CODIB_{\downarrow}$ but elementarily equivalent models of $CODI_{\downarrow}$. Hence, $CODI_{\downarrow}$ is not expressive enough to distinguish those intended structures. For this reason, axiomatizability with respect to the intended structures presented in Chapter 5 cannot be proved for any theory in the primitive language of *CODI* . In the next subsection, we will discuss how we can formalize this idea.

Variants of representation theorems are used in Chapter 4, but instead of giving full representations—which is impossible in the absence of a precise description of what constitutes a *spatially representable* equidimensional mereotopology—we use necessary conditions for spatial representability and show how they restrict the equidimensional mereotopologies, leading to a very small set of potential spatially representable equidimensional mereotopologies.

2.3.2 Expressivity of a theory with respect to the intended structures

As we already discussed, we can compare a theory with respect to its class of intended structures. If there is a sentence in the language of a given theory that discriminates any two non-elementarily equivalent structures in the class of intended structures, then the theory and its language is expressive enough to discriminate the two structures.

Frequently, we are more interested in showing that a certain relation Υ from the intended structures is not definable in a given theory T. We can apply Padoa's method of definability: It suffices to find two intended structures $\mathcal{I}$ and $\mathcal{I}'$ that differ in their extensions of Υ but define models $\varphi(\mathcal{I})$ and $\varphi(\mathcal{I}')$[5] of T such that $\mathbf{\Omega}_{\varphi(\mathcal{I})} = \mathbf{\Omega}_{\varphi(\mathcal{I}')}$ for all symbols $\Omega \in \Lambda(T)$. Then the relation Υ is not definable in T. In other words, no sentence $\sigma \in \mathcal{L}(T)$ can tell the two structures apart because $\varphi(\mathcal{I}_1) \models \sigma \iff \varphi(\mathcal{I}_2) \models \sigma$.

We repeatedly utilize this technique to argue that certain theories are not expressive enough to capture a distinction in the intended class of structures, in particular in Chapters 9 and 10. For example, at the end of Chapter 7 (Figure 7.9) and in our motivation for Chapter 9 (Figure 9.3 and 9.4) we will give intended structures that are not elementarily equivalent but that define equivalent models of $CODI_{\downarrow}$ to show the undefinability of the relation of boundary-containment in *CODI*. Thus we introduce *BCont* as primitive in *CODIB* to capture boundary-containment; therefore the language of *CODIB* is more expressive than that of *CODI*.

2.3.3 Classification of relations within theories

We heavily rely on classification of relations within theories, which essentially shows that in any model of a theory, the extension of some relation symbol Ω_1 is always a subset of the extension of another relation Ω. We then call Ω_1 a *subrelation* of Ω in the theory. More formally, we define a classification of a relation symbol as follows. A classification of a n-place relation symbol Ω in a theory T is a set of n-place relation symbols $\Omega_1, \ldots, \Omega_m$ such that for any model $\mathcal{M} \in \text{Mod}(T)$ with $d_1, \ldots, d_n \in \mathbf{M}$ we have

$$\langle \overrightarrow{d} \rangle \in \mathbf{\Omega_i}_{\mathcal{M}} \text{ for some } i \in [1, m] \Rightarrow \langle \overrightarrow{d} \rangle \in \mathbf{\Omega}_{\mathcal{M}}.$$

We are interested in two properties of sets of subrelations of a relation: *exhaustiveness* and *pairwise disjointness*. A set of n-place relation symbols $\Omega_1, \ldots, \Omega_m$ is an *exhaustive classification* of a relation

[5] Note that we assume a function φ from the class of intended structure to models of T, implying satisfiability of T with respect to the class of intended structures.

symbol Ω if and only if for any model $\mathcal{M} \in \text{Mod}(T)$ with $d_1, \ldots, d_n \in \mathbf{M}$ we have

$$\langle \overrightarrow{d} \rangle \in \mathbf{\Omega_i}_{\mathcal{M}} \text{ for some } i \in [1, m] \Leftrightarrow \langle \overrightarrow{d} \rangle \in \mathbf{\Omega}_{\mathcal{M}}.$$

We usually establish exhaustiveness in the theory by proving

$$T \models \Omega(\overrightarrow{v}) \leftrightarrow \Omega_1(\overrightarrow{v}) \vee \cdots \vee \Omega_m(\overrightarrow{v}).$$

A set of n-place relation symbols $\Omega_1, \ldots, \Omega_m$ are pairwise disjoint if and only if in any model $\mathcal{M} \in \text{Mod}(T)$ with $d_1, \ldots, d_n \in \mathbf{M}$ for every pair $i, j \in [1, m]$ we have

$$\langle \overrightarrow{d} \rangle \notin \mathbf{\Omega_i}_{\mathcal{M}} \text{ or } \langle \overrightarrow{d} \rangle \notin \mathbf{\Omega_j}_{\mathcal{M}}.$$

Again, this can be established proof-theoretically by proving for every pair $i, j \in [1, m]$ the entailment

$$T \models \neg\Omega_i(\overrightarrow{v}) \vee \neg\Omega_j(\overrightarrow{v}).$$

Establishing that a set of relations are a set of jointly exhaustive, pairwise disjoint (JEPD) subrelations of a given relation has several benefits. It defines a hierarchy of successively more fine-grained relations (illustrated in the Figures 6.7, 9.14, and 9.15), from which we can choose the appropriate level of detail for a particular application domain or a specific reasoning task. In other words, we can choose whether we want to distinguish the set of JEPD subrelations of a particular relation or simply lump them together. This way we can switch between different granularities or precision of spatial knowledge within the same theory.

Identifying sets of JEPD subrelations of relations within a theory also helps us to verify the theory by completely characterizing one relation in terms of others. It also restricts the models in ways that are easily verified in the intended structures if we have a fairly good understanding of how entities in the intended structures can be related to another. As an example, we know that any manifolds with boundary either intersects another manifold with boundary in its interior, in its boundary, or not all (i.e., in its exterior). Hence, subrelations of contact between two entities that are supposed to represent manifolds with boundaries must capture those three cases.

The hierarchy over relations defines a subsumption hierarchy (or subsumption lattice) over relations, which can be exploited for more efficient reasoning in at least three different ways. The first, straightforward way uses the lattice directly for subsumption reasoning [Coh+93], but this is restricted to queries about a coarser relation where knowledge about a more fine-grained relation is available.

The second option is to construct a *constraint calculus* following the approach outlined by [Esc01] to obtain the calculus, a kind of semi-automated randomized sampling [Coh+93; LL11] to obtain its composition table, and the methods from [Ren07] to identify tractable subsets for which efficient reasoning is then possible. The constructed spatial calculi can then be used for composition-based reasoning, in which a precomputed composition [BIC97b; RL04; RL05; RN07] allows much more efficient reasoning. Equally, a relation algebra can be defined [Dün05; DSW01; DWM99; DWM01] from a set of JEPD relations and be used with relation-algebraic reasoning method. Reasoning with sets of JEPD relations has been studied extensively in much work on qualitative spatial reasoning; we can readily build on those advances by providing sets of JEPD relations, even if only as refinements of some of the relations of our multidimensional mereotopology.

The third option is to translate the first-order theory with the help of the lattice of relations into a less expressive logical language, such as OWL, an ontology language popular in the Semantic Web Community, or other description logics. We can follow the ideas of [BD05] to translate the lattice of relations into a terminological set of *role inclusion axioms* (sometimes called a *role box*) in a description logic such as $\mathcal{RIQ}$ [HS04] (which belongs to the $\mathcal{SHIQ}$ family of description logics, see [BHS09]) or $\mathcal{ALCH}_{\mathcal{R}^+}$ [HLM99; HG97; Wes01]. This is effectively the reverse translation of the first-order semantic given to the description logic $\mathcal{ALCHI}$ in [Mot09]. If a relation has a JEPD set of subrelations, we know the resulting categories in the a description logic theory are also jointly exhaustive and pairwise disjoint, even though this cannot be expressed in standard description logics. However, neither of OWL, $\mathcal{RIQ}$, or $\mathcal{ALCH}_{\mathcal{R}^+}$ can express exhaustiveness of a set of relations or concepts.

In the thesis, we state classification theorems model-theoretically in terms of a model's extensions, for example, in Theorem 6.1 we say $\mathbf{P}_{\mathcal{M}}$ and $(<_{\mathbf{dim}})_{\mathcal{M}}$ form a partition of $\mathbf{Cont}_{\mathcal{M}}$ in an arbitrary model of *CODI*. The Theorems 6.1, 6.2, 9.1, 9.3, 9.4 are also classification theorems. Theorem 9.4 is the only one amongst them that establishes a jointly exhaustive but not a pairwise disjoint classification. Theorem 7.6 also contains a classification, though it is a more general characterization of the models of $CODI_{\updownarrow}$.

Chapter 11 classifies physical endurants in the DOLCE theory and in our extension by physical voids. Notable are the classification of physical endurants (PED-A1, PED-A2), of features (PED-A7, PED-A8), and the different classification of voids (V-D, V-A9, V-A10, V-A18, V-T2, V-T3, V-T4) along four criteria, resulting in the subcategories of voids depicted in Figure 11.5.

2.3.4 Implementation and semi-automated verification

All of the theories presented in this thesis are specified using Common Logic (ISO 24707) [Int07], a standardized language for the specification of expressive ontologies and for the exchange of knowledge in information systems. Common Logic is not a single logical language but a family of logical languages with a logical expressivity that is a superset of first-order logic. However, when specifying the first-order theories of this thesis in Common Logic, we only use the subset of Common Logic that has a standard first-order semantic. In particular, we do not quantify over relations or functions. Common Logic includes a set of so-called *dialects*, which are syntactic forms that all share the common semantic of Common Logic's *abstract syntax*, which is also specified in the standard. We use the CLIF dialect for the specification of all our theories.

All of our theories are provided in a repository available at `http://code.google.com/p/colore/source/browse/#svn%2Ftrunk%2Ftorsten-phd`. The repository is organized as follows. Each hierarchy, i.e., each set of theories that share a primitive language, is grouped into a folder. For example, the folder `cont` contains the *CO* theories with *Cont* and *ZEX* as only primitives, whereas the folder `codi` contains the theories from the *CODI* hierarchy, i.e., the theories that use *Cont*, *ZEX*, and $<_{\mathrm{dim}}$ as only primitives. Each such folder contains a number of `.clif` files, each specifying a set of axioms and possibly importing other files from the same or other hierarchies. Thereby, each file implicitly specifies a first-order theory as the closure under imported axioms[6], closed again under logical consequence. The list of theories used in the thesis and the name of their axiomatization in Common Logic can be found in Appendix C.

[6] We treat the import as a simple pasting of the axioms into the importing file. This is different from Common Logic's intended semantic of the `cl-imports` statement which specifies that a set of imported axioms are implicitly quantified only over the imported domain and not the entire domain. In such a setting, an imported theory captures submodels the entire theory, amendable by the importing axioms.

Explicit definitions in a hierarchy can be found in the subfolder defs. For example, the folder codi/defs contains definitions of the relations of parthood P (codi/defs/ep.clif) or of internal self-connectedness $ICon$ (codi/defs/icon.clif). The name of a definition file corresponds to the labels of the appropriate definitions, i.e., the definition EP-D of parthood P is defined in codi/defs/ep.clif.

We rely on the Common Logic specifications of our theories for assisting the verification of the theories. In particular, we use them for two types of verification tasks. Both tasks are supported by automated theorem provers (ATPs) and finite model generators (FMGs). Unfortunately, no reasoners are presently available that accept native input in Common Logic syntax. Therefore, we use other tools, in particular the ATPs Prover9 [McC10] and Vampire [RV02] and the FMGs Mace4 [McC10] and version 3.0 of Paradox [CS03]. To utilize them, we first have to translate Common Logic axiomatizations into one of their supported input languages, which is either the TPTP format [Sut09] for Paradox and Vampire or the LADR format native to Prover9 and Mace4. The script clif-to-prover9 from the toolkit cltools developed by Chris Mungall [Mun10] allows translating a single file from the Common Logic syntax to the LADR syntax. We embedded this in a toolkit of our own, called ColoreProver, to automate the translations of entire theories consisting of multiple, sometimes more than two dozen, axiom and definition files to the LADR and TPTP syntaxes. Our toolkit also provides basic functionality to quickly perform verification tasks by feeding the translations of entire theories as input to Prover9 and Mace4.

Let us first explain the two verification tasks in more detail, before we discuss how we utilize the ATPs and FMGs to assist us with those tasks. As already mentioned in Section 2.3.1, verification of a first-order ontology requires us to prove a representation theorem between the models of a specific theory and the class of intended structures. Representation theorems are normally proven in two parts—we first prove every intended structure is a model of the ontology and then prove that every model of the ontology is elementarily equivalent to some intended structure. The first part is rather easy, indeed we are able to provide such results, but the second part is much more difficult, in particular in the absence of a theory characterizing the class of intended models. Thus, we have to rely on other methods to verify a theory. We use three approaches: (1) interpretability in other theories, (2) consistency proofs, and (3) proving key properties of the intended structures directly from the theory. All of those techniques *partially verify* a theory: they help us gain confidence in the axiomatization and identify and fix problems with the axioms or definitions of the theories during its development [KG10]. While we have already elaborated on the first technique in Section 2.2.4, the other two techniques require further explanations.

Consistency is a property required from any theory of practical value. From an inconsistent theory we can prove arbitrary sentences in the theory's language, that is, the logical consequences of an inconsistent theory are vacuous. Consistency is usually proved by explicitly constructing a model of the theory. However, we go a step beyond that and want to prove that all primitive and defined relations in the theories signature are satisfiable, i.e., that their extensions can be nonempty. This is an important property: if a theory only admits models in which a particular relation is false for all tuples, this indicates a problem with the definition of the relation or the theory's axiomatization as whole. To prove this property, we try to prove what we call *nontrivial consistency* of each of our theories: that there exists a model of each theory such that every relation has a nonempty extension[7]. This condition may not be provable for all theories as it is much stronger than the previous condition requiring that the extension

[7] In practise, this condition is often weakened by considering only new relations when a theory extends a theory already known to be nontrivially consistent. Of course, this is only possible for defined relations or when we know that certain relations are unaffected by an extension. Due to high number of relations in some theories, we only explicitly force critical relations to have a nonempty extension, others usually follow automatically in the generated models.

of *each relation* must be nonempty in *some model*. However, if this stronger consistency property is provable, the weaker one follows automatically. We prove the strong property of nontrivial consistency for all our theories. The conditions for non-triviality are specified manually as a partial model, i.e., an extension of the theory by an existential sentence requiring the existence of domain entities for which each of the relations of interest are true. This extension is specified in an axiom file located in the subfolder /consistency of the folder that represents a hierarchy. Notice that we usually only prove nontrivial consistency for the most restricted theories in a hierarchy, if it holds for those, it follows automatically for all theories in the same hierarchy nonconservatively extended by the more restricted theory. For some of the smaller (in particular in the number of nonlogical symbols) and simpler theories, Mace4 is able to construct models. However, for theories with more expressive languages, Mace4 often times out before it ever finds nontrivial models. In such cases, Paradox often successfully constructed a model, its SAT-based approach seems much more scalable than the resolution approach taken by Mace4. However, Paradox requires the theory in the TPTP format as input. To translate our Common Logic theories to that format, we chain clif-to-prover9 with the tool ladr_to_tptp that comes as part of the tools provided by Prover9/Mace4. To conform to the TPTP syntax, nonlogical symbols denoted by non-letters, such as $<$ or $\leq$, that we use in the Common Logic specifications of our theories to denote, e.g., the relations $<_{\text{dim}}$ or $\leq_{\text{dim}}$, are replaced by symbols containing only letters, LESS and LEQ in the example. Each axiom file in the Common Logic notation, such as codi/consistency/codi_down_nontrivial.clif, is translated to an axiom file in the LADR notation, such as codi/consistency/p9/codi_down_nontrivial.p9. To construct a model of the theory $CODI_{\downarrow}$ axiomatized by the axiom file codi/consistency/codi_down_-nontrivial.clif, we also have to include the translations of all imported axiom files, e.g., the translation codi/p9/codi_down.p9 of the Common Logic file codi/codi_down.clif, etc. For the translation of an entire theory to TPTP syntax, the import closure of all its corresponding LADR files is assembled in a single file, which is in the example the LADR file codi/consistency/p9/codi_down_nontrivial.all.p9. This theory is translated to a single file in TPTP syntax, which are found in the subfolder consistency/tptp, in the example the file is called codi/consistency/tptp/codi_down_nontrivial.all.tptp. The models generated by Mace4 and Paradox are found in the consistency/output subfolder, in the example they are called codi/-consistency/output/codi_down_nontrivial.m4.out (which does not contain a model due to the timeout) and codi/consistency/output/codi_down_nontrivial.tptp.out.

We also use the ATPs Prover9 (as our default prover) and Vampire (occasionally as noted individually in Appendix D) to prove properties of specific theories, the sentences labeled as type 'theorem', i.e., with a label of the form XX-T#. Whether the ATP found a proof as well as a link to the theorem file in Common Logic are recorded in Appendix D. All theorems are specified in files located in the theorems subfolder of each hierarchy, one theorem file may contain a set of theorems. Moreover, to simplify proofs and increase our success rate in finding an automated proof, we often break a single theorem into a set of sentences that we fed to the ATPs. For example, we split biconditionals (containing the logical connective $\leftrightarrow$) into its two directions, or we use case-based reasoning by proving a property separately for an exhaustive set of cases. For example, we may prove a property separately for the precondition $ZEX(x)$ and the precondition $\neg ZEX(x)$. Often, we apply case-based reasoning by considering three cases of relative dimension separately: $x <_{\text{dim}} y$, $x =_{\text{dim}} y$, and $x >_{\text{dim}} y$. Equally, we use case-based reasoning by applying previous results about a JEPD set of subrelations (the simplest and most reused classification being the classification of contact into three subtypes of contact from Theorem 6.2). This demonstrates the utility of the classification results for the verification of our theories. A more detailed discussion of the potential role of theorem proving in the lifecycle of ontologies is offered in [KG10].

As an example, consider the verification of the properties CD-T1 – CD-T10 from Chapter 6. They are specified in CommonLogic in the file codi/theorems/codi_theorems.clif. Our toolkit splits such theorem files into individual theorems, resulting in codi/theorems/p9/codi_theorems_x.p9 with x ranging from 1 to 10 (sometimes there are more individual theorems than properties we want to prove due to the splitting of theorems). Each theorem is then proved by Prover9 from the imported set of axioms of $CODI$, resulting in the output codi/theorems/output/codi_theorems_x.p9.

The axiomatization and verification of the proposed theories is part of a larger effort in the context of the COLORE (Common Logic Ontology Repository) project, which aims to build an open repository of first-order ontologies that serves as a testbed for ontology evaluation and integration techniques and that can support the design, evaluation, and application of first-order ontologies [Grü+10; Grü+12]. It is important to remember that the theories we present here are a result of many iterations of ontology development. We used theorem proving and consistency checking throughout this process to improve the theories, to correct axioms, definitions, and theorems and to gain a better understanding of the intended structures as well as the models of the theories. Throughout this process it was often the case that particular properties were not provable, in fact, we could generate a counterexample to a particular property. This forced us to revisit and revise axioms that seemed to pertain to the property in question. Sometimes, we also had to revise a property that had been stated overly generic and could not be provable after closer inspection. For this development process, our toolkit automated many otherwise tedious steps. Automating simple tricks such as proving the consistency of a theory, as well as the consistency of the theory with the set of theorems we want to prove, before the actual proof attempt ensures that the theorems stand a chance to be provable and that a possible proof will less likely be meaningless due to an inconsistency in the theory. Moreover, the toolkit automatically proves several properties in a single run; for each property it concurrently tries to find a proof and a counterexample. This speeds up verification and is easily achievable on modern multicore processors. Once either a proof or a counterexample is found, the other process is aborted immediately. Our simple toolkit demonstrates that the arduous task of verifying a sizable first-order ontology can be made manageable without the need for completely new ATPs by simply automating tedious tasks and by implementing safeguards to prevent misleading results.

However, we were unable to prove some properties—in particular about functions (see the proofs about the difference and sum operations in Appendix D)—automatically, even after tuning the axiomatization, adding lemmas, removing definitions, or splitting theorems into subcases. This could be ascribed to a poor choice of using a resolution-based theorem prover for tasks where term-rewriting systems may have performed better, or it may indicate problems with the way functions are dealt with in Prover9. Alternatively, it could be caused by the way we axiomatized those operations: not as a single definition, but a set of axioms that implicitly define the operations. This is a challenge we generally have to deal with in first-order ontologies: there are many ways to axiomatize equivalent theories, but only some 'canonical' axiomatizations lend themselves to efficient reasoning. This problem could be addressed in the future by syntactic restrictions in the axiomatizations or by the explicit use of syntactic sugar for which standard, schematic translations are provided.

Chapter 3

Mereotopology: theoretical background and applications[1]

Mereotopological theories—which model only topological (of *connection*) and mereological (of *parthood*) aspects of space—are foundational within qualitative representations of space. In the last two decades many first-order theories of mereotopology have been proposed as qualitative representations of space, which has in turn led to fruitful systematic studies exploring their ontological assumptions, their different choices of primitive relations, and their entailed logical properties [CV99a; CV98; CV99b; CV03; Esc07; HG12; Var96]. In this chapter we give an overview of work in mereotopology that is related to this thesis and helps the reader to better understand our work in the context of the research area. The Section 3.1 to 3.4 discuss related theoretical work, whereas Section 3.5 discusses related applications which may benefit from the work in this thesis.

The term *mereotopology* encompasses different but related ideas. The original idea, which we call *equidimensional mereotopology* in the sequel, dates back to descriptions of phenomenological processes in nature in the work of Husserl, Whitehead, and de Laguna [Hus13; Lag22; Whi20; Whi29]. Most work on mereotopology falls into the equidimensional approach, which we will review in Section 3.1. Its main characteristic is that a single primitive binary relation C, called *connection* or *contact*, and a set of "regions", understood to be spatial regions of equal dimension, usually suffice to axiomatize classical mereotopology. Algebraic representations of equidimensional mereotopologies demonstrated that a single primitive binary relation of parthood P is often equally expressive [DW04; HWG09]. Our work in Chapter 4 studies necessary criteria for spatial representability of a very general and particularly interesting class of equidimensional mereotopologies.

Two main variations of equidimensional mereotopology approach are relevant here. The first variation of the equidimensional approach is what we call multidimensional mereotopology, a term coined by Galton [Gal04]. Multidimensional mereotopology differs from the equidimensional approach in that the basic set of regions can include regions of differing dimensions at the same time. A single primitive relation of contact is insufficient to adequately define such a theory. Multidimensional mereotopology loses some of the elegance and simplicity of equidimensional mereotopology. However, it can address

[1] An extended version of this chapter appeared as [HG12] in *Qualitative Spatio-Temporal Representation and Reasoning: Trends and Future Directions*, edited by Shyamanta Hazarika. Copyright 2012, IGI Global, www.igi-global.com. Included by permission of the publisher.

some of the shortcomings of equidimensional mereotopologies and can be considered as a generalization of classical geometries, which are also multidimensional in nature in that they talk about points, lines, and planes. We will review the existing work pertaining to multidimensional mereotopology in Section 3.2.

The second variation extends equidimensional mereotopologies by geometric or similar primitive relations to more expressive spatial theories. As the analysis by Borgo and Masolo [BM10] showed, most of those extensions are equally powerful in that they can define Euclidean geometry. For this reason we will refer to those theories as mereogeometries, the term used in [BM10]. We review geometric extensions to mereotopology in Section 3.3 and mereogeometries in particular in Section 3.3.1 to understand how our work differs from mereogeometries.

In the penultimate Section 3.4, we touch on the issue of boundaries, an issue not restricted to mereotopologies representations of space but closely linked to them. Finally, we give an—by no means exhaustive—overview of some application domains and applications of mereotopologies and qualitative representations of space in general in Section 3.5. Work that is related only to specific chapters or sections of this thesis is briefly discussed at the appropriate time.

3.1 Equidimensional mereotopologies

In this section we will look at two families of equidimensional mereotopology: the Whiteheadean approach and a boundary-based approach. Both families have been studied quite exhaustively [CV99a; CV98; CV99b; CV03; Esc07; HG12; Var96]. Though the focus has been on continuous equidimensional mereotopologies, most of the results readily extend to discrete theories as well [compare HG12].

What we call equidimensional mereotopology comprises pure mereotopological theories restricted to regular equidimensional regions. Notice that there is no restriction to any particular dimension, but instead each model is restricted to regions of equal dimensions. For instance, if a model contains three-dimensional entities like spatial regions, it cannot contain entities of any other dimension, e.g., two-dimensional surfaces, one-dimensional lines, or zero-dimensional points. All regions must be regular in their topological interpretation, sometimes the domain of discourse is further restricted to only regular closed or only regular open regions—the latter approach only taken in Roeper's topological account of mereotopology [Roe97], which we will not discuss here in detail due to the lack of a logical axiomatization. Regularity is a notion rooted in topology.

We assume basic familiarity with topological spaces as covered in standard textbooks such as [Eng77; Mun00]. A topological space $\langle \mathbf{X}, \tau \rangle$ is defined by its universe $\mathbf{X}$ and its topology τ, the set of all open subsets of $\mathbf{X}$. The interior, closure, and complement (with respect to $\mathbf{X}$) of a point set $\mathbf{A}$ are denoted by $\mathrm{int}(\mathbf{A})$, $\mathrm{cl}(\mathbf{A})$, and $\mathbf{X} \setminus \mathbf{A}$. Set intersection, union, and inclusion are denoted by $\cap$, $\cup$, and $\subseteq$. In a topological space (X, τ), a subset $A \subseteq X$ is called *regular* if and only if $\mathrm{cl}(A) = \mathrm{cl}(\mathrm{int}(A))$ and $\mathrm{int}(A) = \mathrm{int}(\mathrm{cl}(A))$. A set is called *regular closed* if $A = \mathrm{cl}(A) = \mathrm{cl}(\mathrm{int}(A))$ and *regular open* if $A = \mathrm{int}(A) = \mathrm{int}(\mathrm{cl}(A))$. Intuitively, regular regions are of uniform dimension, that is, they consist of one or several disconnected point sets of equal dimension. Examples of nonregular regions are given in Figure 3.1.

All equidimensional mereotopologies consists of a single parthood[2] and a single contact relation that satisfy the axioms (P.1) – (P.3) and (C.1) – (C.3) [Var98]. Such theories are commonly referred to as *ground mereotopologies* (**MT**) [CV99a]. If either of C and P or both of them are not explicitly present

[2] In the multidimensional case parthood may be replaced by a suitable multidimensional predicate such as containment.

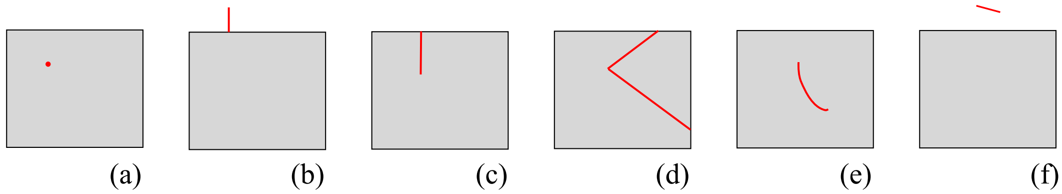


Figure 3.1: Examples of nonregular regions. Each example depicts a single point-set that is nonregular. Regular regions cannot have any kinds of lower-dimensional artefacts, such as missing points as in (a) or internal or boundary cracks as in (c)–(e). Equally, regular regions cannot consist of pieces of different dimensions, whether connected as in (b) or disconnected as in (f).

or are not primitive relations, they still suffice as primitives of a logically equivalent mereotopology. Throughout the chapter we assume that any two regions with identical extensions of parthood *and* contact are identical.

(P.1) $P(x, x)$ (P reflexive)

(P.2) $P(x, y) \wedge P(y, x) \rightarrow x = y$ (P antisymmetric)

(P.3) $P(x, y) \wedge P(y, z) \rightarrow P(x, z)$ (P transitive)

(C.1) $C(x, x)$ (C reflexive)

(C.2) $C(x, y) \rightarrow C(y, x)$ (C symmetric)

(C.3) $C(z, x) \wedge P(x, y) \rightarrow C(y, z)$ (C monotone with respect to P)

Equivalent to (C.3) is the following axiom (C.3′).

(C.3′) $P(x, y) \rightarrow \forall z(C(z, x) \rightarrow C(z, y))$

Any such ground mereotopology allows defining the concepts of *overlap* O, *underlap* U, and *proper part* PP in the following natural way.

(O.D) $O(x, y) \leftrightarrow \exists z[P(z, x) \wedge P(z, y)]$ (overlap)

(U.D) $U(x, y) \leftrightarrow \exists z[P(x, z) \wedge P(y, z)]$ (underlap)

(PP.D) $PP(x, y) \leftrightarrow P(x, y) \wedge \neg P(y, x)$ (proper parthood)

In the sequel, we take these definitions for granted in any mereotopological theory.

3.1.1 Whiteheadean mereotopologies

Whitehead [Whi20; Whi29] pioneered mereotopology in the 1920s by proposing a relation called *extensive connection*, what we now call connection or contact, to qualitatively describe the topological relations between regions of space. Such an economical framework built around a single topological primitive distinguishes his work from the mereological approach of his contemporaries Husserl, Leśniewski, and Leonard and Goodman [Hus13; LG40; Leś27; Lus62]. Wider interest in Whiteheadean space was sparked by Clarke's extensive axiomatic treatment [Cla81; Cla85]. The most prominent mereotopologies are Whiteheadean, among them the Region Connection Calculus (RCC) [Coh+97a; Coh+97b; Got94; GGC96; RCC92] and Asher and Vieu's and Roeper's mereotopologies [AV95; Roe97]. Besides taking regions instead of points as the primitive objects, Whiteheadean theories normally make the following four assumptions [Mor98]. The first two apply to all equidimensional mereotopologies, not just to the Whiteheadean ones.

1. The dimension of all regions coincides with the dimension of space.
2. Regions can be only part of regions and regions have only regions as parts.
3. Regions can be interpreted as point sets (topological representability).
4. The theory is based on a single primitive relation of connection, which is extensional.

Assumptions (a) and (b) are what makes any Whiteheadean theory of space an equidimensional mereotopology. We will not discuss the interpretability of regions as point sets, assumption (c) further here and instead refer to [HG12], it will also be discussed in more detail in Chapter 4. Notice that often another assumption, a requirement for representability by regular regions, is mentioned, but it implicitly follows from (a) and (c).

Assumption (d) captures Whitehead's key motivation to build a theory from contact alone, rendering all Whiteheadean theories extensional with respect to the contact relation C (C.Ext), we say they are C-extensional. For that reason, all Whiteheadean theories are extensions of what Casati and Varzi call a *Strong Mereotopology* (**SMT**) [CV99a].

(C.Ext) $\forall z[C(x,z) \leftrightarrow C(y,z)] \rightarrow x = y$ (C-extensionality)

In the RCC[3], the contact relation induces the parthood relation (or vice versa) through the mapping $\neg C(x,-y) \leftrightarrow PP(x,y)$ where $-$ is the operation of complementation (similar but not equivalent to complements in a topological space). The relationship between contact and parthood in Asher and Vieu's theory [AV95] and Clarke's theory [Cla81] is slightly different, it satisfies $\neg C(x,-y) \leftrightarrow P(x,y)$ [BG91; HWG09] (where $-$ denotes the complement) because in those theories the defined complementation operation is interpretable as true point-set complementation. The choice of a definition for the complementation operation is actually a key distinction between different Whiteheadean mereotopologies. Complementation operations are introduced to ensure that any entity y that is a proper part of x has a supplementing part z of x such that y and z together form the "whole" x. In Chapter 4, we will study ways to define the complementation operations (which automatically force some kind of supplementation) in equidimensional mereotopologies. The different notions of supplementation, in particular the difference between weak and strong supplementation, are discussed in-depth in the multidimensional context in Sections 6.3 and 7.2.1.

A common feature of Whiteheadean mereotopologies is the definition of a relation of external contact (EC), which basically arises from contact without overlap. In addition, Whiteheadean theories define binary topological sums and intersection of regions, as well as complements. We will define those for equidimensional mereotopologies in Section 4.1.1 where we actually need them. Additionally, concepts such as nontangential (proper) part (NTP.D, NTTP.D), and tangential (proper) part (TP.D, TPP.D) can be defined.

(EC.D) $EC(x,y) \leftrightarrow C(x,y) \wedge \neg O(x,y)$ (external contact)

[3] An axiomatization of the RCC is included on page 164 in Chapter 8, at which point we semantically integrate the RCC into our hierarchies of spatial ontologies.

(NTP.D) $NTPP(x, y) \leftrightarrow P(x, y) \wedge \forall z[EC(x, z) \rightarrow O(y, z)]$ (nontangential part)

(TP.D) $TP(x, y) \leftrightarrow P(x, y) \wedge \neg NTPP(x, y)$ (tangential part)

(NTPP.D) $NTPP(x, y) \leftrightarrow PP(x, y) \wedge NTPP(x, y)$ (nontangential proper part)

(TPP.D) $TPP(x, y) \leftrightarrow PP(x, y) \wedge TP(x, y)$ (tangential proper part)

Most importantly, the notion of self-connectedness is definable through axiom Con.S or axiom Con.W, depending on whether the regions in the intended models are all regular closed or simply regular. Intuitively, a region is self-connected if and only if it does not consist of several disconnected, i.e., scattered parts. An even stronger version of self-connectedness is that of s-connection [BGM96], also called internally self-connectedness *ICon* [CV03], meaning the interior of x is a single piece (ICon.D). Notice that the definitions Con.W and ICon may not be definable in a particular axiomatization due to the lack of the operations cl or int; in that case, those definitions should be considered as expressing the intended topological meaning.

(Con.S) $Con(x) \leftrightarrow \forall y, z[y + z = x \rightarrow C(y, z)]$ (strong definition of self-connectedness)

(Con.W) $Con(x) \leftrightarrow \forall y, z[y + z = x \rightarrow C(\mathrm{cl}(y), \mathrm{cl}(z))]$ (weak definition of self-connectedness)

(ICon.D) $ICon(x) \leftrightarrow \forall y, z[y + z = x \rightarrow C(\mathrm{int}(y), \mathrm{int}(z))]$ (strong self-connectedness)

The theories of Whiteheadean mereotopology differ in their axiomatization. Clarke utilizes second-order logical notions (set theory or definite descriptions) to describe infinitary fusions, while Asher and Vieu's theory and the RCC are first-order theories limited to finite sums. Both the RCC and Clarke's theory are atomless, i.e., they include axiom AL forcing every entity to contain a nontangential proper part, while the account of Asher and Vieu allows atoms to exist and thereby admits discrete and finite models. Analyses of the nature of discrete mereotopology can be found, among other works, in [Gal99; MV99; RS02].

(AL) $\forall x \exists y[NTPP(y, x)]$ (atomless)

Changes to the RCC axioms to allow models with atoms have been discussed in [Don08; RS02]. Li and Ying [LY04] suggested a generalization of the RCC to the *Generalized Region Connection Calculus* (GRCC) that allows both discrete and continuous models. The RCC has been further generalized in various ways, particularly for the study of its algebraic counterparts, the contact algebras. Some of the resulting theories are discussed in their algebraic form in Chapter 4. The main difference between these theories lies in their domain of discourse. Clarke and Asher & Vieu allow any kind of regular regions, while the RCC, GRCC, and Roeper's theory only deal with regular closed regions.

3.1.2 Boundary-tolerant equidimensional mereotopologies

Within Whiteheadean and any other classical, i.e., equidimensional, axiomatizations of region-based space, regions are the only entities considered in the domain of interest. Moreover, all regions are of the same dimension, hence boundary elements cannot be in the domain of discourse. For instance, Clarke [Cla81] and Asher & Vieu [AV95] require that all regions have nonempty interiors which must be regions themselves. Hence, boundaries are excluded. In theories restricted to regular closed regions, there is no difference between a regions' interior and closure, i.e., boundary elements cannot be modelled either.

Since boundaries often play an important role, other authors [CV99a; Gal96; Gal04; Got96; Smi96; SV97] have incorporated them into their theories. Two different approaches have been pursued. Most commonly, boundaries are treated as entities of a lower dimension [Gal96; Gal04; Got96], hence dismissing the first Whiteheadean assumption altogether. The resulting theories are what we call multi-dimensional mereotopologies, which we survey in more detail in Section 3.2. A less common approach, taken in [Smi96] does not distinguish boundaries explicitly as lower-dimensional entities, but considers them as special kinds of regions. In Smith's theory [Smi96] every boundary region is part of the region they bound but has an empty interior. Moreover, boundaries are self-bounding. Using an infinitary fusion operation, a maximal boundary $\mathrm{bdy}(x)$ can be defined. Unrestricted fusions and the topological operations sum, intersection, complement, and difference are defined using definite descriptions; thereby avoiding second-order notions but not really giving a first-order axiomatization either. Although Smith does not distinguish boundaries from other regions in the domain, it is clear that boundaries play a special role. It is not clear, however, whether boundaries are understood to be of the same dimension as non-boundaries or whether they are considered of having a lower dimension. On the one hand, every boundary is a part of the region it bounds, but on the other hand, a boundary has an empty interior.

In Section 3.4, we will survey different conceptions of boundaries from a more general perspective as they relate to our work, in particular to Chapter 9 and Chapter 11. Before that, we review multidimensional mereotopologies and geometric extensions to mereotopology.

3.2 Multidimensional mereotopologies

In all equidimensional mereotopologies, lower-dimensional entities can only be defined using higher-order logical constructs. For example, in a three-dimensional spatial configuration, lower-dimensional entities such as points, lines, and surfaces or areas can be reconstructed through the method of extensive abstraction that has already been discussed by Whitehead and de Laguna [Lag22; Whi20; Whi29]. The idea dates back to Lobačevskij's work [Lob34] from 1834. Indeed, most proposals of equidimensional mereotopologies include a definition of so-called abstract points as limits of infinitely many nested regions or sets of regions, compare for example [Cla85; Esc94; Men40; Tar56a].

Galton [Gal96] argues that we should neither assume regions nor points, nor any other kind of spatial entities as more fundamental than the other. In this spirit and irrespective of the philosophical or cognitive adequacy of regions or points, Galton [Gal96] and Gotts [Got96] have proposed frameworks that accommodate entities of any kind of dimension (in particular points, lines, surfaces) by using a binary predicate of equidimensionality and two separate parthood relations, one between equidimensional entities and another one between entities of different dimensions. This comes close to what Hayes [Hay85] envisioned for a commonsense representation of physics. Points and indivisible atomic regions can then theoretically co-exist [Gal96]. Furthermore, it lends itself to an elegant definition of boundaries: boundaries are defined in [Gal96; Got96] as entities of a dimension one lower than the entities they bound. Key to the axiomatization in [Gal96] is the distinction between parthood P as relation that exclusively applies to entities of equal dimension, and the relation $IN(x, y)$ that relates a lower-dimensional entity x to a higher-dimensional y. We would say "a point lies in a region" instead of "a point is part of a region". In a similar framework, The primitive nonsymmetric relation $INCH(x, y)$ from [Got96], meaning that 'x includes a chunk of y', effectively combines both: y can be of a lower dimension than x, in which case $\neg INCH(y, x)$, whereas $INCH$ is symmetric for equidimensional entities x and y. The use of cell complexes, that is, collections of discrete objects of different dimensions, is another solution which can accommodate objects of different dimensions [RS02; WF00], cell complexes are frequently used in geographic information systems [BF95; Fra05].

In strictly topological theories that define mereotopological relations, such as the work by Egenhofer and associates [Ege89; Ege91; EF91; EH91; ES93], the relations between one-, two-, and three-dimensional entities (points, curves and curve segments and hence also boundaries, and two-dimensional areas) have been investigated. As result, classifications of the mereotopological relations between pairs restricted to specific dimensions, such as between curves and areas, or between points and curves, have been proposed [CDFO93; Ege91; EH91; McK+05]. However, these frameworks employ full point-set topology avoided by the previously mentioned logical theories. In our work, we therefore attempt to construct a logical theory in which entities of different dimensions can co-exist within a single model without resorting to the use of full topology. Moreover, we want to find general relations between entities of any finite dimension, without being restricted to two- or three-dimensional space only. We will discuss the relationship to the topological approaches in more detail in Chapter 9.

3.3 Geometric extensions to mereotopology

Mereotopology, whether equidimensional or multidimensional, does not deal with any geometric notions at all. Geometries, on the other sides, inherently deal with geometric aspects, that is, metrics are an essential part of geometries. Usually, those metrics involve distances between points or angles between intersecting lines. Classical geometries such as Hilbert's axiomatization of Euclidean geometry [Hil71] are based only on primitive spatial entities that have no curvature, such as straight lines, planar planes, etc. In this restriction, a metric is inherent: that of the shortest distance between two points, which happens to be the length of the line segment formed by the two points. This metric is usually captured by a *congruence* relation: two line segments are congruent if and only if they have the same length. More generally, two figures $\mathbf{A}$ and $\mathbf{B}$ (treated as point sets) are congruent if there is an isometry between them, that is, there is an injective mapping $\varphi : \mathbf{A} \rightarrow \mathbf{B}$ such that the straight-line distance between $x, y \in \mathbf{A}$ is equivalent to the straight-line distance between $\varphi(x)$ and $\varphi(y)$ in $\mathbf{B}$. Congruence is more commonly said to preserve shape (morphology) and size; two figures are congruent if one can be obtained from the other by a series of translations, rotations, and reflections.

3.3.1 Mereogeometries

Clearly, congruence is a very powerful concept. There have also been attempts at extending mereotopologies, in particular equidimensional ones, with geometric notions while still only dealing with regions. Those are called mereogeometries, a term coined for them by Borgo and Masolo [BM10]. Tarski's categorical *geometry of solids* [Tar56a] is probably the best known mereogeometry, later incorporated into Bennett's *region-based geometry* (RBG) [Ben01; Ben+00], a categorical first-order theory. Borgo, Guarino, and Masolo [BGM96] have proposed an alternative first-order theory (in the sequel we refer to it as BGM) using three primitives: binary parthood $P(x, y)$ for the mereological part, the unary, quasi-topological predicate *simple region* $SR(x)$ with an intended meaning of 'x is an s-connected region' (compare the earlier definition of $ICon$), and the morphological binary primitive of congruence $CG(x, y)$. In style, the axiomatization of BGM is closest to the axiomatizations of Whiteheadean mereotopology we saw before, its exact relationship to other Whiteheadean mereotopologies has been studied by Eschenbach [Esc07]. On the geometrical (or morphological) part, BGM uses the primitive relation of congruence, CG, to define spheres as special kinds of simple regions (SPH.D).

This enables the theory to reuse Tarski's [Tar56a] defined relations among spheres, such as *externally tangent*, *internally tangent*, *externally diametrical*, and *internally diametrical*. Most importantly, it allows defining when two spheres are *concentric*, which in turn allows defining the ternary betweenness relation $Btw(x, y, z)$ for spheres meaning 'sphere x is in between the spheres y and z'. The core notion of two points being *equidistant* to a third point can then be defined by two pair of congruent spheres having equidistance centers [Tar56a]. The first-order theory BGM does not reconstruct points, but Tarski [Tar56a] does using a limit construction, allowing him to define equidistance of two points from a third. Nevertheless, the theory BGM [BM10] can be used to define equidistance of two congruent spheres from a third sphere, which implies that the center of the two spheres are equidistance from the center of the third sphere, effectively defining equidistance of points without mentioning points (points are second-order construct: ultrafilters of nested spheres). The standard betweenness relation (which includes linear alignment) or the equidistance relation can subsequently be used to define a metric system and to reconstruct elementary geometry [Tar59].

(SPH.D) $SPH(x) \leftrightarrow SR(x) \wedge \forall y[CG(x, y) \wedge PO(x, y) \rightarrow SR(x - y))]$

(a sphere is a simple region that cannot be disconnected by congruent simple regions)

Due to the results of [BM10] other mereogeometries can be reformulated using the primitive relations and axioms from [BM10] and [Tar56a]. For example, the primitive relation $CCon(x, y, z)$ meaning 'x can connect y and z' from [Don01; Lag22] can be defined in terms of CG and P [BM10]. For more detailed discussions of the full mereogeometries we refer to [Ben01; Ben+00; BGM96; GP08; Nic24; Tar56a].

Slightly different, but related work includes so-called *pointless* or *point-free geometries* [Ger95] as attempts to axiomatize Euclidean geometry from regions or solids without a primitive or defined notion of a point. Earlier work in this direction includes [Hun13; Lob34].

3.3.2 Convexity

Convexity is usually a morphological concept that is weaker than congruence, though it may still be sufficient to define a full mereogeometry [BM10]. The only theory including a notion of convexity but not constructing a full mereogeometry that we know of is the RCC extended by a convex hull primitive [Coh95; CRC95; Coh+97b; RCC92]. The resulting theory is strictly weaker than full mereogeometry [CR08]. It has been conjectured that the RCC together with a convex hull (or convexity) primitive is a point-free equivalent of affine ordered geometry. We prove an analogue relationship for our multidimensional mereotopology in Chapter 10, namely that the language of *CODI* is expressive enough to define affine spaces (see Theorems 10.5 and 10.6). Together with a primitive relation of multidimensional betweenness, we can also reconstruct affine ordered geometry as a consequence of the results in Section 10.3.3. Betweenness and convexity are closely related concepts, with convexity being definable using betweenness in a sufficiently restricted spatial theory as we briefly discuss in Section 10.3.5.

3.3.3 Other extensions

Recently, combinations of qualitative properties have received increased attention. Among them convexity and relative size (or distances) [Bit09; BD07b; Gah95] are some attempts to supplement mereotopological relations, but all result in full mereogeometries: region-based theories of space with essentially the same expressive power as Euclidean geometry. Qualitative theories about relative positions, directions and orientation have also been combined with the RCC-5 [Che+07b] and a rectangular cardinal direction calculus [NS06; SK04; SK05]. The proposed theories are problematic since relative positions of extended objects either rely on some center of each region (centre of mass, geometrical centre, or similar) or are expressed in terms of minimal bounding objects, such as rectangles, blocks, cubes [BCC98; Che+07a], or spheres. Neither of them are dimension-independent.

We investigate a relation of betweenness as a more adequate qualitative relation of relative position. It applies to entities of different dimensions, but can be weakly axiomatized so that congruence or relative size are still indefinable—thereby avoiding a reconstruction of full mereogeometry. Not surprisingly, integrating region-based theories of space with other qualitative properties is a challenge closely linked to the exploration of qualitative spatial theories with an expressivity between mereotopology and mereogeometry leading to a hierarchy that resembles the relationships between the different strengths of geometries. Our extension of multidimensional mereotopology by betweenness is related closest to the ordering of points on oriented lines [KE99], which uses the line's orientation to get by with a binary ordering relation instead of a ternary or quaternary relation.

3.4 Boundaries in mereotopology

Boundaries are a key concept in topology and mereotopology. Various notions of precise[4] boundaries have been discussed in the literature that are relevant to our work [CV99a; Chi83; Kac09; SV97; Str88; Var08]. All acknowledge the difficulty of capturing boundaries in a theory of space, proposing different classification schemes for boundaries. A common distinction is between boundaries as being dependent on a single object [Chi83] and boundaries as interfaces between two objects [Str88]. In [Var08], the former are called *owned* and the latter *non-owned* boundaries. The view of boundaries as dependent on a single object, its host, considers the boundary as an integral feature of the bounded object. As a consequence, two objects that touch each other may have boundaries that coincide as discussed in [BH11; Chi96]. In the view of boundaries as interfaces the boundary is not dependent on any particular object, but is characterized by how it delineates two objects [Var08]. Then, a single object has no such thing as a boundary unless it touches other objects. A slightly different perspective is offered by Kachi [Kac09] distinguishing symmetrical ("Brentanian") from asymmetrical ("Bolzanian") boundaries. Both are dependent particulars, the difference lies in that in the Brentanian view boundaries of two adjacent objects can coincide, whereas in the Bolzian view we have to choose to which of two bounding objects the boundary belongs.

Another distinction is between *bodiless* and *bulky* boundaries [Var08]. Bodiless boundaries are said to occupy no space and are in that sense abstract *A-surfaces* [Str88] (not to be confused with boundaries in abstract vs. physical space). More precisely, they are of a lower dimension than the object they bound and thereby do not occupy a region of space of the dimension of the bounded object as opposed to bulky boundaries that occupy space of the same dimension as the bounded object. The notion of a bodiless spatial boundary manifests itself in the continuous model of space commonly used for topology and geometry, such as Euclidean geometry, wherein lower-dimensional boundaries are defined as limits. The notion of a bulky boundary is based on a discrete model of space, which seems particularly appropriate

[4] We do not deal with vague boundaries here at all. See [BF96; CG96; Sch+08] for different treatises of vague boundaries.

for capturing material objects. Bulky surfaces are referred to by Stroll as physical surfaces (*P-surfaces*) for that reason [Str88]. Contrary to the limit construction possible when defining bodiless boundaries, material objects are usually divisible only to a certain extent based on the granularity of interest (such as its matter, its atoms, or its elementary particles such as quarks). Such material objects have bulky, material surfaces—the outermost thin layer of its material—which rely on a discrete representation of space, wherein we stop dividing objects or spatial regions into smaller ones at some level of granularity. There, limits in the traditional sense do not apply. Because this understanding of space is usually tied to a physical reality, it is often used to model physical space as compared to the continuous conceptions of space that are primarily used to model abstract space.

Yet another distinction is between *fiat* and *bona-fide* boundaries [CV99a; SV97; Var08]. Loosely speaking, bona-fide boundaries require a physical discontinuity, such as a change in material (e.g., the water surface) or a physical disconnection which may or may not be a change in material at the same time (e.g., a book laying on a table or a stack of wooden shelves before assembling a book shelf). A fiat boundary is one that requires an underlying physical discontinuity, though it is left open whether it may have such a discontinuity. Fiat boundaries are usually treated as bodiless boundaries. For example we divide the space of the world into regions that we call countries: how we draw the boundaries between countries is often arbitrary—especially boundaries that have been established during colonial times—from a purely spatial point of view (with the exception of, e.g., countries that occupy an entire island, which have a physically meaningful boundary, namely the bona-fide boundary of the island) and may not be based on physical boundaries. On the other hand, so-called bona fide boundaries of physical objects can be treated either as bodiless interfaces between objects or as "thin" layers of material at the surface of an object. Bona-fide boundaries include all perceived surfaces such as a table top or the walls of a room, but may also include other proper objects that are perceived as boundaries, such as a river separating two pieces of land or two countries.

For more thorough discussions of the various conceptions of boundaries and the intricate philosophical issues involving boundaries we invite the reader to consult, e.g., [Chi83; SV97; Str88; Var08].

3.5 Applications of mereotopology

Though theoretical work on mereotopology is often motivated by practical applications, these applications remain sparse. Only recently specific applications of mereotopology and mereogeometry have emerged and been used to test the viability of mereotopology in practise. Most of the known work customizes mereotopology to fit the application domain, but it has turned out that mereotopology by itself is rather limited in its usefulness. Instead, it usually must be integrated into more expressive ontologies or reasoning frameworks.

Among the main areas applying mereotopologies in one way or another are geographic information systems (GIS), computer-aided design and manufacturing (CAD and CAM), navigation, computer vision applications, biological and medical ontologies, product and assembly modelling and engineering, and applications in (computational) linguistics, e.g., for language understanding.

Apart from these specific areas of applications, all upper ontologies need to incorporate spatial and spatio-temporal concepts to be of use for representing physical reality. For that reason, for example, the upper ontologies BFO [Gre03; Smi+12], DOLCE [Mas+03], SUMO [NP01], and the upper ontology of openCyc [Cyc12] all include some mereotopological component. See [BF05] for an overview of the categories and relations pertaining to space that are present in the various upper ontologies. However, the relations in upper ontologies are only sparsely axiomatized, leaving the interpretation wide open. We show in Chapter 11 how our work in this thesis can be used to restrict the interpretations of certain space-related categories in DOLCE and how we can obtain more fine-grained categories.

Geographic information systems Traditionally, the GIS community has been a driving force in the advancement of qualitative theories of space with the objectives of formalizing spatial relations used by humans and of applying high-level reasoning to those spatial relations. This is demonstrated by the large body of work on qualitative relations concerning geographic space and geographic information systems [CDF97; CDF98; CDFO93; Ege91; Ege94; EH91; EM95; Fra96; HCDF95; ME94]. The role of ontologies, including of spatial ontologies, for semantic integration of spatial information in GIS has been discussed extensively in [Fon+00; Fon+02]. In the context of built environments, Bittner [Bit00] demonstrated how a mereotopological theory with rough location relations can be used to model a parking lot. He explored the necessity of boundaries in general, but also the necessary distinction of different kinds of boundaries, in particular bona-fide and fiat boundaries, to naturally capture the built space.

Recently, *GeoSPARQL* [Ope12] has been proposed by the Open Geospatial Consortium as a standard for geospatial data in the syntax of the *Resource Description Framework* (RDF) [RDF04]. It includes some qualitative spatial relations, in particular mereotopological relations, such as the RCC relations, the 9-intersection relations, and relations capturing the dimensionality and the boundary of spatial entities. Moreover, GeoSPARQL distinguishes simple (not self-intersecting) from complex entities (similar to how we define atomic and composite manifolds in Chapter 5). It has the main disadvantage that it provides not much more than a vocabulary: the semantic is only verbally explained, thereby limiting its use for semantic integration. Our work here could be used to formalize the meaning of those mereotopological relations in first-order logic, thereby basing the standard on a rigorous axiomatic foundation, and eventually allowing the standard to be used for exchanging spatial knowledge across spatial information systems.

Computer-aided design and manufacturing One particularly promising field for applications of mereotopologies are ontologies for CAD and CAM software, which allow the exchange of architectural or manufacturing blueprints without the loss of critical semantics. For example, a “hole” in a product part or assembly has in some CAM systems a much more specialized meaning than generally in space: it must be something that is obtained by the process of drilling (and thus is of round or oval shape). A similar system that models the layout of an operation exploiting natural resources will have a very different notion of a “hole”. There are also much more subtle differences. E.g. one software system treating a closed set of lines or curves as a region, i.e., the boundary defining a region, whereas another system treats them as a linear feature that just happens to enclose a region. On a higher level, we want to know under what circumstances a translation between the spatial representations of two different theories is possible: What specific knowledge will be lost and what will be preserved?

Mereotopological and mereogeometrical relations have been used for representing assemblies of parts by Kim et al. [KMY06; KYK08; Kim+09], though only in a logical language with very limited expressiveness, namely the Semantic Web Rule Language (SWRL). For example, they distinguish different kinds of assembly joints obtained by welding, gluing, brazing, fastening, soldering, stitching, stapling, etc. based on Smith’s boundary-tolerant mereotopology [Smi96]. However, they introduce additional geometrical predicates such as angles and offsets of the joined objects. The basic mereotopological definitions are translated into SWRL rules. The contact is further refined to distinguish the morphology of the contact. This documents that for practical applications, mereotopology is usually only a basis and needs to be extended by domain-specific terminology. The different kinds of joints can be more precisely captured in a first-order ontology. Moreover, many distinctions can be captured more easily in a truly multidimensional mereotopology.

Bio-ontologies Biological, biomedical, and medical research has shown considerable interest in ontologies to represent various relations, e.g., anatomical, genetical, or simple spatial and spatio-temporal relations for describing medical images (X-rays, tomographic images, etc.). Many relations occurring in these fields are of mereological and mereotopological nature. The ontologies in the Open Biomedical Repository (OBO) use basic spatial and spatio-temporal relationships defined in the BFO and the OBO relation ontology. The mereotopological and mereogeometrical concepts of the OBO Relation Ontology (RO) have been explored in [Bit09; Smi+05]. The RO also contains location relations, while an explicit distinction between contact and adjacency (external connection) is made. All mereotopological and mereogeometrical relations in RO are temporal, thus allowing for change over time. This is an aspect we will not model in our work, we strictly work with a static view of space. One focus in bio-ontologies has been on capturing the spatial structure of anatomy, see e.g., [Bit09; Don04; Don05; RMJ03]. However, most of the bio-ontology community has focused on ontologies formalized in description logics instead of using the more expressive first-order logic, which could capture the semantics of the various relations much more precisely.

Robot navigation Robot navigation through unknown or partially unknown territory can benefit from qualitative, in particular from mereotopological, representations of space. Examples include exploring the connectivity of rooms in an unknown building to learn which rooms, hallways, and staircases are connected [KB91; KL88; LL90; RK04] or which rooms belong to certain floors. This provides a high-level spatial model for a robot to search for things in a building (e.g., search-and-rescue robots), find their way out again, or backtrack once trapped in a dead end. Learning topological maps directly from the environment can be achieved used mereotopological representations where the maps usually consist of entities of multiple dimensions including regions, lines, and points, supplemented by orientation information about the robot. Learnt topological maps can subsequently be refined by geometrical information. However, the topological information is not directly used for qualitative reasoning. Instead, graph-based approaches such as Voronoi diagrams or connectivity graphs, sometimes in connection with region partitioning, e.g., in [Thr98], are dominant. The use of mereotopologies or mereogeometries is much less prevalent in practical navigation applications. Nevertheless, the multidimensional theories we develop in this thesis may be a suitable representation for such tasks, helping to better separate the layout of a single floor from the layout of the building.

Interesting problems in a similar direction include qualitative route finding where traditional graph-based route finding is combined with region properties. For example, instead of finding the shortest or fastest route between some points, we might be interested in the most scenic route (going through forests, along a lake, outside a city) where the different properties are represented as regions (from geographic maps) instead of assigning each link in the network an individual value for such properties.

Human navigation can also be supported by qualitative representations of space, in particular in built environments and cities where distances are usually of a lesser concern. In particular within three-dimensional environments, simple descriptions of routes are crucial, but differ from traditional two-dimensional navigation [TDZ11]. In such settings, our multidimensional theory may be a good foundation for a representation of space that is equally usable for humans and information systems.

Natural language processing Many navigation problems such as translating a route description into a map are directly linked to natural language processing of spatial relations. Because of the variability and ambiguity of language in expressing mereological and topological relations, understanding of mereotopological relations or spatio-temporal relations in general is more of an extraction challenge. We need to identify the proper interpretation of terms such as “is a part of”, “adjacent”, etc. to correctly build mereotopological models. This has been done for temporal relations [Ver+05], but the methodology should be applicable to spatial relations as well. For instance, the language presented in [Cha04]—an extension of the event calculus with mereotopological relations—can help track epidemics by capturing and understanding the language of epidemic outbreak reports. More recently, there has been some interest in sketch map understanding. Recent work [WL12; WS09] shows that topology and the spatial order among spatial objects are most consistently preserved in human sketch maps. Consequently, those are the relations we need to include in qualitative representation of space useful for human navigation. To that respect, the theories in the hierarchy *OMTB* developed in Chapter 10 may be suitable for formally representing sketch maps drawn by humans, since they preserves those two kinds of relations. In the other direction, qualitative spatial ontologies could be used to automatically general sketch maps through an abstraction process from more detailed maps often found in navigation software. Such generated sketch maps can help prevent confusion through information overload.

Chapter 4

Equidimensional mereotopologies with mereological closures[1]

Closure Mereotopology (**CMT**: [CV99a]) is widely accepted as the most restricted mereotopology that does not contain any controversial ontological assumptions. Though some specific extensions of **CMT** have been studied in great detail, the question of what constitutes a mereotopology that adequately represents the abstract space underlying physical space has been largely neglected. In particular, the existing work on specific mereotopologies suggests that still new combinations of axioms could yield yet unexplored theories of closure mereotopology. We give strong evidence why this is not the case. We do so by focusing on the spatial representability of the models of a mereotopology. Though many concrete embeddings of mereotopological models in topological spaces have been constructed [see BD07a; DV06; DV07; DW04; DW05a; Dün+06; Dün+08; Vak07], the question of whether these topological representations adequately reflect the intended structure of physical space has not be addressed[2]. As it turns out, the key in this pursuit is the necessary strength of the complementation operation. We show that assuming the existence of some kind of uniquely defined complements and requiring a weak form of spatial representability restrict the algebraic structure arising from mereotopologies to an extent that only a few particular theories remain. Only two distinct minimal classes of ontologically coherent mereotopologies (we define C-closure in that regard) are conceivable—distinguished by the presence or absence of unique complements. Our analysis further identifies the algebraic properties that correspond to the various closure operations and other ontological assumptions of the mereotopologies.

For our investigation we treat mereotopology algebraically as first proposed by Stell [Ste00; SW97] and Düntsch and Winter [DW04; DW05b]. The systematic studies of *algebraic counterparts*[3] of mereotopologies in [HG12; Vak07] offer many insights that help us understand the different mereotopological

[1]This chapter and Appendix F have been originally published as [HG13] in Notre Dame Journal of Formal Logic.

[2]Our notion of *spatial representability* deviates from standard topological representations in the sense that we are interested in whether all regions of an algebraic theory of mereotopology can be represented by adequately sized point sets so that notions such as contact (sharing a point), overlap (sharing a region), and complementation have intuitive spatial semantics. This understanding of spatial representations is similar to what are called a "faithful interpretations" by [For96; Mor98]. It is more stringent than the standard notion of topological representability of algebraic structures in pure mathematics.

[3]*Algebraic counterpart* refers to the class of contact algebras that can be constructed according to Theorem 4.2 Equally, a mereotopology whose models can be mapped to structures of a certain class of contact algebras is referred to as *logical counterpart* of the class of contact algebras.

theories and the relationships among them. The study of algebraic theories of mereotopology is, for example, most convincing in separating the mereological component from the topological component as pointed out by [LY04]. We are particularly interested in *Unique Closure Mereotopology* (**UCMT**) and *Unique Infinitary Closure Mereotopology* (**UGMT**), subclasses of **CMT** of which *all* models have algebraic counterparts. **UCMT** includes many prominent mereotopologies as subtheories, such as the theories of Whitehead [Whi29], of Clarke [Cla81], the Region-Connection Calculus (RCC: [GGC96; RCC92]), and its generalization (GRCC: [LY04]) which admits discrete models. The theory **UCMT** is introduced in Section 4.1; it assumes closure under (binary) sums and intersections just as **CMT** does, but additionally assumes closure under complementation with respect to a universal region and that all these closure operations are unique. In Section 4.2 we show that the algebraic counterparts of models of **UCMT** are orthocomplemented contact algebras (OCA). Thus, the spatial representability of **UCMT**s can be studied through the spatial representability of OCAs—a task we are much more comfortable with. In Section 4.3 we look at spatially representable OCAs, but lacking a complete definition of spatial representability we resort to a weaker form thereof, MT-representability. We can show that every MT-representable complete OCA is pseudocomplemented and satisfies the Stone identity, i.e., is a SPOCA. For this result, we rely on the lattices being complete. However, this is only a minor restriction since we can reasonably expect all spatially representable contact algebras to be complete. For discrete MT-representable mereotopologies, it is no restriction at all.

Section 4.4 contains our key contribution: We identify algebraic conditions that are necessary and sufficient for the closure operations sums, intersections, complements, and universal to be defined mereologically or topologically in SPOCAs. In particular, we show that the ontological choice between a mereological or topological complementation in a mereotopology is reflected in the algebraic structure: The algebras of mereologically closed mereotopological models are uniquely complemented and thus distributive while those of topologically closed models are only pseudo- and orthocomplemented but potentially non-distributive. This confirms how central complementation, and thus supplementation, is in mereotopology—as already emphasized by [Ste04].

We identify the two minimal classes that emerge as MT-representable and ontologically coherent (a notion formalized later) algebraic structures from those two classes of SPOCAs in Section 4.5. The first class, namely weak Boolean contact algebras (WBCA), defines all closure operations mereologically; though only the more restricted generalized Boolean contact algebras (GBCA) are guaranteed to have intuitive spatial representations. The second class, namely SPOCAs with contact defined as $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ or as $x\mathbf{C}y \leftrightarrow x \nless y^{\perp}$, defines all closure operations topologically or quasi-topologically. These two classes are also the weakest ones that could axiomatize space as intended by Whitehead [Whi29]. However, neither of them satisfies all conditions discussed by Whitehead. As a further consequence of our work, we can verify algebraically that the assumptions of Whiteheadean mereotopology as outlined in [For96; Mor98] are not compatible with the connectivity axiom Con, stating $\forall x[C(x, -x)]$. Ways to overcome this problem are discussed in Section 4.6. Furthermore, we prove that no "true mereotopology", that is, no MT-representable MT-closed mereotopology, with atoms can exist. Only if we allow coherently closed (C-closed) instead of MT-closed mereotopologies, exactly two theories (among all combinations of mereological and topological definitions of each of the closure operation sum, intersection, complement, and universal), namely the GBCAs and the SPOCAs with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$, admit both continuous and discrete models.

From a methodological point of view, this chapter demonstrates that the duality between algebraic

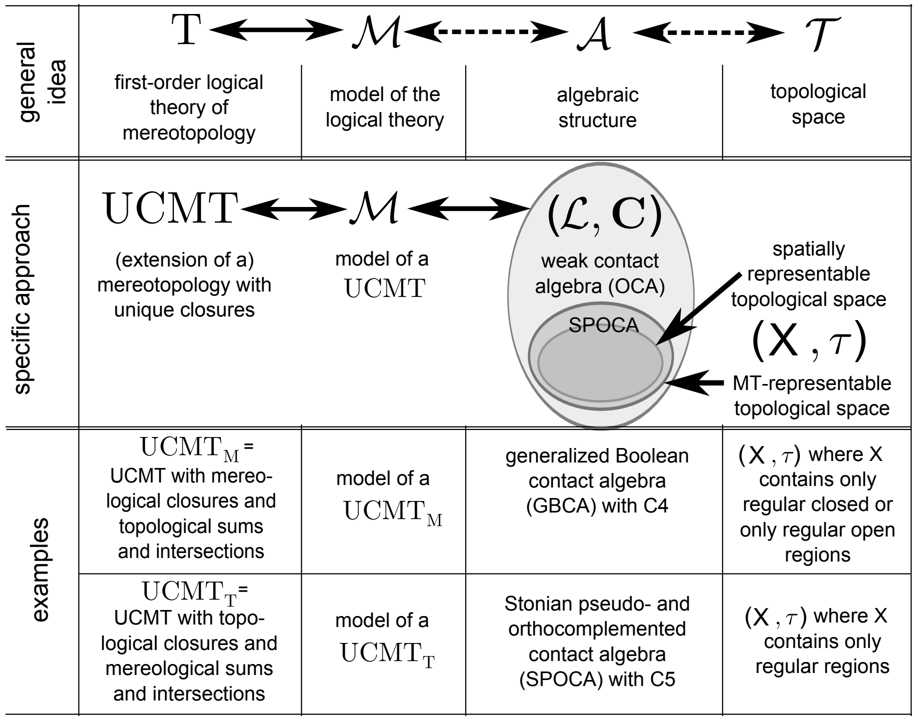


Figure 4.1: An overview of our approach in Chapter 4. The correspondence between a logical theory of mereotopology and its model on the left-hand side are fairly standard. Representation results between some classes of algebraic structures, specifically lattices, and topological spaces as indicated on the right-hand side are known for some specific cases. In order to establish a subset of the logical theories of mereotopologies that have spatially representable topological interpretations, we need to, first and foremost, establish a correspondence between the models of **CMT**s and contact algebras. Since we cannot achieve this in general (indicated by the dashed arrow), we resort to the restriction of **CMT**s to **UCMT**s as shown in the second row. Every model of a **UCMT** has an algebraic counterpart in the class of orthocomplemented contact algebras (OCAs) as indicated by the solid arrow in the middle. As the second crucial step (the right dashed arrow in the top row), we try to reduce the class of OCAs to a smaller class that still includes all spatially representable contact algebras. This is a subclass of the Stonian pseudo- and orthocomplemented contact algebras (SPOCAs), which are at least MT-representable. However, as indicated by the uni-directed solid arrow on the right, not all SPOCAs are spatially representable or even topologically representable.

structures and topological spaces is not a mere theoretical exercise only of mathematical interest, but helps us understand the diversity of theories of qualitative space and select an axiomatization according to any given set of desirable ontological assumptions. Our methodology is outlined in Figure 4.1: We leverage the knowledge about duality between certain lattices and topological spaces to the understanding of mereotopology. The models of all mereotopologies satisfying the discussed closure assumptions can be represented algebraically in a straightforward manner. With the introduced notion of MT-representability we are then able to reduce the contact algebras resulting from **UCMT**s to a much more restricted set of contact algebras, namely SPOCAs, that includes all spatially representable and ontologically coherent algebraic counterparts to models of **UCMT**. Two examples of such contact algebras arising from **UCMT**s, which are representative of the only two C-closed MT-representable contact algebras with discrete models, are given in Figure 4.1. The figure also describes their logical counterparts as well as the common spatial interpretation of their models.

4.1 Mereotopologies with complements

In this chapter, we only consider so-called equidimensional mereotopologies, i.e., unsorted mereotopological theories whose domain elements can be interpreted as all being of equal dimension. For example, the domain elements could be interpreted all as 1D regions (such as time intervals or intervals on a line) as in Allen's Interval Algebra [All83], or as spatial regions which are all 2D or all 3D, or as spatio-temporal regions of either all 3D or all 4D. The language of those theories consist of a parthood and a contact relation that satisfy P.1 – P.3 and C.1 – C.3 as discussed in Chapter 3. We also take the definitions of overlap (O.D), underlap (U.D), and proper parthood (PP.D) from Chapter 3 for granted in any equidimensional mereotopology.

4.1.1 Closure mereotopology with unique closures (UCMT)

A common requirement for mereotopological theories is the existence of closure operations. These require an intersection for any two overlapping entities and a sum for any two underlapping entities, compare, for example, closure meoreotopology (**CMT**: [CV99a]). Here, we go beyond **CMT** in three ways in order to define *unique closure mereotopology* (**UCMT**).

First, we require a greatest entity to exist, i.e., something that everything else is a part of UCMT.4. The existence of such a universal entity, denoted by u, is plausible in any restricted domain of interest, such as the earth, a specific country, building, or an even smaller experimental domain (such as a closed "blocks world" consisting of a finite number of blocks).

(UCMT.4) $\forall x[P(x,\mathrm{u})]$ (unique universal entity)

Secondly, we require all closure operations to be uniquely defined. The universal u is already unique by P.2. Because we want sums and intersections also to be uniquely defined for all pairs of entities, we denote them by function symbols, namely $\oplus$ and $\odot$. We need to ensure that the sum $x \oplus y$ is the smallest element which has both x and y as parts (*supremum*) and that anything that overlaps the sum must also overlap either x or y. Likewise, the intersection $x \odot y$ is the greatest element that is both part of x and y (*infimum*) if x and y overlap at all. If x and y do not overlap, the intersection $x \odot y$ is meaningless and may be assigned an arbitrary entity without further logical consequences. These conditions are reflected in the axioms UCMT.1 and UCMT.2, which entail $\forall x, y\, P(x, x \oplus y)$ and $\forall x, y\, P(x \odot y, x)$. It follows that every pair of elements has a sum and intersection so that $P(x_1, x_2)$ and $P(y_1, y_2)$ together imply $P(x_1 \oplus y_1, x_2 \oplus y_2)$ and $P(x_1 \odot y_1, x_2 \odot y_2)$, the latter only if $O(x_1, y_1)$. We do not require a similar precondition in UCMT.1 because in the presence of UCMT.4 any two entities automatically underlap.

(UCMT.1) $\forall z[(O(x,z) \vee O(y,z)) \leftrightarrow O(x \oplus y, z)]$ (sum is supremum)

(UCMT.2) $O(x,y) \rightarrow \forall z[(P(z,x) \wedge P(z,y)) \leftrightarrow P(z, x \odot y)]$ (intersection is infimum)

Finally, we require models not only to be closed under intersections and sums, but also to be closed under complementation. Given that a universal entity exists, complements are a natural concept motivated by human perception of physical space: if we are given a restricted physical space, we can easily identify the complement with respect to the universal entity. Again, the complement shall be uniquely defined for every entity, hence we denote it by a function, namely $\ominus$. Note, however, that the complement of the universal u is not meaningful because in a moment we will specifically prohibit a null (empty) entity to exist. We can choose, for example, $\ominus \mathrm{u} = \mathrm{u}$. The complement function shall be involutary UCMT.5—a reasonable assumption for uniquely defined complements. Additionally, UCMT.6 and UCMT.7 ensure that entities and their complement interact correctly with respect to sums and intersections (overlap). Though $\ominus$ is a total function, the universal's complement is not meaningful. For this reason, UCMT.5, UCMT.6, and UCMT.7 explicitly do not apply to the universal u.

(UCMT.5) $x \neq \mathrm{u} \rightarrow x = \ominus(\ominus x)$ (complements involutary)

(UCMT.6) $x \neq \mathrm{u} \rightarrow x \oplus (\ominus x) = \mathrm{u}$ (sum of complements)

(UCMT.7) $x \neq \mathrm{u} \rightarrow \neg O(x, \ominus x)$ (complements nonoverlapping)

We do not restrict $\oplus$, $\odot$, and $\ominus$ any further at this point. Instead, we consider in Section 4.4 two plausible definitions, a mereological and a topological one, of each of these functions.

Contrary to the existence of a universal entity, a *null entity* (also called *zero region*) is often deemed cognitively undesirable. The null entity would be part of every entity, thus it would also be in contact to every entity. On the other side, the null entity is empty, i.e., not really existent and thereby not in contact to anything at all. To avoid this paradox, we postulate UCMT.3 to ensure the cognitive adequacy of the mereotopological theories.

(UCMT.3) $\forall x \exists y \neg P(x, y)$ (no null entity)

However, it is not an essential assumption here because the algebraic counterparts of these mereotopologies explicitly introduce a null entity. That means our subsequent analysis extends to mereotopologies with unique closures that allow or even require a null entity, such as Roeper's mereotopology [Roe97]. In fact the multidimensional theories constructed in later chapters all use a zero region to simplify their axiomatizations.

We use the term **UCMT** in the following broader sense:

Definition 4.1. *Let* MT *be a consistent, unsorted first-order theory with two distinguished binary predicates C and P, two binary functions $\oplus$, $\odot$, a unary function $\ominus$, and a constant* u*. If* MT *entails the sentences P.1 – P.3, C.1 – C.3, and UCMT.1 – UCMT.7 with the definitions O.D, U.D, and PP.D, we call* MT *a* **UCMT**.

The domain elements in a model of **UCMT** are often called *regions*.

Any **UCMT** has a mereological component that is restricted to a closed mereology **CM** where sums, intersections, complements, and the universal are unique but is noncommittal with respect to other mereotopological principles. These mereotopological principles, their corresponding axioms, and the properties of the resulting logical theories have been studied in much detail in [CV99a; Esc07]. We will show later that the requirement of unique closures including unique complements does not leave many choices with respect to other mereotopological principles if we require spatial representability and ontological coherence at the same time.

4.1.2 General mereotopology with unique infinitary closures (UGMT)

Many mereotopologies go beyond **CMT** by requiring sums and intersections of arbitrarily many—possibly infinitely many—entities to exist. Axioms postulating such infinitary closures or unrestricted fusions either require axiom schemas or sets or classes, [CV99a]. For better readability we use a set notation here: **X** denotes an arbitrary set of domain entities.

(UGMT.1) $\forall \mathbf{X}\Big[\forall z\big[\exists x \in \mathbf{X}[O(x,z)] \leftrightarrow O(\bigoplus \mathbf{X}, z)\big]\Big]$ (unrestricted sum)

(UGMT.2) $\forall \mathbf{X}\Big[\exists z\big[\forall x \in \mathbf{X}[P(z,x)]\big] \rightarrow \forall z\big[\forall x \in \mathbf{X}[P(z,x)] \leftrightarrow P(z, \bigodot \mathbf{X})\big]\Big]$ (unrestricted intersection)

A **UCMT** that satisfies these axioms is a *general mereotopology* (**GMT**) with unique infinitary closures (including complements).

Definition 4.2. *A* **UGMT** *is a* **UCMT** *that satisfies UGMT.1 and UGMT.2.*

Only in Section 4.2.2 we will briefly discuss the subclass **UGMT** and how their algebraic counterparts yield complete lattices.

4.2 The algebraic structures arising from models of UCMT

We now introduce a class of algebraic structures called contact algebras and show that the models of **UCMT** correspond to orthocomplemented contact algebras (OCA) while the models of **UGMT** correspond to complete OCAs. First let us define what we mean by a contact algebra. Contact algebras are not a new concept, various classes thereof have been studied as algebraic counterparts of specific mereotopological theories, e.g., by [BD07a; DW04; DW05b; Ste00; SW97; Vak07]. Our definition here encompasses the weakest common properties and is based on bounded lattices, which we will define in a moment in Definition 4.4. Within contact algebras we denote the lattice operations meet and join using the symbols $\cdot$ and $+$.

Definition 4.3. *A* <u>*contact algebra*</u> $(\mathcal{L}, \mathbf{C})$ *consists of a bounded lattice* $\mathcal{L}$ *which defines a partial order* $\leq$ *and a contact relation* $\mathbf{C}$ *that satisfies the axioms C0 – C3.*

(C0) $0 \neg \mathbf{C} x$ (null disconnectedness)

(C1) $x \neq 0 \rightarrow x\mathbf{C}x$ (C reflexive)

(C2) $x\mathbf{C}y \leftrightarrow y\mathbf{C}x$ (C symmetric)

(C3) $x\mathbf{C}y \wedge y \leq z \rightarrow x\mathbf{C}z$ (C monotone with respect to $\leq$)

Thus, the contact relation must satisfy the axioms of a ground mereotopology. The axioms C1 – C3 are algebraic versions of the axioms C.1 – C.3 of **MT**s while C0 deals with the the newly introduced smallest element 0 that is necessary to construct a lattice from a mereotopological model. The assumption that 0 is not connected to any other entities is merely a convenient choice without deeper implications. To distinguish the contact relation in a mereotopological theory from the contact relation in its algebraic counterpart, we write $C(x,y)$ to refer to the former and $x\mathbf{C}y$ to refer to the latter. The latter has nothing to do with the bold notation used in later chapters to refer to the extension of C.

4.2.1 Relevant classes of lattices

Before we show how to construct the algebraic counterparts of **UCMT**s, we review the various classes of lattices necessary for the later sections of this chapter. These are used to define more restricted classes of contact algebras. Most of these classes of lattices are defined in standard references such as [Bly05; Grä98], while more specialized classes are covered in [Ste99]. Each class allows nondistributive models

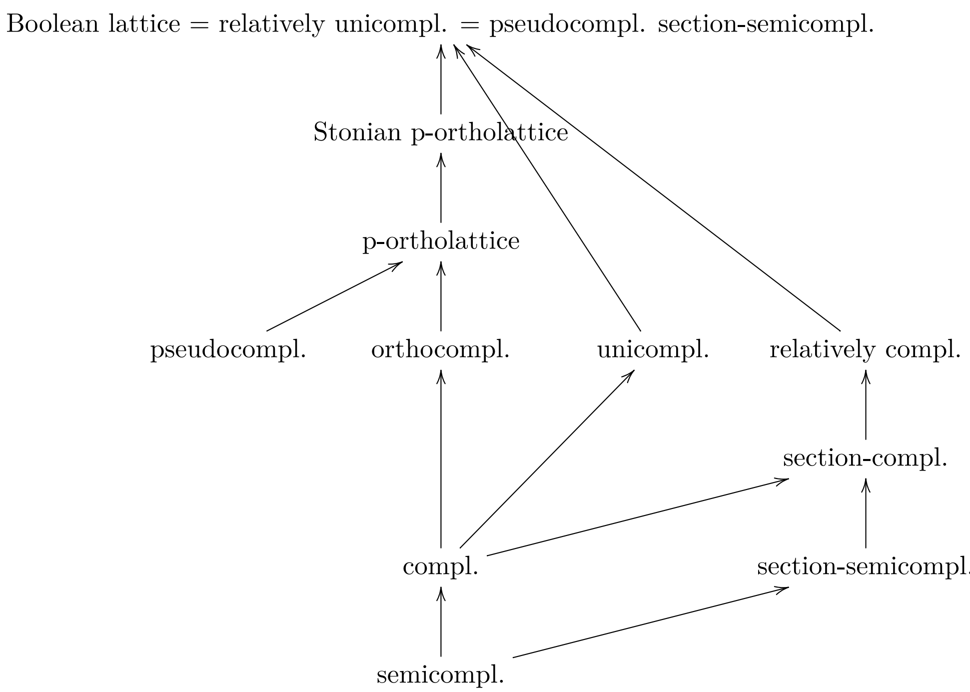


Figure 4.2: Relationships between bounded lattices with varying kinds of complementation; adapted from [Hah08; Ste99]. The arrows indicate refinement, e.g., every p-ortholattice is also a pseudocomplemented and an orthocomplemented lattice. These refinements are transitive. In the case of distributive bounded lattices many of these classes of lattices collapse.

unless explicitly ruled out. The relations between these classes of bounded lattices are illustrated in Figure 4.2.

One remark upfront: Any lattice can be treated as an algebraic structure $\langle \mathbf{L}, \cdot, + \rangle$ as well as a partially ordered set $\langle \mathbf{L}, \leq \rangle$ with unique supremum $+$ and unique infimum $\cdot$ for any pairs of entities. We can define $x \leq y \leftrightarrow x \cdot y = x$ for any $x, y \in \mathbf{L}$. We depict lattices as Hasse diagrams which are transitive reductions of the partial order of the lattice. That means only the direct, i.e., covering, order relations are depicted while transitive closure is implied. $x \leq y$ holds if and only if there is a path consisting of one or multiple line segments strictly leading upwards from x to y.

Definition 4.4. *A* _bounded lattice_ *is a structure* $\langle \mathbf{L}, \cdot, +, 0, 1 \rangle$ *of arity* $\langle 2, 2, 0, 0 \rangle$ *such that*

(L.B0) $\langle \mathbf{L}, \cdot, + \rangle$ *is a lattice, i.e.,* $a + b$ *and* $a \cdot b$ *are uniquely defined for all* $a, b \in \mathbf{L}$*;*

(L.B1) *there exists an element* $1 \in \mathbf{L}$ *so that* $1 \cdot a = a$ *(and* $1 + a = 1$*) for all* $a \in \mathbf{L}$*;*

(L.B2) *there exists an element* $0 \in \mathbf{L}$ *so that* $0 \cdot a = 0$ *(and* $0 + a = a$*) for all* $a \in \mathbf{L}$.

Definition 4.5. *A* _bounded distributive lattice_ *is a structure* $\langle \mathbf{L}, \cdot, +, 0, 1 \rangle$ *such that*

(L.D0) $\langle \mathbf{L}, \cdot, +, 0, 1 \rangle$ *is a bounded lattice,*

(L.D1) *the distributive law holds, i.e.,* $a \cdot (b + c) = a \cdot b + a \cdot c$ *for all* $a, b, c \in \mathbf{L}$.

The structures in the following Definitions 4.6 to 4.10 are all of type $\langle 2, 2, 1, 0, 0 \rangle$ equipped with a unary function of complementation or pseudocomplementation.

Definition 4.6. *A* *complemented lattice* *is a structure* $\langle \mathbf{L}, \cdot, +, ', 0, 1 \rangle$ *such that*

(L.C0) $\langle \mathbf{L}, \cdot, +, 0, 1 \rangle$ *is a bounded lattice,*

(L.C1) a' *is a complement of* a*, i.e.,* $a + a' = 1$ *and* $a \cdot a' = 0$*.*

Definition 4.7. *An* *orthocomplemented lattice* *(short:* *ortholattice**) is a structure* $\langle \mathbf{L}, \cdot, +, {}^{\perp}, 0, 1 \rangle$ *such that*

(L.O0) $\langle \mathbf{L}, \cdot, +, 0, 1 \rangle$ *is a bounded lattice,*

(L.O1) $a^{\perp}$ *is an orthocomplement of* a*, i.e., for all* $a, b \in \mathbf{L}$ *we have*

(a) $a^{\perp\perp} = a$*,*

(b) $a \cdot a^{\perp} = 0$*,*

(c) $a \leq b$ *implies* $b^{\perp} \leq a^{\perp}$*.*

Notice that orthocomplemented lattices are complemented.

Definition 4.8. *A* *pseudocomplemented lattice* *is a structure* $\langle \mathbf{L}, \cdot, +, {}^{*}, 0, 1 \rangle$ *such that*

(L.P0) $\langle \mathbf{L}, \cdot, +, 0, 1 \rangle$ *is a bounded lattice,*

(L.P1) a^{*} *is the pseudocomplement of* a*, i.e., for all* $b \in \mathbf{L}$*,* $a \cdot b = 0 \iff b \leq a^{*}$*.*

Definition 4.9. *A* *quasicomplemented lattice* *is a structure* $\langle \mathbf{L}, \cdot, +, {}^{+}, 0, 1 \rangle$ *such that*

(L.Q0) $\langle \mathbf{L}, \cdot, +, 0, 1 \rangle$ *is a bounded lattice,*

(L.Q1) a^{+} *is the quasicomplement of* a*, i.e., for all* $b \in \mathbf{L}$*,* $a + b = 1 \iff b \geq a^{+}$*.*

Quasicomplemented lattices are also known as dually pseudocomplemented lattices.

Definition 4.10. *A* *uniquely complemented lattice* *(short:* *unicomplemented lattice**) is a structure* $\langle \mathbf{L}, \cdot, +, ', 0, 1 \rangle$ *such that*

(L.U0) $\langle \mathbf{L}, \cdot, +, ', 0, 1 \rangle$ *is a complemented lattice,*

(L.U1) a' *is the unique complement of* a*, i.e., for all* $b \in \mathbf{L}$*,* $b + a = 1$ *and* $b \cdot a = 0$ *imply* $b = a'$*.*

Clearly, every uniquely complemented lattice is orthocomplemented, but not necessarily pseudocomplemented or quasicomplemented. On the other side, Figure 4.3 gives an example of a orthocomplemented, pseudocomplemented, and quasicomplemented lattice which is not unicomplemented. Pseudo- or quasicomplemented lattices do not even have to be complemented. Lattices that are both pseudocomplemented and orthocomplemented (and thus also complemented and quasicomplemented) but not unicomplemented were introduced in [HWG09] as p-ortholattices.

Definition 4.11. *A* *p-ortholattice* *is a structure* $\langle \mathbf{L}, \cdot, +, {}^{+}, {}^{\perp}, 0, 1 \rangle$ *such that*

(L.PO0) $\langle \mathbf{L}, \cdot, +, {}^{+}, 0, 1 \rangle$ *is a quasicomplemented lattice,*

(L.PO1) $\langle \mathbf{L}, \cdot, +, {}^{\perp}, 0, 1 \rangle$ *is an ortholattice.*

An ortholattice is pseudocomplemented if and only if it is quasicomplemented. For a given p-ortholattice $\langle \mathbf{L}, \cdot, +, {}^{+}, {}^{\perp}, 0, 1 \rangle$, the structure $\langle \mathbf{L}, \cdot, +, {}^{*}, 0, 1 \rangle$ is a pseudocomplemented lattice if we define $x^{*} = x^{\perp+\perp}$. P-ortholattices in which the orthocomplementation and pseudocomplementation operations coincide (unlike Figure 4.3) are unicomplemented. Unicomplemented ortholattices are Boolean [Bir67].

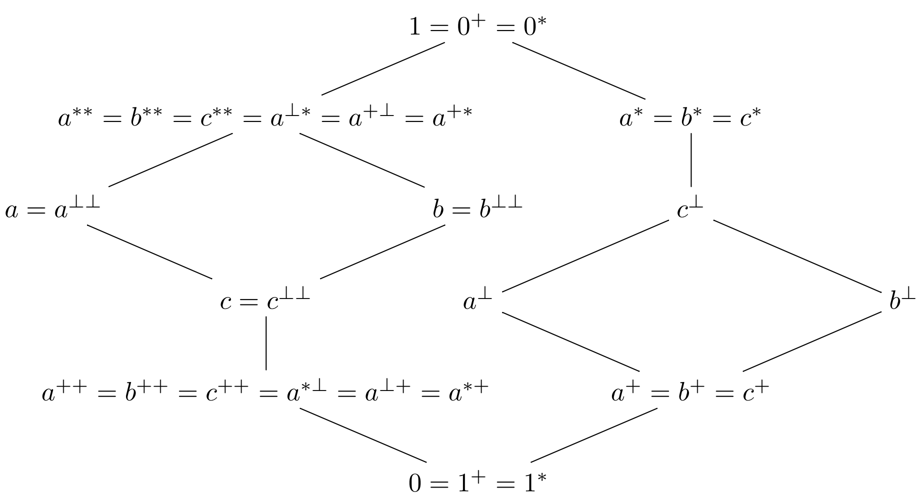


Figure 4.3: A p-ortholattice with ortho-, pseudo-, and quasicomplements indicated.

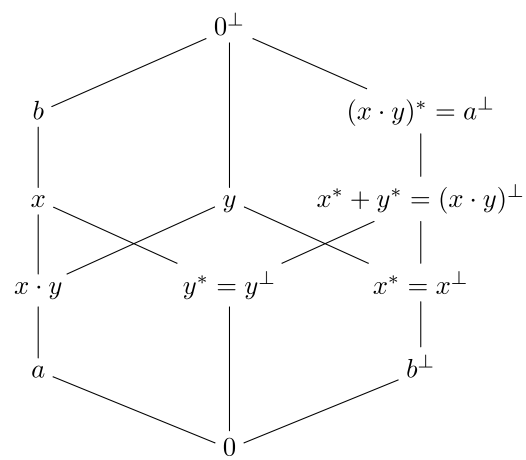


Figure 4.4: A p-ortholattice that violates $(x \cdot y)^* = x^* + y^*$ and is therefore not Stonian [HWG09].

Definition 4.12. *A* <u>*Boolean lattice*</u> *is a structure* $\langle \mathbf{L}, \cdot, +, ', 0, 1 \rangle$ *such that*

(L.BO0) $\langle \mathbf{L}, \cdot, +, ', 0, 1 \rangle$ *is an orthocomplemented lattice,*

(L.BO1) *the distributive law holds, i.e.,* $a \cdot (b + c) = a \cdot b + a \cdot c$ *for all* $a, b, c \in \mathbf{L}$.

But there are other interesting subclasses of p-ortholattices that are not distributive and thus not Boolean. Stonian p-ortholattices were introduced in [HWG09] to algebraically capture the structure of the mereotopology of [AV95]. A Stonian p-ortholattice is a p-ortholattice that satisfies the Stone identity (L.SPO1). Figure 4.4 illustrates that not all p-ortholattices are Stonian.

Definition 4.13. *A* <u>*Stonian p-ortholattice*</u> *is a structure* $\langle \mathbf{L}, \cdot, +, ^+, ^\perp, 0, 1 \rangle$ *such that*

(L.SPO0) $\langle \mathbf{L}, \cdot, +, ^+, ^\perp, 0, 1 \rangle$ *is a p-ortholattice,*

(L.SPO1) *The Stone identity holds, i.e.,* $(a + b)^+ = a^+ \cdot b^+$ *for all* $a, b \in \mathbf{L}$.

Again, a Stonian p-ortholattice $\langle \mathbf{L}, \cdot, +, ^+, ^\perp, 0, 1 \rangle$ is equivalently defined as $\langle \mathbf{L}, \cdot, +, ^*, ^\perp, 0, 1 \rangle$ using pseudocomplementation if we choose $x^* = x^{\perp + \perp}$. We use both structures interchangeably. Stonian p-ortholattice generalize the (distributive) Stone lattices to non-distributive lattices.

The Stone identity was originally proposed by Marshall Stone as an immediate generalization of Boolean algebras to so-called Stone lattices—pseudocomplemented distributive lattices which satisfy the Stone identity. Several other ways of stating the Stone identity in p-ortholattices are known, as the following theorem from [HWG09] showed.

Theorem 4.1. *Let* $\langle \mathbf{L}, \cdot, +, ^{+}, ^{\perp}, 0, 1\rangle$ *be a p-ortholattice with* $x^* = x^{\perp+\perp}$ *for all* $x \in \mathbf{L}$*. Then the following statements are equivalent:*

1. $(x \cdot y)^* = x^* + y^*$ *for all* $x, y \in \mathbf{L}$*;*
2. $(x + y)^+ = x^+ \cdot y^+$ *for all* $x, y \in \mathbf{L}$*;*
3. $(x \cdot y)^{++} = x^{++} \cdot y^{++}$ *for all* $x, y \in \mathbf{L}$*;*
4. $(x + y)^{**} = x^{**} + y^{**}$ *for all* $x, y \in \mathbf{L}$.

We will later use the properties 4.1(3) and (4); (4) is captured by the axiom S in the axiomatization of SPOCA's.

Notice that the dual of L.SPO1, $(a \cdot b)^+ = a^+ + b^+$, holds for all quasicomplemented lattices and, equally, $(a + b)^* = a^* \cdot b^*$ holds for all pseudocomplemented lattices. Moreover, (L.SPO1) and its dual hold for orthocomplements in ortholattices, that is $(a + b)^\perp = a^\perp \cdot b^\perp$ and $(a \cdot b)^\perp = a^\perp + b^\perp$ for all $a, b \in \mathbf{L}$ if $\mathcal{L}$ is orthocomplemented [HWG09]. Finally, it is easily verifiable that Boolean lattices are Stonian p-ortholattices.

4.2.2 Orthocomplemented contact algebras (OCA)

We now show that all the models of a **UCMT** can be viewed algebraically as contact algebras in which the lattice is orthocomplemented[4].

Definition 4.14. *An* <u>*orthocomplemented contact algebra*</u> *(OCA) is an algebraic structure* $\mathcal{A} = (\mathcal{L}, \mathbf{C})$ *consisting of an ortholattice* $\mathcal{L} = \langle \mathbf{L}, \cdot, +, ^{\perp}, 0, 1\rangle$ *and a contact relation* $\mathbf{C}$ *that satisfies C0 – C3.*

The theory

$$OCA = \{\mathrm{L2}^{\vee} - \mathrm{L6}^{\vee},\ \mathrm{L2}^{\wedge} - \mathrm{L4}^{\wedge},\ \mathrm{O1}' - \mathrm{O3}',\ \mathrm{C0} - \mathrm{C3}\}$$

axiomatizes OCAs, see Appendix F.1 for the algebraic axioms we use. Notice that OCAs are not necessarily distributive. We only consider nontrivial OCAs which contain an element apart from 0 and 1. Now we show how to construct an OCA from an arbitrary model of **UCMT**.

Theorem 4.2. *Let* $\mathcal{M}$ *be a model of* **UCMT** *with domain* $\mathbf{M}$.
Then $\mathcal{M}$ *with the extended domain* $\mathbf{L} = \mathbf{M} \cup \{0\}$ *where* $0 \notin \mathbf{M}$ *and with the definitions*

$$x \cdot y = \begin{cases} 0 & \text{if } x = 0 \text{ or } y = 0 \text{ or } \langle x, y\rangle \notin \mathbf{O}_{\mathcal{M}} \\ x \odot y & \text{otherwise} \end{cases}$$

[4]Orthocomplemented lattices have first been used by Biacino and Gerla [BG91] as an algebraic theory of Clarke's axiomatization of mereotopology.

$$x + y = \begin{cases} 0 & \textit{if } x = 0 \textit{ and } y = 0 \\ y & \textit{if } x = 0 \textit{ and } y \neq 0 \\ x & \textit{if } x \neq 0 \textit{ and } y = 0 \\ x \oplus y & \textit{otherwise} \end{cases}$$

$$x^{\perp} = \begin{cases} 0 & \textit{if } x = \mathrm{u} \\ 1 & \textit{if } x = 0 \\ \ominus x & \textit{otherwise} \end{cases}$$

$$1 = \mathrm{u}$$

$$x\mathbf{C}y = \begin{cases} \textit{false} & \textit{if } x = 0 \textit{ or } y = 0 \\ \langle x, y\rangle \in \mathbf{C}_{\mathcal{M}} & \textit{otherwise} \end{cases}$$

defines an structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle \mathbf{L}, \cdot, +, ^{\perp}, 0, 1\rangle, \mathbf{C})$ *that is an OCA.*

Proof. In order to show that $\mathcal{A}$ is an OCA it suffices by Definition 4.14 to prove that

(i) $\mathcal{L}$ is an ortholattice, and

(ii) $\mathbf{C}$ satisfies C0 to C3.

(i): Since $\oplus$ and $\odot$ define supremum and infimum for every pair of elements (infimum is defined as 0 for all nonoverlapping pairs), $\mathcal{L} = (\mathbf{M} \cup \{0\}, +, \cdot, ^{\perp}, 0, 1)$ is a lattice with the partial order defined as $x \leq y \Leftrightarrow \big(\mathbf{P}(x, y) \text{ or } x = 0\big)$. This definition of $\leq$ follows from our construction by the following two derivations:

$$\begin{aligned} x \leq y &\Rightarrow x \cdot y = x \\ &\Rightarrow \big(x \odot y = x \text{ and } \mathbf{O}(x, y)\big) \text{ or } x = 0 \\ &\Rightarrow \forall z[\mathbf{P}(z, x) \wedge \mathbf{P}(z, y) \leftrightarrow \mathbf{P}(z, x)] \text{ or } x = 0 \\ &\Rightarrow \forall z[\mathbf{P}(z, y) \leftarrow \mathbf{P}(z, x)] \text{ or } x = 0 \\ &\Rightarrow \big(\mathbf{P}(x, y) \leftarrow \mathbf{P}(x, x)\big) \text{ or } x = 0 \\ &\Rightarrow \mathbf{P}(x, y) \text{ or } x = 0 \end{aligned}$$

and

$$\begin{aligned} \mathbf{P}(x, y) \text{ or } x = 0 &\Rightarrow \big(\mathbf{P}(x, y) \wedge \forall z[\mathbf{P}(z, x) \rightarrow \mathbf{P}(z, y)]\big) \text{ or } x = 0 \\ &\Rightarrow \big(\mathbf{O}(x, y) \wedge \forall z[\mathbf{P}(z, x) \rightarrow \mathbf{P}(z, y)]\big) \text{ or } x = 0 \\ &\Rightarrow \big(\forall z[\mathbf{P}(z, x) \wedge \mathbf{P}(z, y) \leftrightarrow \mathbf{P}(z, x \odot y)] \wedge \forall z[\mathbf{P}(z, x) \rightarrow \mathbf{P}(z, y)]\big) \text{ or } x = 0 \\ &\Rightarrow \forall z[\mathbf{P}(z, x) \leftrightarrow \mathbf{P}(z, x \odot y)] \text{ or } x = 0 \\ &\Rightarrow x = x \odot y \text{ or } x = 0 \\ &\Rightarrow x = x \cdot y \\ &\Rightarrow x \leq y \end{aligned}$$

By the definition of the meet operation $\cdot$, the lattice has in 0 a lower bound. By the definition of the join operation + and UCMT.4, the lattice has in 1 an upper bound. Thus, $\mathcal{L}$ is a bounded lattice.

$\mathcal{L}$ further satisfies the properties L.O1(a) − L.O1(c) of ortholattices:

L.O1(a): for all $a \in \mathbf{L}$ with $a \neq 0, 1$, $a^{\perp\perp} = a$ follows from the definition of $\ominus$ in UCMT.5. For $a = 0$ we have $a^{\perp\perp} = (a^{\perp})^{\perp} = 1^{\perp} = 0$ and for $a = 1$ we have $a^{\perp\perp} = (a^{\perp})^{\perp} = 0^{\perp} = 1$.

L.O1(b): for all $a \in \mathbf{L}$, $a \cdot a^{\perp} = 0$ follows immediately if $a = 0$ or $a = 1$. For all $a \in \mathbf{L}$ with $a \neq 0, 1$ we have $\langle x, (\ominus x)\rangle \notin \mathbf{O}_{\mathcal{M}}$ by UCMT.7, which results in $a \cdot a^{\perp} = 0$ by our definition of $\perp$ and $\cdot$.

L.O1(c): Assume $a, b \in \mathbf{L}$ with $a \leq b$. We will prove that $b^{\perp} \leq a^{\perp}$ in any case.
If $a = 0$, then $b^{\perp} \leq 1 = 0^{\perp} = a^{\perp}$, which is trivially satisfied.
If $a = 1$, then $a \leq b$ implies $b = 1$ and therefore $b^{\perp} = 1^{\perp} \leq 1^{\perp} = a^{\perp}$ is trivially satisfied.
If $b = 0$, then $a \leq b$ implies $a = 0$ and therefore $b^{\perp} = 0^{\perp} \leq 0^{\perp} = a^{\perp}$ is trivially satisfied.
If $b = 1$, then $b^{\perp} = 1^{\perp} = 0 \leq a^{\perp}$, which is trivially satisfied.
Now consider the following computation for the case $a, b \neq 0, 1$ with the assumption $a \leq b$:

$$
\begin{aligned}
a \leq b &\Rightarrow a \cdot b = a && \text{definition of } \leq \\
&\Rightarrow a \odot b = a && \text{definition of } \odot \text{ for } a, b \neq 0, 1 \\
&\Rightarrow \forall z[\mathbf{P}(z, a) \wedge \mathbf{P}(z, b) \leftrightarrow \mathbf{P}(z, a)] && \text{UCMT.2} \\
&\Rightarrow \forall z[\mathbf{P}(z, b) \leftarrow \mathbf{P}(z, a)] && \\
&\Rightarrow \forall z[\neg\mathbf{P}(z, b) \rightarrow \neg\mathbf{P}(z, a)] && \\
&\Rightarrow \forall z[\mathbf{P}(z, \ominus b) \rightarrow \mathbf{P}(z, \ominus a)] && \text{UCMT.7} \\
&\Rightarrow \forall z[\mathbf{P}(z, \ominus a) \wedge \mathbf{P}(z, \ominus b) \leftrightarrow \mathbf{P}(z, \ominus b)] && \\
&\Rightarrow \ominus b \odot \ominus a = \ominus b && \text{UCMT.2} \\
&\Rightarrow \ominus b \cdot \ominus a = \ominus b && \text{definition of } \odot \text{ for } a, b \neq 0, 1 \\
&\Rightarrow b^{\perp} \cdot a^{\perp} = b^{\perp} && \text{definition of } {}^{\perp} \text{ for } a, b \neq 0, 1 \\
&\Rightarrow b^{\perp} \leq a^{\perp} && \text{definition of } \leq
\end{aligned}
$$

Then $b^{\perp} \leq a^{\perp}$ follows as well. Hence the structure $\mathcal{L}$ is an ortholattice.

(ii) The contact relation **C** satisfies C0 by definition and C1 − C3 follow directly from C.1 − C.3 of a **UCMT**.

Thus, the structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle \mathbf{M} \cup \{0\}, \cdot, +, {}^{\perp}, 0, 1\rangle, \mathbf{C})$ is an OCA. □

We can obtain an analogous result for **UGMT** in terms of complete OCAs. First, we define what it means for a lattice to be complete—a second-order property similar to the fusion operator in **UGMT**.

Definition 4.15. *Let $\langle \mathbf{L}, \cdot, +, 0, 1\rangle$ be lattice. It is complete if and only if it is closed under arbitrary meets, that is*

$$\forall \mathbf{S} \subseteq \mathbf{L}\ \exists x \in \mathbf{L} : x = \prod \mathbf{S}$$

A complete lattice is also complete under arbitrary joins, i.e.,

$$\forall \mathbf{S} \subseteq \mathbf{L}\ \exists x \in \mathbf{L} : x = \sum \mathbf{S}.$$

These so-called fusion operators $\sum$ and $\prod$ are often alternatively denoted as $\bigvee$ and $\bigwedge$, respectively. We call a contact algebra complete if its underlying lattice is complete.

Then, the next corollary immediately follows.

Corollary 4.1. *Let $\mathcal{M}$ be a model of* **UGMT** *with domain* **M**.
Then $\mathcal{M}$ with the extended domain $\mathbf{L} = \mathbf{M} \cup \{0\}$ where $0 \notin \mathbf{M}$ and with the definitions

$$\prod \mathbf{S} = \begin{cases} 0 & \text{if } 0 \in \mathbf{S} \text{ or } \neg\exists z \big(\forall x \in \mathbf{S}\ (\langle z, x\rangle \in \mathbf{P}_{\mathcal{M}})\big) \\ \bigodot \mathbf{S} & \text{otherwise} \end{cases}$$

$$\sum \mathbf{S} = \begin{cases} 0 & \text{if } \forall x \in \mathbf{S}\ (x = 0) \\ \bigoplus (\mathbf{S} \setminus \{0\}) & \text{otherwise} \end{cases}$$

$$x^{\perp} = \begin{cases} 0 & \text{if } x = u \\ 1 & \text{if } x = 0 \\ \ominus x & \text{otherwise} \end{cases}$$

$$1 = u$$

$$x\mathbf{C}y = \begin{cases} false & \text{if } x = 0 \text{ or } y = 0 \\ \langle x, y\rangle \in \mathbf{C}_{\mathcal{M}} & \text{otherwise} \end{cases}$$

defines an structure $\mathcal{A} = (\mathcal{L}, \mathbf{C}) = (\langle \mathbf{L}, \cdot, +, ^{\perp}, 0, 1\rangle, \mathbf{C})$ that is a complete OCA.

Proof. With $\mathcal{M}$ being a model of **UGMT**, it is also a model of **UCMT**. Hence we can construct an OCA $(\mathcal{L}, \mathbf{C})$ from $\mathcal{M}$ according to Theorem 4.2 because the binary definitions for intersection, $\cdot$, and sum, $+$, follow from the fusion definitions. Because the lattice $\mathcal{L}$ is complete by the very existence of the fusions of arbitrary sets of entities, the OCA is also complete. □

Theorem 4.2 and Corollary 4.1 enable us to focus on the topological representability or embeddability of (complete) OCAs in order to study representability of all the models of **UCMT** and of **UGMT**.

4.3 Mereotopologically representable complete OCAs

The study of topological representability of algebraic structures has a long tradition established in the seminal work by [Sto36] on the duality between Boolean algebras and the topological spaces now known as Stone spaces. Since then, many generalizations thereof have been found. Here, we are not interested in full duality, but rather in embeddings of OCAs (with lattices as core) in a topological space in a way that preserves the mereotopological structure, i.e., gives point-set interpretations to all lattice elements so that parthood and contact also have point-set interpretations that reflect their intended spatial meaning. If an OCA has such a topological representation or embedding, we call it *spatially representable*. But instead of giving a complete definition of spatial representability, we only partially define it by giving a few necessary conditions that must hold in a spatially representable OCA. Every OCA that satisfies these conditions is called *mereotopologically representable* (MT-representable). Then we have for all OCAs

$$\text{spatially representable} \Rightarrow \text{MT-representable}$$

but not its converse, i.e.

$$\text{MT-representable} \not\Rightarrow \text{spatially representable}$$

Nevertheless, by showing that MT-representable complete OCAs are pseudocomplemented and satisfy the Stone identity we can conclude the same for spatially representable complete OCAs. Thus, MT-representability restricts the behaviour of complementation in the lattice structure of the algebraic counterparts resulting from models of **UGMT**. Translated into the realm of the logical theories, we essentially show that all models of **UGMT** that have some spatial representation must have an algebraic structure whose lattice is a Stonian p-ortholattice. This defines a weakest class of equidimensional mereotopologies with unique closures under arbitrary sums, arbitrary joins, and under complementation.

A few words on our notation: Sets are denoted by capital letters to distinguish them from lattice elements. $h(a)$ denotes the set that a lattice element a is represented by. We already introduced our notation for topological spaces on page 35. In addition, the following set-theoretic equivalences in topological spaces are used without further mentioning.

Lemma 4.1. *Let $\langle \mathbf{X}, \tau \rangle$ be a topological space. Then for all sets $\mathbf{S} = \{\mathbf{A} : \mathbf{A} \subseteq \mathbf{X}\}$,*

$$\mathrm{int}\Big(\bigcap_{\mathbf{A}\in\mathbf{S}} \mathbf{A}\Big) = \bigcap_{\mathbf{A}\in\mathbf{S}} \mathrm{int}(\mathbf{A}) \quad \text{and} \quad \mathrm{cl}\Big(\bigcup_{\mathbf{A}\in\mathbf{S}} \mathbf{A}\Big) = \bigcup_{\mathbf{A}\in\mathbf{S}} \mathrm{cl}(\mathbf{A}),$$
$$\mathrm{int}\Big(\bigcup_{\mathbf{A}\in\mathbf{S}} \mathbf{A}\Big) \supseteq \bigcup_{\mathbf{A}\in\mathbf{S}} \mathrm{int}(\mathbf{A}) \quad \text{and} \quad \mathrm{cl}\Big(\bigcap_{\mathbf{A}\in\mathbf{S}} \mathbf{A}\Big) \subseteq \bigcap_{\mathbf{A}\in\mathbf{S}} \mathrm{cl}(\mathbf{A}).$$

4.3.1 MT-representability

For an OCA to be spatially representable, we require that a lattice homomorphism h into a set of subsets of $\mathbf{X}$ of a topological space $(\mathbf{X}, \tau)$ exists. The binary lattice operations $\cdot$ and $+$ correspond to binary operations $\sqcap$ and $\sqcup$ defined over the subsets of $\mathbf{X}$. They may map to standard set intersection $\cap$ and union $\cup$ in the topological space, though this is not required. The infinitary versions of $\cdot$ and $+$ that must exist in complete lattices then map to infinitary version of $\sqcap$ and $\sqcup$, which we denote as $\bigsqcap$ and $\bigsqcup$. Notice that as a lattice homomorphism, h must preserve joins and meet, i.e., $h(x \cdot y) = h(x) \sqcap h(y)$ and $h(x + y) = h(x) \sqcup h(y)$. In particular, we must have $h(x) \subseteq h(y) \iff x \cdot y = y$. In other words, the lattice order $\leq$ and thus the inherent parthood order, P, is preserved as subset inclusion $\subseteq$ in the representing topological space.

We are now ready to define MT-representability of a complete OCA.

Definition 4.16. *Let $\mathcal{A} = (\langle \mathbf{L}, +, \cdot, ^{\perp}, 0, 1\rangle, \mathbf{C})$ be a complete OCA.*
It is called <u>MT-representable</u> iff there is some topological space $\langle \mathbf{X}, \tau \rangle$ and an injective lattice homomorphism h from $\mathbf{L}$ into the structure $\langle \mathcal{T}, \sqcap, \sqcup \rangle$ where $\mathbf{T} \subseteq \mathbf{X}$ for each $\mathbf{T} \in \mathcal{T}$ and the following conditions are satisfied:

1. *$h(1) = \mathbf{X}$ and $h(0) = \emptyset$;*
2. *for all sets $\mathbf{S} \subseteq \mathbf{L}$ we have*

$$\bigcap_{x\in\mathbf{S}} \mathrm{int}(h(x)) \subseteq h\Big(\prod \mathbf{S}\Big) = \bigsqcap_{x\in\mathbf{S}} h(x) \subseteq \bigcap_{x\in\mathbf{S}} \mathrm{cl}(h(x)) \text{ and}$$

$$\bigcup_{x\in\mathbf{S}} \mathrm{int}(h(x)) \subseteq h\Big(\sum \mathbf{S}\Big) = \bigsqcup_{x\in\mathbf{S}} h(x) \subseteq \bigcup_{x\in\mathbf{S}} \mathrm{cl}(h(x));$$

3. *any* $x \in \mathbf{L}$ *is regular, i.e. satisfies* $\text{int}(x) = \text{int}(\text{cl}(x))$ *and* $\text{cl}(\text{int}(x)) = \text{cl}(x)$*;*

4. *for all* $x, y \in \mathbf{L}$*, if* $\text{int}(h(x)) \cap \text{int}(h(y)) \neq 0$ *then* $x\mathbf{C}y$*;*

5. *for all* $x, y \in \mathbf{L}$*, if* $\text{cl}(h(x)) \cap \text{cl}(h(y)) = 0$ *then* $x\neg\mathbf{C}y$*.*

Condition (1) ensures that the embedding topological space is not larger than necessary, while condition (2) ensures that the set that represents the meet (or join) of a set of entities differs only in boundaries from the point-set intersection (union) of their representing sets. More specifically, the representation of the meet (join) of a set of entities is not smaller (greater) than the intersection (union) of the interiors of their representations and not larger (smaller) than the intersection (union) of the closures of their representations. Condition (3) ensures that all lattice elements are represented by regular sets, so that all elements apart from 0 have a nonempty interior, that is, an MT-representable complete OCA satisfies

3′. for all $x \in \mathbf{L}$, $\text{int}(h(x)) = \emptyset$ if and only if $x = 0$.

To prove the direction $\rightarrow$ of this implication, assume $\text{int}(h(x)) = \emptyset$ for some $x \in \mathbf{L}$. Then from

$$h(x) \subseteq \text{cl}(h(x)) = \text{cl}(\text{int}(h(x))) = \text{cl}(\emptyset) = \emptyset$$

it immediately follows that $x = 0$ because h is injective and satisfies condition (1).

The conditions (4) and (5) ensure that contact is adequately interpreted so that any two entities whose representations share a point are indeed in contact, while if the closures of their representations do not share a point, they are not in contact. Finally, if $x \cdot y = 0$ and $x + y = 1$ then $h(x) \sqcap h(y) = \emptyset$ and $h(x) \sqcup h(y) = \mathbf{X}$. Then from conditions (2) and (3) of Definition 4.16 we deduce the following additional condition,

6. for all $x, y \in \mathbf{L}$, if $x \cdot y = 0$ and $x + y = 1$ then $\text{int}(\mathbf{X} \setminus h(x)) \subseteq h(y) \subseteq \text{cl}(\mathbf{X} \setminus h(x))$,

which follow immediately from the next two derivations:

$$\begin{aligned}
\text{int}(\mathbf{X} \setminus h(x)) &= \mathbf{X} \setminus \text{cl}(h(x)) && \text{int}(\mathbf{X} \setminus \mathbf{A}) = \mathbf{X} \setminus \text{cl}(\mathbf{A}) \\
&\subseteq \text{cl}(h(y)) && \text{Definition 4.16(2) implies } \text{cl}(h(y)) \supseteq \mathbf{X} \setminus \text{cl}(h(x)) \\
&\subseteq \text{int}(\text{cl}(h(y))) && \text{apply int() on both sides} \\
&= \text{int}(h(y)) && \text{Definition 4.16(3)} \\
&\subseteq h(y)
\end{aligned}$$

$$\begin{aligned}
\text{cl}(\mathbf{X} \setminus h(x)) &= \mathbf{X} \setminus \text{int}(h(x)) && \text{cl}(\mathbf{X} \setminus \mathbf{A}) = \mathbf{X} \setminus \text{int}(\mathbf{A}) \\
&\supseteq \text{int}(h(y)) && \text{Definition 4.16(2) implies } \text{int}(h(y)) \subseteq \mathbf{X} \setminus \text{int}(h(x)) \\
&\supseteq \text{cl}(\text{int}(h(y))) && \text{apply cl() on both sides} \\
&= \text{cl}(h(y)) && \text{Definition 4.16(3)} \\
&\supseteq h(y)
\end{aligned}$$

In other words, any complement of an entity x in an OCA is mapped by h to a point set that is at least as large as the interior of the topological complement of x and no larger than the closure of the topological complement of x.

Special versions of MT-representability are representability by regular closed (or regular open) sets or by regular sets as for the Boolean Contact Algebras (BCAs) with $x\neg\mathbf{C}y \leftrightarrow x < y^{\perp}$ or the Stonian p-ortholattices with $x\neg\mathbf{C}y \leftrightarrow x \leq y^{\perp}$. In other words, lattices representable by regular closed sets of a topological space, such as BCAs, satisfy all conditions of Definition 4.16. Key is that conditions (2) to (5) are satisfied if we use $\cap$ as $\sqcap$ and if we have $\mathrm{cl}(x) = x$ for all $x \in \mathbf{L}$; (2) then simplifies (in the binary case) to $\mathrm{int}(h(x)) \cap \mathrm{int}(h(y)) \subseteq h(x) \cap h(y) \subseteq h(x) \cap h(y)$ which is trivially true. Conditions (4) and (5) then amount to $\mathrm{cl}(h(x)) \cap \mathrm{cl}(h(y)) \neq 0 \Leftrightarrow x\mathbf{C}y$ which is satisfied once we define $x\neg\mathbf{C}y \leftrightarrow x < y$ as in BCAs, compare the definition of contact in [DW05a]. For the representation of Stonian p-ortholattices by regular sets, we can choose $x \sqcap y = x \cap y \cap \mathrm{int}(\mathrm{cl}(x \cap y))$ to satisfy condition (2) while the conditions (4) and (5) are satisfied if we define $h(x) \cap h(y) \neq 0 \Leftrightarrow x\mathbf{C}y$, compare the definition of contact in [AV95; HWG09]. Now we can prove the first property of MT-representable complete OCAs.

Theorem 4.3. *An MT-representable complete OCA is pseudocomplemented.*

Proof. Suppose $\mathcal{A} = (\langle \mathbf{L}, +, \cdot, ^{\perp}, 0, 1\rangle, \mathbf{C})$ is an MT-representable complete OCA. Let $x \in \mathbf{L}$ be an arbitrary lattice element. We will show that it must have a pseudocomplement in $\mathbf{L}$.
Let $\mathbf{S}_x = \{x_i^* : x_i^* \in \mathbf{L}$ and $x \cdot x_i^* = 0\} \subseteq \mathbf{L}$ denote the set of meet-complements of x in $\mathbf{L}$. Because $\mathcal{A}$ is a complete lattice, we have $\sum \mathbf{S}_x \in \mathbf{L}$. We will now show that $x \cdot \sum \mathbf{S}_x = 0$ and thus $\sum \mathbf{S}_x \in \mathbf{S}_x$.

Note that all $x_i^* \in \mathbf{S}_x$ not only satisfy $x \cdot x_i^* = 0$ but also $x + x_i^* \geq x + x^{\perp} = 1$, allowing us to utilize Definition 4.16(6) in the following computation:

$$
\begin{aligned}
\mathrm{int}\big(h(x \cdot \textstyle\sum \mathbf{S}_x)\big) &\subseteq \mathrm{int}\big(\mathrm{cl}\big[h(x) \cap h\big(\textstyle\sum \mathbf{S}_x\big)\big]\big) && \text{Def. 4.16(2)}\\
&\subseteq \mathrm{int}\big(\mathrm{cl}\big[h(x) \cap \mathrm{cl}\big(\bigcup_{y \in \mathbf{S}_x} h(y)\big)\big]\big) && \text{Def. 4.16(2)}\\
&\subseteq \mathrm{int}\big(\mathrm{cl}\big[h(x) \cap \bigcup_{y \in \mathbf{S}_x} \mathrm{cl}(h(y))\big]\big) && \text{Lemma 4.1}\\
&\subseteq \mathrm{int}\big(\mathrm{cl}\big[h(x) \cap \bigcup_{y \in \mathbf{S}_x} \mathrm{cl}(\mathrm{cl}(\mathbf{X} \setminus h(x)))\big]\big) && \text{Def. 4.16(6)}\\
&= \mathrm{int}\big(\mathrm{cl}\big[h(x) \cap \mathrm{cl}[\mathbf{X} \setminus h(x)]\big]\big) && \mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(\mathbf{A})\\
&\subseteq \mathrm{int}\big(\mathrm{cl}(h(x)) \cap \mathrm{cl}[\mathbf{X} \setminus h(x)]\big) && \text{Lemma 4.1}\\
&= \mathrm{int}(\mathrm{cl}(h(x))) \cap \mathrm{int}\big(\mathrm{cl}[\mathbf{X} \setminus h(x)]\big) && \text{Lemma 4.1}\\
&= \mathrm{int}(h(x)) \cap \mathrm{int}\big(\mathrm{cl}[\mathbf{X} \setminus h(x)]\big) && \text{Def. 4.16(3)}\\
&= \mathrm{int}(h(x)) \cap \mathrm{int}\big(\mathbf{X} \setminus \mathrm{int}(h(x))\big) && \mathrm{cl}(\mathbf{X} \setminus \mathbf{A}) = \mathbf{X} \setminus \mathrm{int}(\mathbf{A})\\
&= \mathrm{int}(h(x)) \cap \big(\mathbf{X} \setminus \mathrm{cl}(\mathrm{int}(h(x)))\big) && \mathrm{int}(\mathbf{X} \setminus \mathbf{A}) = \mathbf{X} \setminus \mathrm{cl}(\mathbf{A})\\
&= \mathrm{int}(h(x)) \cap \big(\mathbf{X} \setminus \mathrm{cl}(h(x))\big) && \text{Def. 4.16(3)}\\
&= \big(\mathrm{int}(h(x)) \cap \mathbf{X}\big) \setminus \big(\mathrm{int}(h(x)) \cap \mathrm{cl}(h(x))\big) && \mathbf{A} \cap (\mathbf{B} \setminus \mathbf{C}) = (\mathbf{A} \cap \mathbf{B}) \setminus (\mathbf{A} \cap \mathbf{C})\\
&= \mathrm{int}(h(x)) \setminus \big(\mathrm{int}(h(x)) \cap \mathrm{cl}(h(x))\big) && \mathrm{int}(h(x)) \subseteq \mathbf{X}\\
&= \mathrm{int}(h(x)) \setminus \mathrm{int}(h(x)) && \mathrm{int}(h(x)) \subseteq \mathrm{cl}(h(x))\\
&= \emptyset
\end{aligned}
$$

By Definition 4.16(1) and (3′) we conclude that $x \cdot \sum \mathbf{S}_x = 0$. Hence, $\sum \mathbf{S}_x$ is the pseudocomplement of x, i.e., a meet-complement of x greater than or equal to any $x_i^* \in \mathbf{S}_x$. Thus any element in $\mathcal{A}$ must have a pseudocomplement. Consequently, $\mathcal{A}$ is pseudocomplemented. □

The restriction to complete lattices essentially shifts the focus from **UCMT** to **UGMT**. Notice, however, that all discrete models of **UCMT** are trivially complete.

Now we prove that in an MT-representable OCA the Stone identity must also hold. First, recall that a pseudocomplemented ortholattice is also quasicomplemented, which also applies to contact algebras defined over those lattices. In the following, we utilize the fact that MT-representable complete OCAs are quasicomplemented to prove that they satisfy the Stone property. We exploit the fact that $h(x) \rightarrow h(x^{++})$ is an interior mapping in the topological sense for a quasicomplemented OCA given condition 2 of Definition 4.16, see [HWG09] for details, that is,

$$h(x^{++}) = \text{int}(h(y)) \qquad (+)$$

This is well known for Boolean lattices which are representable by the regular open sets of a topological space. More generally, it can be justified by considering that by the definition of a quasicomplement, x^+ is the smallest entity so that $x + x^+ = 1$. We then have

$$h(x^+ + x^{++}) = h(x^+ + x^{\perp +}) = h(x^+) \cup h(x^{\perp +}) = h(1) = \mathbf{X},$$

which is an open set in every topological space.

Analogously, $h(x) \rightarrow h(x^{**})$ is a closure mapping in the representation of a pseudocomplemented OCA given condition 2 of Definition 4.16, that is,

$$h(x^{**}) = \text{cl}(h(y)) \qquad (*)$$

We further need the following result from [HWG09].

Lemma 4.2. *Let $\langle \mathbf{L}, +, \cdot, ^*, ^\perp, 0, 1\rangle$ be a p-ortholattice. Then we have*

1. $a^{**} = (a^{++})^{**}$
2. $a^{++} = (a^{**})^{++}$

We are now ready to prove the Stone identity for MT-representable, quasicomplemented OCAs.

Theorem 4.4. *An MT-representable OCA $\mathcal{A} = \langle(\mathbf{L}, +, \cdot, ^\perp, ^+, 0, 1), \mathbf{C}\rangle$ satisfies $(x \cdot y)^{++} = x^{++} \cdot y^{++}$ for all $x, y \in \mathbf{L}$.*

Proof. Suppose $\mathcal{A} = \langle(\mathbf{L}, +, \cdot, ^\perp, ^+, 0, 1), \mathbf{C}\rangle$ is an MT-representable quasicomplemented OCA. Let $x, y \in \mathbf{L}$ denote two arbitrary lattice elements. We prove the two directions $(x \cdot y)^{++} \subseteq x^{++} \cdot y^{++}$ and $(x \cdot y)^{++} \supseteq x^{++} \cdot y^{++}$ individually.

First $(x \cdot y)^{++} \subseteq x^{++} \cdot y^{++}$ follows from

$$\begin{aligned}
h((x \cdot y)^{++}) &= \text{int}(h(x \cdot y)) && (+)\\
&\subseteq \text{int}(\text{cl}(h(x)) \cap \text{cl}(h(y))) && \text{Def. 4.16(2)}\\
&= \text{int}(h(x^{**}) \cap h(y^{**})) && (*)\\
&= \text{int}(\text{int}(h(x^{**}) \cap h(y^{**}))) && \text{int}(\text{int}(\mathbf{A})) = \text{int}(\mathbf{A})\\
&= \text{int}(\text{int}(h(x^{**})) \cap \text{int}(h(y^{**}))) && \text{Lemma 4.1}\\
&= \text{int}(h(x^{**++}) \cap h(y^{**++})) && (+)
\end{aligned}$$

$$\begin{aligned} &\subseteq h(x^{**++} \cdot y^{**++}) && \text{Def. 4.16(2)} \\ &= h(x^{++} \cdot y^{++}) && \text{Lemma 4.2} \end{aligned}$$

For the other direction, $(x \cdot y)^{++} \supseteq x^{++} \cdot y^{++}$, suppose $(x \cdot y)^{++} \not\supseteq x^{++} \cdot y^{++}$. Then $h((x \cdot y)^{++}) \not\supseteq h(x^{++} \cdot y^{++})$ and there must exist a nonempty set z so that $h(z) \subseteq h(x^{++} \cdot y^{++})$ but $h(z) \not\subseteq h((x \cdot y)^{++})$. By Def. 4.16(3), we know that $\text{int}(h(z))$ is nonempty; hence we assume

$$\text{int}(h(z)) \subseteq \text{int}(h(x^{++} \cdot y^{++})) \qquad \text{assumption}$$

while $\text{int}(h(z)) \not\subseteq \text{int}(h((x \cdot y)^{++}))$ is contradicted by the following computation:

$$\begin{aligned} \text{int}(h(z)) &\subseteq \text{int}(h(x^{++} \cdot y^{++})) && \text{assumption} \\ &\subseteq \text{int}(\text{cl}(h(x^{++}) \cap h(y^{++}))) && \text{Def. 4.16(2)} \\ &= \text{int}(\text{cl}(\text{int}(h(x) \cap \text{int}(h(y)))) && (+) \\ &\subseteq \text{int}(\text{cl}(h(x \cdot y))) && \text{Def. 4.16(2)} \\ &= \text{int}(\text{int}(\text{cl}(h(x \cdot y)))) && \text{int}(\text{int}(\mathbf{A})) = \text{int}(\mathbf{A}) \\ &= \text{int}(\text{int}(h((x \cdot y)^{**}))) && (*) \\ &= \text{int}(h((x \cdot y)^{**++})) && (+) \\ &= \text{int}(h((x \cdot y)^{++})) && \text{Lemma 4.2} \end{aligned}$$

With h being an injective lattice homomorphism, we conclude $(x \cdot y)^{++} = x^{++} \cdot y^{++}$. □

We thereby proved that one version of the Stone property, namely condition (3) from Theorem 4.1, is satisfied in any MT-representable OCA. This leads us to the definition of SPOCAs as a subclass of OCAs which contains all complete OCAs that are MT-representable. SPOCAs can be axiomatized algebraically by the theory

$$\mathit{SPOCA} = \{\text{L2}^{\vee}-\text{L6}^{\vee},\ \text{L2}^{\wedge}-\text{L4}^{\wedge},\ \text{O1}'-\text{O3}',\ \text{PC1}',\ \text{PC2}',\ \text{PC}'',\ \text{S},\ \text{C0}-\text{C3}\},$$

see Appendix F.1 for the axioms, and see [WHG12] for more explanations and a reduction of this nonminimal theory.

Definition 4.17. *A Stonian pseudocomplemented and orthocomplemented contact algebra (SPOCA) is a structure* $(\langle \mathbf{L}, \cdot, +, ^{\perp}, ^{+}, 0, 1\rangle, \mathbf{C})$ *such that*

1. $\langle \mathbf{L}, \cdot, +, ^{+}, ^{\perp}, 0, 1\rangle$ *is a Stonian p-ortholattice;*
2. $\mathbf{C}$ *satisfies C0 to C3.*

The following corollary summarizes our result of this section:

Corollary 4.2. *An MT-representable complete OCA is a complete SPOCA.*

As a consequence, from now on we can focus our attention to SPOCAs without worrying that other spatially representable classes of contact algebras may be overlooked. The only case we have not accounted for are lattices that are not complete. It is, however, unlikely that any such class is of relevance for a spatially representable mereotopology.

4.4 Closure operations in SPOCAs

In this section, we give a mereological and a topological definition of each of the closure operations sum, intersection, complement, and universal; closely adhering to the definitions presented in [CV99a]. We investigate whether each of the four closure operations are defined in either (or in both) ways in general SPOCAs. For those mereological or topological closure operations that are not entailed, we identify equivalent algebraic properties. Surprisingly, very few such additional properties are necessary; the necessary ones primarily arise from complements being defined mereologically or topologically. If we define complements mereologically, the arising SPOCAs are distributive, while defining complements topologically allows SPOCAs whose underlying Stonian p-ortholattices are non-distributive. In the later case, the contact relation must be more restricted. The resulting two main types of SPOCAs are explored in detail in Section 4.5.

Generally, we expect each of the closure operations to be defined at least mereologically or topologically. But from an ontologically sound theory of mereotopology, we expect further that all closure operations are defined consistently, e.g., either all are defined mereologically or all are defined topologically. We use the following terminology, the axioms follow shortly.

Definition 4.18. *A* **UCMT** *is* <u>*M-closed*</u> *iff it satisfies* $M\text{-}I_{\text{UCMT}}$, $M\text{-}S_{\text{UCMT}}$, *and* $M\text{-}C_{\text{UCMT}}$.

Definition 4.19. *A* **UCMT** *is* <u>*T-closed*</u> *iff it satisfies* $T\text{-}I_{\text{UCMT}}$, $T\text{-}S_{\text{UCMT}}$, $T\text{-}C_{\text{UCMT}}$, *and Dis.*

Definition 4.20. *A* **UCMT** *is* <u>*T′-closed*</u> *iff it satisfies* $T\text{-}I_{\text{UCMT}}$, $T\text{-}S_{\text{UCMT}}$, $T\text{-}C'_{\text{UCMT}}$, *and Dis.*

A **UCMT** is then *coherently closed* (C-closed) if it is defined in one of those three ways.

Definition 4.21. *A* **UCMT** *is* <u>*C-closed*</u> *iff it is M-closed, T-closed, or T′-closed.*

Ideally, the closure operations can be defined mereologically and topologically at the same time. Then we call it *MT-closed* (mereotopologically-closed).

Definition 4.22. *A* **UCMT** *is* <u>*MT-closed*</u> *iff it is*

1. *M-closed, and*
2. *T-closed or T′-closed.*

We use all of these properties for both the logical theories and their corresponding algebraic theories.

4.4.1 Mereological closure operations

The closure operations intersection, sum, and complementation can be defined mereologically as follows. It is easily verified that these are consistent with UCMT.6 and UCMT.7.

(M-I$_{\text{UCMT}}$) $\forall w[P(w, x \odot y) \leftrightarrow (P(w,x) \wedge P(w,y))]$ (intersection)

(M-S$_{\text{UCMT}}$) $\forall w[O(w, x \oplus y) \leftrightarrow (O(w,x) \vee O(w,y))]$ (sum)

(M-C$_{\text{UCMT}}$) $\forall w[O(w, \ominus x) \leftrightarrow \neg P(w,x)]$ (complement)

In the sequel we will exclusively use the algebraic equivalents of these axioms as found in Appendix F.1. These differ only slightly from the above axioms to account for the additional bottom element 0 in a contact algebra, see Lemma F.1 in Appendix F.2 for the proof of the equivalence of the two versions.

Notice that the universal u (denoted by 1 in the algebraic counterpart) is always defined mereologically as $\forall x\ [P(x, \text{u})]$. Moreover, we can easily prove that the algebraic equivalents of M-I$_{\text{UCMT}}$ and M-S$_{\text{UCMT}}$, i.e., M-I and M-S, are theorems in SPOCAs.

Lemma 4.3. *SPOCA* $\models$ *M-I*

Lemma 4.4. *SPOCA* $\models$ *M-S*

M-C does not necessarily hold in SPOCAs. Defining complementation mereologically requires the SPOCA to be uniquely complemented and thus distributive and Boolean.

(Uni) $(x \cdot y = 0 \wedge x + y = 1 \wedge x \cdot z = 0 \wedge x + z = 1) \rightarrow y = z$ (unicomplemented)

Lemma 4.5. *SPOCA* $\models$ *M-C* $\leftrightarrow$ *Uni*

Proof. Since unicomplemented ortholattices are Boolean and vice versa it suffices to show that a unicomplemented SPOCA satisfies the algebraic equivalence of M-C: $z \cdot x^{\perp} \neq 0 \leftrightarrow z \nleq x$ and that a SPOCA satisfying this property is unicomplemented. This has been done using the automated theorem prover. □

For the sums and complements to be unique, we further need extensionality of O postulated as O-Ext. Recall that $\neg O(x, y) \iff x \cdot y = 0$.

(O-Ext) $\forall z(z \cdot x = 0 \leftrightarrow z \cdot y = 0) \leftrightarrow x = y$ (O-extensionality)

But from M-C we can already prove extensionality of O.

Lemma 4.6. *SPOCA* $\cup$ *M-C* $\models$ *O-Ext*

We obtain the following corollary on the effects of mereological closures in SPOCAs.

Corollary 4.3. *A SPOCA is M-closed iff it is unicomplemented. An M-closed SPOCA is O-extensional.*

4.4.2 Topological closure operations

The closure operations intersection, sum, and complementation can be defined topologically as following. Again, their algebraic versions are found in Appendix F.1 with Lemma F.2 in Appendix F.2 proving the equivalence of both versions. It is easily verified that these are consistent with UCMT.6 and UCMT.7. There are two slightly distinct ways of defining topological complements, denoted by T-C and T-C′.

(T-I$_{\text{UCMT}}$) $\forall w[C(w, x \odot y) \rightarrow (C(w, x) \wedge C(w, y))]$ (intersection)

(T-S$_{\text{UCMT}}$) $\forall w[C(w, x \oplus y) \leftrightarrow (C(w, x) \vee C(w, y))]$ (sum)

(T-C$_{\text{UCMT}}$) $\forall w[P(w, \ominus x) \leftrightarrow \neg C(w, x)]$ (complement)

(T-C$'_{\text{UCMT}}$) $\forall w[PP(w, \ominus x) \leftrightarrow \neg C(w, x)]$ (alternative complement)

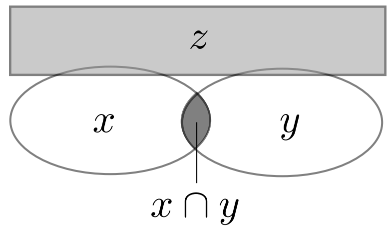


Figure 4.5: Two regions x and y connected to z whose set intersection is not connected to z due to the nontransitive nature of contact.

Notice that since the universal is always defined mereologically as $\forall x\ P(x, \mathrm{u})$, it is also automatically defined topologically as $\forall x\ C(x, \mathrm{u})$. However, this does not guarantee the topological uniqueness of the universal, i.e., that $\forall y\ [(\forall x\ C(x, y)) \rightarrow y = \mathrm{u}]$ holds. Therefore, we introduce Dis, which has been previously used to study contact algebras. Recall that the algebraic equivalent of u is 1.

(Dis) $\forall x[x \neq 1 \rightarrow \exists y(x \neg \mathbf{C} y)]$ (only the top element is connected to all entities)

Intersections are always defined topologically in SPOCAs. Notice however that T-I only contains a simple implication and not a biconditional. The reverse direction is not desirable as Figure 4.5 illustrates.

Lemma 4.7. *SPOCA* $\models$ *T-I*

Proof. Follows directly from C3. □

Moreover, SPOCAs satisfy one direction of the implication in the axiom T-S, namely the direction T-S$^{\leftarrow}$ (the algebraic version of T-S$^{\leftarrow}_{\mathrm{UCMT}}$).

(T-S$^{\leftarrow}_{\mathrm{UCMT}}$) $\forall w[C(w, x \oplus y) \leftarrow (C(w, x) \vee C(w, y))]$

Lemma 4.8. *SPOCA* $\models$ *T-S*$^{\leftarrow}$

Proof. Follows directly from C3. □

Since the reverse direction of T-S does not always hold, we use C4 to guarantee that sums are defined topologically in SPOCAs, i.e., if an element x is connected to another element z, it is also connected to one of the parts of z that make up z.

(C4) $x\mathbf{C}(y + z) \rightarrow x\mathbf{C}y \vee x\mathbf{C}z$ (topological sum)

Lemma 4.9. *SPOCA* $\cup$ *C4* $\models$ *T-S*

Proof. Immediately follows from Lemma 4.8. □

Topological complement operation

Now we turn to the complement. We have two options, using either T-C$_{\mathrm{UCMT}}$ or T-C$'_{\mathrm{UCMT}}$. We first study T-C$_{\mathrm{UCMT}}$ and then proceed with T-C$'_{\mathrm{UCMT}}$. In SPOCAs, T-C$_{\mathrm{UCMT}}$ is captured algebraically by C5 which requires an element to be in contact to all elements that are not parts of its orthocomplement. In particular, for any x, $x \neg \mathbf{C} x^{\perp}$.

(C5) $z\mathbf{C}x \leftrightarrow z \not\leq x^{\perp}$ (topological complement)

Interestingly, C5 alone is sufficient to ensure that T-S holds and that C is extensional, i.e., C4 and C-Ext are satisfied in all SPOCAs which satisfy C5. C-Ext expresses extensionality of C, that is, two elements are considered identical if they are in contact to exactly the same elements. C-extensionality is equivalent to requiring that a mereotopology can be reconstructed from contact as the only primitive relation. It further ensures topological uniqueness of the universal (Dis).

(C-Ext) $\forall z(z\mathbf{C}x \leftrightarrow z\mathbf{C}y) \leftrightarrow x = y$ (C-extensionality)

Lemma 4.10. *SPOCA* $\cup$ *C5* $\models$ *C4*

Lemma 4.11. *SPOCA* $\cup$ *C5* $\models$ *{C-Ext, Dis}*

Moreover, Int must hold in SPOCAs satisfying C5. This seems, however, coincidental and owed to the fact that elements are disconnected from their complements, that is, ¬Con holds. Despite its name, ¬Con is not the negation of Con but the exact opposite assumption. Con is inconsistent with a nontrivial SPOCA satisfying C5. Int and Con have previously only been used in the context of contact algebras with Boolean lattices, but easily generalize to SPOCAs. In our study we include Int only for completeness purposes, it is not motivated by or directly related to the closure operations.

(Con) $\forall x \neq 0, 1[x\mathbf{C}x^{\perp}]$ (connected complements)

(¬Con) $\forall x[x\neg\mathbf{C}x^{\perp}]$ (disconnected complements)

(Int) $\forall x, y[x\neg\mathbf{C}y \rightarrow \exists z(x\neg\mathbf{C}z \wedge y\neg\mathbf{C}z^{\perp})]$ (interpolation)

Lemma 4.12. *SPOCA* $\cup$ *C5* $\models$ *¬Con*

Proof. Choose $y = x^{\perp}$ in C5 to obtain $x\neg\mathbf{C}x^{\perp}$. □

Lemma 4.13. *SPOCA* $\cup$ *C5* $\models$ *Int*

Proof. Choosing $z = x^{\perp}$ in Int always evaluates to true: we obtain $x\neg\mathbf{C}y \rightarrow (x\neg\mathbf{C}x^{\perp} \wedge y\neg\mathbf{C}x^{\perp\perp})$. By Lemma 4.12 it is sufficient to prove $\forall x, y[x\neg\mathbf{C}y \rightarrow y\neg\mathbf{C}x^{\perp\perp}]$ which is with $x = x^{\perp\perp}$ the trivially true inverse of C2. □

We obtain the following corollary on the effect of topological closures in SPOCAs.

Corollary 4.4. *A SPOCA is T-closed iff it satisfies C5. A T-closed SPOCA is C-extensional and satisfies C4, ¬Con, and Int.*

Finally, we verify that C5 and Uni (unique complementation) are independent of one another, i.e., that there exist SPOCAs that satisfy C5 but are not uniquely complemented and that there exist SPOCAs with a Boolean lattice that do not satisfy C5. Both results are not very surprising.

Lemma 4.14. *SPOCA* $\cup$ *Uni* $\not\models$ *C5*

Proof. Counterexample provided in ca/theorems/spoca_Uni_C5.clif. □

Lemma 4.15. *SPOCA* $\cup$ *C5* $\not\models$ *Uni*

Proof. Counterexample provided in ca/theorems/spoca_C5_Uni.clif. □

Quasi-topological complement operation

Now we turn to T-C'_{UCMT} as an alternative to the axiom T-C_{UCMT} for defining complements topologically. T-C'_{UCMT} is captured algebraically by C5′.

(C5′) $(x \neq 0 \vee z \neq 1) \wedge (x \neq 1 \vee z \neq 0) \rightarrow [z\mathbf{C}x \leftrightarrow z \not< x^{\perp}]$ (alternative topological complement)

Obviously, C5 and C5′ are mutually inconsistent but what are the consequences of using C5′ instead of C5 to define complements? First, C5′ is in SPOCAs not sufficient to entail C4.

Lemma 4.16. *SPOCA* $\cup$ *C5′* $\not\models$ *C4*

Proof. Counterexample provided in ca/theorems/spoca_C5prime_C4.clif. □

Subsequently, we will focus on SPOCA together with C4 and C5′. It requires an element to be connected to all other elements that are not proper parts of its (ortho-)complement, in other words, Con is a theorem.

Lemma 4.17. *SPOCA* $\cup$ *C5′* $\models$ *Con*

By Lemma 4.17 C5′ is not really a topological definition of complementation since complements are connected, i.e., $x\mathbf{C}x^{\perp}$. Truly topological complements are complementary with respect to their extension of contact. In a SPOCA that satisfies C5′, none of C-Ext, Dis, Int, or Uni necessarily hold. Let us start with C-Ext: we can have models in which

$$\exists x, y\ [x \neq y \wedge \forall z\ (x\mathbf{C}z \wedge y\mathbf{C}z)].$$

Then, the universal is no longer topologically unique; this would require Dis in addition. For that reason, we refer to C5′ as a *quasi-topological complement*.

Lemma 4.18. *SPOCA* $\cup$ *{C4, C5′}* $\not\models$ *C-Ext*

Proof. Counterexample provided in ca/theorems/spoca_C4_C5prime_C-Ext.clif. □

In the presence of C5′, Int is a also theorem of SPOCAs.

Lemma 4.19. *SPOCA* $\cup$ *{C4, C5′}* $\not\models$ *Int*

Proof. Counterexample provided in ca/theorems/spoca_C4_C5prime_Int.clif. □

Finally, neither C5′ together with C4 entails Uni in SPOCAs, nor vice versa.

Lemma 4.20. *SPOCA* $\cup$ *{C4, C5′}* $\not\models$ *Uni*

Proof. Counterexample provided in ca/theorems/spoca_C4_C5prime_Uni.clif. □

Lemma 4.21. *SPOCA* $\cup$ *C5′* $\not\models$ *Uni*

Proof. Counterexample provided in ca/theorems/spoca_C5prime_Uni.clif. □

Therefore the class of SPOCAs satisfying C5′ do not necessarily have a Boolean lattice structure. Those that additionally satisfy C4 have all closure operations defined mereologically and topologically except for the complement which is defined mereologically but only quasi-topologically. The following corollary summarizes the effect of quasi-topological closures in SPOCAs.

Corollary 4.5. *A SPOCA is T′-closed iff it satisfies C4 and C5′. A T′-closed SPOCA satisfies Con.*

4.5 Coherently closed MT-representable UCMTs

We already mentioned that a **UCMT** is only ontologically coherent if it is M-closed, T-closed, or T′-closed. Now we can use the Corollaries 4.3, 4.4, and 4.5 to identify the weakest theories of C-closed MT-representable **UCMT**s and explore the theories with stronger topological or mereological closure conditions. A particular emphasis will be on theories that admit discrete models, i.e., theories allowing models that contain atomic entities.

4.5.1 M-closed MT-representable UCMTs

Because M-closed SPOCAs are unicomplemented, they must have a Boolean lattice.

Corollary 4.6. *The algebraic counterpart of an M-closed* **UCMT** *has a Boolean lattice.*

Proof. Follows from unicomplemented ortholattices being Boolean [Bir67]. □

Many of the contact algebras previously studied in the literature have Boolean lattices and satisfy C0 – C3 [see DW04; LY04; Ste00]. The most important ones are the following.

Definition 4.23. *A contact algebra* $(\mathcal{L}, \mathbf{C})$ *in which* $\mathcal{L}$ *is a Boolean lattice is a*

1. *Generalized Boolean contact algebra (GBCA) if* $\mathbf{C}$ *satisfies C4;*
2. *Boolean contact algebra (BCA) if* $\mathbf{C}$ *satisfies C4 and C-Ext;*
3. *RCC algebra (RBCA) if* $\mathbf{C}$ *satisfies C4, C-Ext, and Con;*
4. *Proximity BCA (PBCA) if* $\mathbf{C}$ *satisfies C4, C-Ext, and Int.*

BCAs are the algebraic counterparts of the theory *RCC* presented in Section 8.1.1 and RBCAs are the algebraic counterparts of the theory *RCC* ∪ {RCC4′, RCC8} also presented in Section 8.1.1. For a more comprehensive overview of the different classes of contact algebras and their relationships to one another we refer to [HG12]. Contact algebras that have Boolean lattices but do not satisfy C4 are even weaker than GBCAs; we call them *weak Boolean contact algebras* (WBCA), axiomatizable as

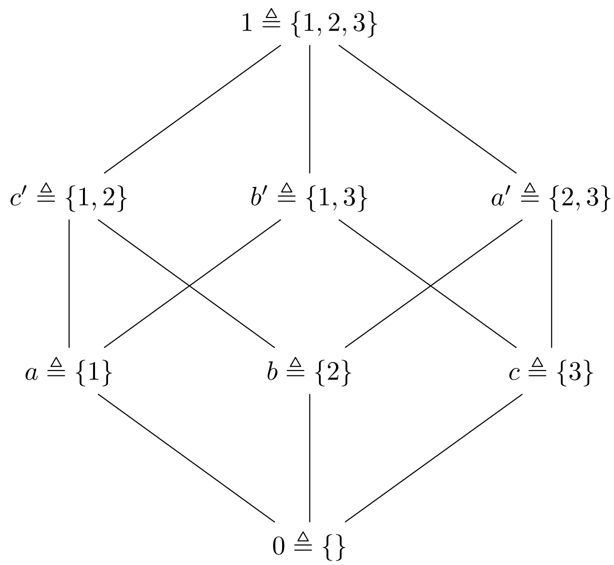
$$WBCA = SPOCA \cup \text{Uni}.$$

Definition 4.24. *A weak Boolean contact algebra (WBCA) is a contact algebra* $(\mathcal{L}, \mathbf{C})$ *in which* $\mathcal{L}$ *is a Boolean lattice.*

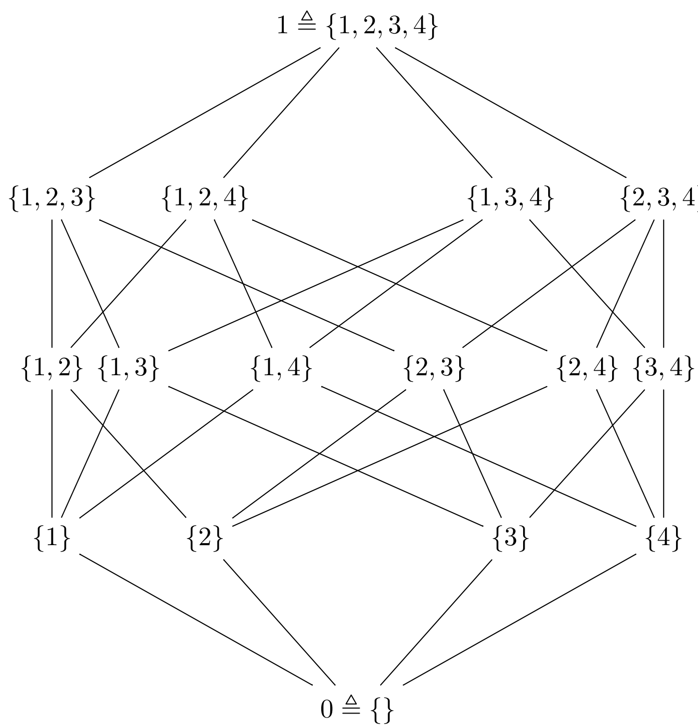
As illustrated by the model in Figure 4.6(a), there do exist WBCAs that satisfy neither C4 nor C-Ext. Thus, the class of WBCAs is strictly more general than both EWBCAs (to be introduced shortly) and GBCAs. WBCAs are the weakest algebraic structures resulting from an MT-representable **UCMT** that is M-closed. WBCAs admit atoms and in particular finite models as Figure 4.6(a) shows.

Theorem 4.5. *An M-closed MT-representable* **UCMT** *has an algebraic structure* $(\mathcal{L}, \mathbf{C})$ *whose lattice* $\mathcal{L}$ *is Boolean and whose contact relation satisfies C0 to C3.*

This is a more general perspective of the results from [DW08] in which the different contact relations definable on Boolean algebras have been studied. The weakest contact relation in [DW08] already



(a) The Boolean lattice B_3 with 3 atoms



(b) The Boolean lattice B_4 with 4 atoms

Figure 4.6: The Boolean lattices B_3 and B_4 as WBCA and EWBCA that do not satisfy C4.
(a) B_3 with $\{\{\langle 0, x\rangle : x \in \mathbf{L}\} \cup \{\langle a, b\rangle, \langle a, c\rangle, \langle b, c\rangle\}\} \notin \mathbf{C}$ (and symmetric tuples) defining disconnection is a WBCA which does not satisfy C-Ext or C4. B_3 with $\{\{\langle 0, x\rangle : x \in \mathbf{L}\} \cup \{\langle a, c\rangle\}\} \notin \mathbf{C}$ (and symmetric tuples) is a non-extensional GBCA. The elements a', b', c', and 1 are indistinguishable with respect to the contact relation.
(b) B_4 with $x < y' \rightarrow x\neg\mathbf{C}y$ defining disconnection except for $\{\langle 1, 2\rangle, \langle 3, 4\rangle\} \in \mathbf{C}$ results in an EWBCA not satisfying C4.

satisfies C4, while Figure 4.6(a) shows that there are weaker contact relations definable on a contact algebra with a Boolean lattice which may arise from M-closed **UCMT**s whose sums are not topologically closed, i.e., which violate C4. We do not argue for the usefulness of these structures; in practice C4 seems like a reasonable assumption. We only explore weaker M-closed contact algebras by showing what other contact relations are theoretically definable on a Boolean lattice.

WBCAs can be extended by C-Ext to obtain *extensional weak Boolean contact algebras* (EWBCA) or by C4 to obtain the already defined GBCAs. EWBCAs are axiomatizable as

$$EWBCA = SPOCA \cup \{\text{Uni, C-Ext}\}$$

Definition 4.25. *An extensional weak Boolean contact algebra (EWBCA) is a WBCA* $(\mathcal{L}, \mathbf{C})$ *in which the contact relation* $\mathbf{C}$ *satisfies C-Ext.*

Again, there exist EWBCAs whose contact relations do not satisfy C4 (compare Figure 4.6a). However, in the following we show that in all nontrivial EWBCAs not satisfying C4, $x\mathbf{C}x'$ holds for some elements, while for atoms it cannot hold. In other words, the theory of EWBCAs extended by the negation of C4 (¬C4), by ¬Triv, and by Atom is inconsistent with either of ¬Con and Con. For the proof we rely on the following result from [DW05b] stating that Dis implies C-Ext in contact algebras and thus in WBCAs. This results extends to SPOCAs.

(¬C4) $\exists x, y, z[x\mathbf{C}(y + z) \wedge x\neg\mathbf{C}y \wedge x\neg\mathbf{C}z]$ (some y is connected to $y + z$ but neither to y nor to z)

(¬Triv) $\exists y[y \neq 1 \wedge y \neq 0]$ (some entity besides 0 and 1 exists)

(Atom) $\exists a[a \neq 0 \wedge \forall x(x = 0 \vee x = a \vee x \cdot a \neq x)]$ (existence of an atom)

Lemma 4.22. *SPOCA* ∪ *C-Ext* ⊨ *Dis*

Lemma 4.23. *EWBCA* ∪ *{¬C4, ¬Triv, ¬Con}* ⊨ ⊥

Proof. We give an automatic proof showing that *SPOCA* ∪ {Uni, Dis} ∪ {¬C4, ¬Triv, ¬Con} ⊨ ⊥. Since by Lemma 4.22 *EWBCA* ⊨ *SPOCA* ∪ {Uni, Dis}, ¬Con is then inconsistent with any nontrivial EWBCA that does not satisfy C4. □

That does not mean that *EWBCA* ∪ {¬C4, ¬Triv} entails Con because ¬Con is not the simple negation of Con but states that *all* entities are disconnected from their complement. We will next prove that EWBCAs that contain an atom are inconsistent with Con as well because the atom must be connected to its complement. This is generally true for all SPOCAs that satisfy Atom and Con.

Lemma 4.24. *SPOCA* ∪ *{Dis, Atom, Con}* ⊨ ⊥

Proof. Let a be an atom in $\mathbf{L}$. Then $a^\perp$ is a dual atom, i.e., 1 is the only element greater than $a^\perp$. By overlap, $a^\perp$ is in contact to all elements except for a and 0. Suppose Con would hold, then $a\mathbf{C}a^\perp$ holds and $\forall y[a^\perp\mathbf{C}y \leftrightarrow 1\mathbf{C}y]$ but $1 \neq a^\perp$, a violation of Dis. This does not hold for a trivial model in which 1 is the only atom. □

It immediately follows that EWBCAs that with atoms cannot satisfy Con.

Lemma 4.25. *EWBCA* ∪ *{Atom, Con}* ⊨ ⊥

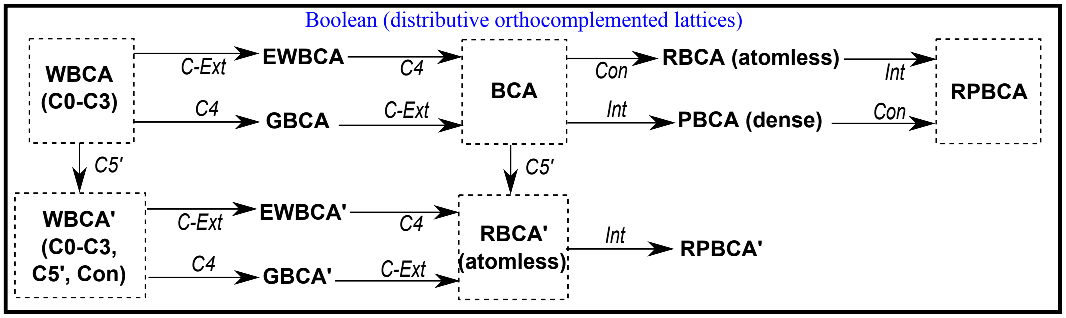


Figure 4.7: The classes of M-closed contact algebras and the logical extension relations among them, which are indicated by arrows. For each class more than a single contact relation may exist. For example, on BCAs contact can be defined as overlap, $x\mathbf{C}y \leftrightarrow x \cdot y \neq 0$, or as the standard contact relation, $x\neg\mathbf{C}y \leftrightarrow x < y'$. These are two distinct extensional contact relations. On the other side, there are Boolean algebras that only allow strictly nonweak and/or extensional contact relations.

Proof. By Lemma 4.22, $EWBCA \models Dis$ and by Lemma 4.24 $EWBCA \cup \{\text{Atom}, \text{Con}\} \models \bot$ follows. □

Therefore, all models of EWBCAs which do not satisfy C4 but contain an atom suffer from a nonuniform interpretation of the contact relation — in particular all atomic, all atomistic, and all finite models of EWBCAs and, more generally, of WBCAs with Dis. That $x\mathbf{C}x^{\perp}$ for atoms x is inconsistent with extensionality had already been observed for BCAs in [RS02]. Our proofs are slightly stronger and show that this problem persists in the weaker theory WBCAs extended by Dis requiring a topologically unique universal element. The failure of $x\mathbf{C}x^{\perp}$ for some elements is not by itself a concern; in a disconnected model one element may be isolated from the remaining space. However, the failure of $x\mathbf{C}x^{\perp}$ for *all* atoms is a serious issue hinting to a weakness in the theory. Although it can be overcome by enforcing C4, this creates other problems since C4 and C-Ext together disallow any discrete models unless contact is reduced to overlap (which in turn reduces the theory to a pure mereology). The problem does not persist in WBCAs; for those we can prove that Con is consistent.

Lemma 4.26. $WBCA \cup \neg C4 \cup Atom \cup Con \not\models \bot$

Proof. Figure 4.6(a) provides a counterexample. □

What extensions of WBCAs are obtained if some of the closure operations are also defined topologically? Intersections are already defined topologically in WBCAs. If sums are defined topologically, we require C4 and obtain GBCAs. If we define complements topologically by C5, we obtain PBCAs. Its discrete models again reduce contact to overlap. Finally, if we require neither sums nor complements to be defined topologically, but instead enforce C-extensionality, we obtain BCAs whose discrete models have overlap as the only feasible contact relation. Hence, among the different strengths of closure operations, the two classes WBCAs and GBCAs are the only algebraic theories of M-closed MT-representable **UCMT**s that admit non-atomless models with a contact relation different from overlap.

The extensions of WBCAs with the quasi-topological complements require adding C5′ and Con, which results in MT-representable contact algebras that parallel those without C5′, see Figure 4.7. Those in the classes WBCA′ and GBCA′ that do not satisfy Dis admit finite models, but those that satisfy Dis and, in particular C-Ext, do not admit any models with atoms. In any of those models Con is satisfied and thereby $\mathbf{C}_{\mathcal{M}} \neq \mathbf{O}_{\mathcal{M}}$.

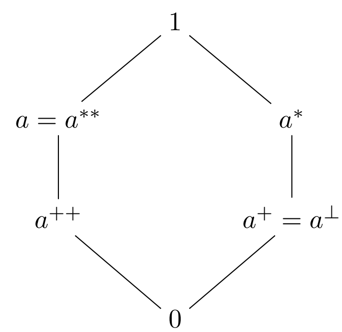


Figure 4.8: A contact algebra with a six-element Stonian p-ortholattice. Let $\forall x, y[x \neq 0 \wedge x \leq y \rightarrow x\mathbf{C}y]$ and $a\mathbf{C}a^*$ (and symmetric tuples) define contact. Then the displayed lattice $\mathcal{C}_6$ together with $\mathbf{C}$ defines a SPOCA $(\mathcal{C}_6, \mathbf{C})$ that satisfies C5 and is thereby T-closed.

4.5.2 T-closed and T′-closed MT-representable UCMTs

T-closed MT-representable **UCMT**s have SPOCAs as algebraic counterparts which may be non-distributive as long as C5 is satisfied.

Theorem 4.6. *A T-closed MT-representable* **UCMT** *has an algebraic structure* $(\mathcal{L}, \mathbf{C})$ *in which* $\mathcal{L}$ *is a Stonian p-ortholattice and* $\mathbf{C}$ *satisfies C0 – C5, C-Ext, ¬Con, and Int.*

This nondistributive class of SPOCAs has been studied in depth in [HWG09]; it is the algebraic equivalent of the subtheory RT^- of the mereotopology of [AV95].

Because such T-closed SPOCAs satisfy C4, intersections and sums are implicitly defined mereologically as well. The only real extension in terms of additional mereological closure operations requires complements to be mereologically defined, which in turn by Lemma 4.5 makes the lattice Boolean and thus results in a PBCA: C-Ext as well as Int are already entailed in all T-closed SPOCAs. This also means C-Ext extends T-closed SPOCAs nonconservatively, while Con is altogether inconsistent with T-closed SPOCAs. We already know that PBCAs are always atomless, hence the theory SPOCA ∪ C5 is—among all possible extensions of T-closed MT-representable **UCMT**s by additional mereological closure operations—the only theory that admits atoms. Figure 4.8 gives such a model.

T′-closed MT-representable **UCMT**s also have SPOCAs as algebraic counterparts which may be non-distributive as long as C5′ is satisfied. They differ from the T-closed ones in that they satisfy Con instead of ¬Con but do not necessarily satisfy Dis, C-Ext or Int. However, those that have a universal that is topologically defined, i.e., those that satisfy Dis are always atomless by Lemma 4.24.

Theorem 4.7. *A T′-closed MT-representable* **UCMT** *has an algebraic structure* $(\mathcal{L}, \mathbf{C})$ *in which* $\mathcal{L}$ *is an atomless Stonian p-ortholattice and* $\mathbf{C}$ *satisfies C0 – C4, C5′ and Con.*

4.5.3 MT-closed MT-representable UCMTs

Sections 4.5.1 and 4.5.2 let us conclusively answer the question whether MT-closed MT-representable **UCMT**s exist and what their structure is. Such structures must be M-closed and either T-closed or T′-closed. For the first case (a M- and T-closed theory) the intersection of the respective minimal theories, i.e., of WBCAs and SPOCAs satisfying C5, results in PBCAs that satisfy C5 and which are necessarily atomless. In these structures ¬Con is entailed; it requires that the contact relation be defined as overlap $x\mathbf{C}y \leftrightarrow x \cdot y \neq 0$ which reduces the theory to a pure mereology. For the second case (a M- and T′-closed theory) we get the RBCA′s as minimal theory, which are RBCAs with contact defined as C5′ and which are also atomless. Hence, we have the following result.

Theorem 4.8. *Every MT-closed MT-representable* **UCMT** *has an algebraic structure that is an atomless BCA.*

We also have negative results on the existence of MT-representable **UCMT**s with $\mathbf{C}_{\mathcal{M}} \neq \mathbf{O}_{\mathcal{M}}$ or with atoms.

Corollary 4.7. *No M-closed and T-closed MT-representable* **UCMT** *with* $\mathbf{C}_{\mathcal{M}} \neq \mathbf{O}_{\mathcal{M}}$ *exists.*

Corollary 4.8. *No MT-closed MT-representable* **UCMT** *with atoms exists.*

Corollary 4.9. *No MT-closed MT-representable* **UCMT** *with discrete models exists.*

4.6 Conclusions

Our exploration revealed three weakest classes of potentially spatially representable complete OCAs that correspond to extensions of **UCMT**. The first are WBCAs, the weakest class in which all closure operations are defined mereologically. The second class are SPOCAs with C5, the weakest class in which all closure operations are defined topologically. The third class are SPOCAs with C4, C5′, and Dis (which further implies C-Ext). We are not aware of full embedding theorems for these three weakest classes of contact algebras. This remains to be investigated in the future.

4.6.1 Spatially representable contact algebras with discrete models

Among the spatially representable OCAs, the classes allowing discrete models are of particular interest. Although space is potentially infinitely divisible according to Aristotle, in practical applications any concrete model of space will have “atoms” at some level, i.e., there is some finest granularity. This granularity is usually determined by the precision of available data or measurement devices (think of satellite images vs. microscopic pictures) or the precision we want to reason at (think of a car navigation system vs. the accurate description of surface chemistry). For a generic Ontology (in the philosophical sense) of space discrete models might not be that important, but for any specific domain we want to be able to specify models completely e.g., by explicitly listing a finite set of regions and the primitive relations (such as connection and parthood) among them. Such a specification should be consistent with the theory and not a mere approximation thereof. Many mereotopologies, e.g., the RCC (corresponding to RBCAs), prevent the existence of atomic regions by including a divisibility axioms, i.e., requiring the existence of an interior part for each region. Such theories do not allow us to list all atomic regions of a specific model. Of course, approximations of such models are possible, but these approximations have different model-theoretic properties. This has an important consequence: the construction of and the reasoning with specific models using a theory consistent with discrete, and especially finite, models can be achieved using standard theorem provers, which is not possible for mereotopological theories that only admit infinite models.

Which extensions of the three weakest classes of MT-representable C-closed OCAs allow discrete models, i.e., are not atomless? We showed that non-atomless WBCAs and EWBCAs have contact relations that behave erratically with regard to contact among complements. While the stronger BCAs and extensions thereof do not suffer from this problem, their discrete models are only of mereological nature, i.e., $\mathbf{C}_{\mathcal{M}} = \mathbf{O}_{\mathcal{M}}$ for any discrete model $\mathcal{M}$ [DW05a]. Similarly, SPOCAs satisfying C5′ and Dis rule out discrete models by Lemma 4.24. This leaves GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ as the only (among all combinations of mereological and topological or quasi-topological closure operations) MT-representable OCAs that admit discrete models. These two classes can be characterized as following:

1. GBCAs in which all closure operations are defined mereologically while sums and intersection are also defined topologically. In general, GBCAs are consistent with either of Con or ¬Con. The entities in such algebraic structures are representable by either (1) only regular open, (2) only regular closed, or (3) unrestricted point sets (with point-set intersections, unions, and complements). In the second case Con must hold while in the other cases ¬Con must hold. The lattices underlying this class are distributive, i.e., parthood is distributive with respect to sum and intersections.

2. The subclass of SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ as weakest contact algebras defining all closure operations topologically while sums and intersections are also defined mereologically. Due to the topological nature of complements, ¬Con must hold. The representation of such algebraic structures must include both regular open and regular closed sets, since each regular closed set has a regular open set as complement and vice versa. In this class, the underlying lattices—and thus the parthood relation—may be non-distributive.

Indeed, GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ exemplify the two ways of constructing discrete mereotopologies discussed in [MV99]. SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ constitute a C-extensional theory with classical topological operators in which each entity, in particular each atom, is “duplicated” as an open and as a closed set, while GBCAs define an O-extensional theory without classical topological operations, i.e., that do not distinguish regions with identical closures.

4.6.2 Spatially representable Whiteheadean mereotopology

In [Whi29], Whitehead originally proposed a C-extensional mereotopology and defined atoms as regions without proper parts. We can interpret this as an implicit endorsement of the existence of atoms. Unfortunately, as Corollary 4.8 shows, no MT-closed MT-representable mereotopology with atoms can exist. In fact, the only theory that (1) allows atoms, (2) is C-extensional, and (3) is MT-representable are the SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ defining contact—assuming that this class of SPOCAs can be further strengthened to a class of spatially representable SPOCAs; see [WHG12] for work in this direction. From [HWG09] we know that such theory is also definable by a single mereological primitive P (the partial order relation $\leq$ in the lattices) or by a single topological primitive C; it seems to seamlessly bridge the gap between mereology and topology. But at the same time, Whitehead never distinguished sets with identical closures. We can understand this as an implicit condition for representations by closed regions (or, dually, by only open regions); in fact many researchers followed this understanding of Whitehead’s intentions. He entices us to believe that the two assumptions, namely existence of atoms and representability by closed regions, are consistent. However, SPOCAs with $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$ as the only remaining candidate for true Whiteheadean mereotopology do rely on this difference between interiors and closures. If the distinction between interiors and closures is removed, these models collapse into Boolean contact algebras, compare [WHG09], and thereby prevent a meaningful definition of contact apart from overlap in discrete models. With this stricter requirement of representability by only closed sets, no discrete region-based theory in the intention of Whitehead is definable [see also For96; Mor98]. Further research on theories of qualitative discrete space must therefore concentrate on non-topological, such as graph-based, approaches or on multidimensional approaches that accommodate regions of various dimensions. We explore the latter path in the remainder of this thesis.

There are others ways out of this dilemma as demonstrated in the literature. If we do not insist on discrete models, RBCAs and the equivalent logical theory RCC, provide a truly Whiteheadean account of *continuous* space. One spatial representation thereof is the complemented disk algebra as described in detail in [LL06] that consists of all simple closed regions of, e.g., $\mathbb{R}^2$. RBCA′s,which are RBCAs with a particular definition of contact, admit continuous models in which entities can be connected to their complement. If we abandon C-extensionality instead we can rely on GBCAs. Non-extensional theories have also been used for defining multidimensional mereotopologies [Gal99; RS02]. The rationale for giving up C-extensionality is simple [RS02]: C-extensionality is a principle that holds in the perfect world where we can always find smaller parts that distinguish two distinct entities. If finite models are considered as models with limited accuracy, i.e., as approximations of continuous models, C-extensionality may be violated because the distinction in the contact between two entities may be too small a part so that it is lost in the approximation.

An alternative parsimonious way out of this dilemma is to abandon $\forall x\,[C(x, \ominus x)]$ (Con) instead. The nondistributive SPOCAs with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ allow such choice. At first sight it seems to be a surprising choice since well-behaviour of lattices is usually associated with distributivity. But as we have shown in [HWG09], the non-distributive lattices in question (Stonian p-ortholattices and restrictions) behave nicely even without distributivity. In particular, these structures also satisfy the DeMorgan laws and stop only short of being Boolean. We thereby are able to answer the question posed in [Dün+08] asking what kind of structures should be considered the standard model of a non-distributive contact algebra for the case of spatially representable contact algebras. The standard (and only) models of spatially representable complete non-distributive contact algebras are the regular sets of a topological space.

Notice that there is no need to completely abandon Con. If we define an additional *attachment* relation A from C as

$$A(x, y) \leftrightarrow [C(x, y^{**}) \vee C(x^{**}, y)] \wedge \neg C(x, y),$$

we can prove $\forall x\,[A(x, \ominus x)]$ in a connected space even if $\forall x\,[\neg C(x, \ominus x)]$. Attachment is a stronger relation than contact defined in SPOCAs as $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$, but weaker than *weak contact* WCont as defined in [AV95]. Moreover, C and A make the distinction between the intended interpretations of 'sharing a point' and 'overlapping neighbourhoods' clear.

4.6.3 Summary

This chapter treated mereotopology with unique closure operations algebraically and studied the arising contact algebras that may yield spatial representations for all their models. In particular, this is the first time that non-distributive contact algebras are included and studied comprehensively as algebraic counterparts of mereotopologies. We showed that SPOCAs defined over Stonian p-ortholattices with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ as contact are a good candidate for an ontologically coherent region-based theory of space. In fact, these are the least constrained algebraic structures that admit discrete C-extensional models among all of the algebraic theories satisfying the conditions of MT-representability which are at the same time necessary conditions for spatial representability. The other candidates for spatially representable contact algebras are SPOCAs with $x\mathbf{C}y \leftrightarrow x \not< y^{\perp}$, BCAs, in particular its atomless extension RBCA, and the weaker GBCAs. The latter two correspond to the logical theories RCC and GRCC known from the literature. While RCC models are C-extensional and always continuous, the models of GRCC can be discrete but are not C-extensional. We demonstrated that the main difference between GBCAs and SPOCAs with $x\mathbf{C}y \leftrightarrow x \nleq y^{\perp}$ or $x\mathbf{C}y \leftrightarrow x \not< y^{\perp}$ is whether complements are defined mereologically or topologically. Mereological complements require distributive contact algebras such as GBCA, BCA, or RBCA, while topological complements allow non-distributive contact algebra based on Stonian p-ortholattices. The remaining closure operations sum, intersections, and universal are in either case defined mereologically; topological sums require C4 while a topological universal requires Dis or C-Ext. As one of our key contributions in this chapter, all mereological and topological closure operations are directly attributed to properties of the parthood lattice or the contact relation. Mereological complements manifest themselves in unique complementation in the algebraic counterparts while topological complements require C5 which binds the contact relation to the orthocomplementation operation. Contact algebras with topological complements can be non-distributive, but are required to satisfy Con, C-Ext, and C4. Thus the ontological choice of defining complements topologically is directly associated with other, more implicit, ontological choices.

We have established in GBCAs and SPOCAs with C5 two weakest, potentially spatially representable, theories that allow atoms and that define all closure operations either mereologically or topologically. As natural next steps (**Question 1**) concrete topological embeddings theorems for these two classes of contact algebras need to be established analogously to the topological embeddings for BCAs [DW05a]. For the SPOCAs with C5, we know that non-representable models exist [WHG12]. Extending the theory of SPOCAs with axioms that rule out some of the non-representable models, [WHG12] is a first step towards such an embedding theorem.

On a separate note, all equidimensional mereootologies have a limited expressivity by relying only on contact and parthood, often even defining one relation in terms of the others. For the remainder of the thesis, we will look at ways to increase this expressivity in a qualitative representation of space that captures mereotopological spatial relations.

Chapter 5

The intended structures of multidimensional qualitative space

The aim of the subsequent chapters is to develop a logical theory that qualitatively captures arbitrary arrangements of idealized, i.e., uniform-dimensional, spatial entities in an abstract space. Any such spatial arrangement is called an *intended structure*. Any single intended structure may contain idealized entities of different dimensions at the same time, but each idealized spatial entity in an intended structure must have a uniform dimensions, i.e., must not contain artefacts of lower dimensions. The purpose of this chapter is to formally characterize the class of intended structures, which can be considered as qualitative abstractions of n-dimensional simplicial complexes. First, we review *simplicial complexes* and their building blocks—*simplices*—and then analogously define how to construct *composite manifolds* from *m-manifolds with boundaries* as building blocks. Finally, we introduce the notion of *complex manifolds*[1] which correspond to the intended structures. This will also shed some light on how composite and complex manifolds generalize simplicial complexes by abstracting away distinctive vertices and by ignoring whether spatial entities are curved or not. Essentially, we replace simplices by the more general m-manifolds with boundaries as basic entities.

5.1 Simplices and simplicial complexes

In this section we maintain the definitions and terminology from [Lee11]. Any *n-simplex* is spanned by $n+1$ *vertices*—which are 0-simplices (points)—that are not all contained in a single $(n-1)$-simplex. The number n denotes the dimension of a simplex. For example, a 2-simplex is spanned by three points that are not collinear and a 3-simplex is spanned by four points that are not coplanar. A 0-simplex is a single point, a 1-simplex a line segment, a 2-simplex a triangle, a 3-simplex a solid tetrahedron.

Simplices are always closed in the topological sense: every n-simplex includes all bounding $(n-1)$-simplices. For example, a 2-simplex includes the three vertices as 0-simplices and three 1-simplices, namely the line segments that connect each pair of vertices. In other words, each subset of a simplex' vertices is itself a simplex; we call those the *faces* of the simplex. The $(n-1)$-simplices spanned by

[1]The term *complex manifold* used throughout the thesis refers to a collection of composite manifolds. It is not related to another notion of *complex manifold* used frequently [compare MK71] to denote a manifold that is embeddable in the field of complex numbers.

vertices of a simplex are called its *boundary faces*.

Simplicial complexes are collections of nicely arranged simplices. More precisely, a collection of simplices K is a simplicial complex if and only if it satisfies the following three conditions [Lee11]:

1. if $\sigma \in K$, then every face of σ is in K;
2. the intersection of two simplices in K is either empty or a face of each;
3. every point in a simplex $\sigma \in K$ has a neighbourhood that intersects at most finitely many simplices in K.

The maximal dimension of any simplex contained in a simplicial complex is called the dimension of the simplicial complex. Essentially, simplices can only be fused together in their boundaries (faces), but cannot overlap. To capture a spatial model with two overlapping simplices, we would need to break the overlapping region into simplices again. However, two different simplicial complexes may overlap, their overlap could be modelled again as a simplicial complex.

Notice that condition (3) considers all points, not just the spanning points, the vertices, of a simplex. For example, a 3-simplex has only three vertices, but an infinite number of points (consider the simplex as a point set). The condition ensures that each point (including each vertex) is included only in a finite number of simplices within a simplicial complex. However, this does not require a simplicial complex to be a finite collection of simplices; an infinite set of simplices that are all disjoint would be a simple counterexample. Moreover, it does not guarantee that a simplex is spanned by a finite number of vertices, i.e., is finite-dimensional.

Despite their simplicity, simplicial complexes can model many complex geometric objects, such as polygons (as a collection of 2-simplices) in 2D or 3D and other kinds of surfaces as well as polyhedra as collections of 3-simplices. For this reason, simplicial complexes have been widely used as basis for models of geographic space, of built environments (e.g., houses, airports, subway systems), of parts and assemblies in manufacturing (e.g., cars, machinery), of virtual objects (rendering of objects and characters in movies, video games, virtual realities), etc.

What constitutes now a qualitative abstraction of simplicial complexes? Note that the definition of simplices and simplicial complexes implicitly use a metric - namely the distance in Euclidean space to define simplices as the regions bounded by the shortest lines that connect its vertices. That also ensures that simplices are always convex (though simplicial complexes such as polyhedra can be nonconvex). In the desired generalization, we will remove these restrictions and allow nonconvex entities bounded by arbitrary curves or "folded" curves (with sharp points) and their higher-dimensional equivalents (curved planes, etc.) as generalizations of n-simplices. We want to maintain the following properties from the definition of simplicial complexes for our intended models:

1. Every entity is a collection of simple entities.
2. The intersection between two simple n-dimensional entities in a collection that constitutes an entity is either empty or of a dimension $< n$.
3. Every entity is topologically closed, i.e., includes its topological boundaries (faces) of the next-lowest dimension. But we do not require faces to be in the entity's collection of simple entities.
4. All entities are decomposable into the simple entities (such as a polygon or polyhedra can be decomposed into sets of simplices).

5. The simple entities are m-manifolds with boundaries. We will explain shortly what that means.

To describe the class of intended structures, we employ a methodology very similar to the definition of simplicial complexes: we first review a generalization of simplices, known as m-manifolds with boundaries in mathematics, and subsequently use those as the building blocks for constructing composite m-manifolds, which in many ways generalize simplicial complexes.

A remark about our terminology is in order. Throughout this thesis, we are often a bit imprecise in the use of the terms "line", "curve", "line segment", and "curve segment". Usually, all refer to the most general class, which is most accurately denoted as "linear feature", i.e., a "curve segment", which may be arbitrarily curved and may or may not bounded by endpoints. In that sense this class includes straight lines, curves, straight line segments, curve segments, straight rays, and curved rays. An "area" means a two-dimensional piece of space that may be curved, folded, or similarly transformed in a higher-dimensional space. Equally, the geometric entities of higher dimensions can also be bent, curved, or warped in any imaginable way (unless otherwise stated) as long as its topology is maintained. In general, we make no assumptions about the curvature of spatial entities unless we are explicit about, i.e., when we say "straight line" or "flat area". In our reconstruction of classical geometries in Chapter 10 the distinction will become important. Up to, but not including Chapter 9 we also do not discriminate bounded regions, so-called "segments", from unbounded regions such as curves or lines without endpoints. Up to where we formally define closed entities, all spatial entities may have a boundary or may be closed, that is, have an empty boundary such as a sphere.

5.2 Manifolds with boundaries

As the most primitive building blocks for the spatial arrangement we intend to capture we use *m-manifolds with boundaries*, a well-known mathematical concept [Lee11]. Note that we use here the more general notion of topological manifolds, not the restricted version of smooth manifolds (which is implicitly assumed in the majority of work on manifolds). m-Manifolds with boundaries are point sets that are locally Euclidean. We reuse the following formal definition from [Lee11]:

Definition 5.1. *An m-manifold with boundary,* MF*, is a second-countable Hausdorff space in which every point* $p \in \mathrm{MF}$ *has a neighbourhood homeomorphic to an open subset of the m-dimensional upper half space* $\mathbb{H}^m = \{(x_1, ..., x_m) \in \mathbb{R}^m : x_m \geq 0\}$.
We call m the dimension of MF *and write* $dim(\mathrm{MF}) = m$.

Second-countable Hausdorff spaces are topological spaces that have a countable cover of open subsets and any two points are distinguishable by some open set. The last condition of the above definition says that manifolds are locally Euclidean, that is, they behave locally like n-dimensional Euclidean space, i.e., every point (except boundary points) has a neighbourhood in the manifold that is homeomorphic to $\mathbb{R}^m$. Every boundary point has a neighbourhood in the manifold that is homeomorphic to the upper half space of $\mathbb{R}^m$. Most importantly, this disallows topological singularities (see Figure 5.2): any kind of self-intersection (both X-intersection and T-intersection), also known as double points, and any kind of isolated or missing entities of lower dimension such as an isolated point or a missing point in a manifold of a dimension 1 or greater (a line, area, etc.), or an isolated or missing line in a manifold of dimension 2 or greater (an area, volume, etc.), and so on. It also prevents constructs such as space-filling curves [Sag94], first constructed by Peano and Hilbert [Hil91; Pea90], which self-intersect at any point [Sag94]. Space-filling curves are the reason why many definitions of dimensions are extremely complicated. However, in our setting where self-intersection of manifolds is not allowed, we can clearly identify the dimension of any manifold by the dimension of the homeomorphic Euclidean space.

Notice that we do not rule out so-called non-topological singularities (compare Figure 5.1), such as sharp points ("cusps"), which are not defects with regard to the topology, but only with regard to the smoothness (or differentiability). In particular, we do not require the manifolds in this thesis to be smooth manifolds, differentiable manifolds, orientable manifolds, or triangulable manifolds; for definitions of those see e.g., [Lee11]. While all topological manifolds of dimensions two and three admit a unique piecewise-linear triangulation and are therefore triangulable (with triangulations that are simplicial complexes) [compare the Triangualation Theorems in Lee11], some manifolds of dimension four or greater are not piecewise-linear triangulable. For example, the 4-manifold E8 is not piecewise-linear triangulable [Fre82].

For brevity we will use the term *m-manifolds* to refer to a topological m-manifold with (possibly empty) boundary throughout the thesis. This deviates from standard mathematical usage, with the term m-manifolds more commonly referring to smooth m-manifolds without boundary. Unless we speak of manifolds of a particular dimension, we will often drop the 'm-' as well.

We are interested in both the interior and boundary of manifolds. We define the interior of a manifold as follows.

Definition 5.2. *Let* MF *be an m-manifold. The interior of* MF *is the set* MF° *of all points $p \in \mathrm{MF}$ that have a neighbourhood homeomorphic to $\mathbb{R}^m$.*

Then the (manifold) boundary of an m-manifold is the point-set difference between the manifold and its interior.

Definition 5.3. *The boundary of an m-manifold* MF *is the point set* $\delta\mathrm{MF} = \mathrm{MF} \setminus \mathrm{MF}^\circ$.

Note that the concept of a *manifold with boundary* admits empty boundaries; if the boundary is empty, the manifold is called closed.

Definition 5.4. *An m-manifold* MF *is closed if and only if* $\delta\mathrm{MF} = \emptyset$.

Closed manifolds are either boundaries of a higher-dimensional manifold, e.g., the sphere is a 2-manifold that bounds a 3-manifold solid ball, or are unbounded manifolds that stretch into infinity such as a line (as opposed to a line segment), a plane, or any entity homeomorphic to $\mathbb{R}^m$ for some finite m. Non-closed manifolds also include entities that are only partially bounded such as a ray, a half-plane, or a half-sphere.

Notice that any 0-manifold, i.e., any isolated point or any discrete space, is completely contained in its own interior, that is, the boundary of a 0-manifold is always empty [Lee11, p.43].

The notion of a closed manifold should not be confused with the notion of a closed set in a topological space, they are quite different concepts. Foremost, the notion of closure in a topological space applies to arbitrary subsets, whereas the notion of closure discussed here only applies to manifolds. The topological closure of a point set is the smallest closed set defined by the topology—it may not be the smallest manifold that does embed the point set. Next, we define the manifold-closure of a set; which is the smallest manifold with boundary (closed or not) containing the set.

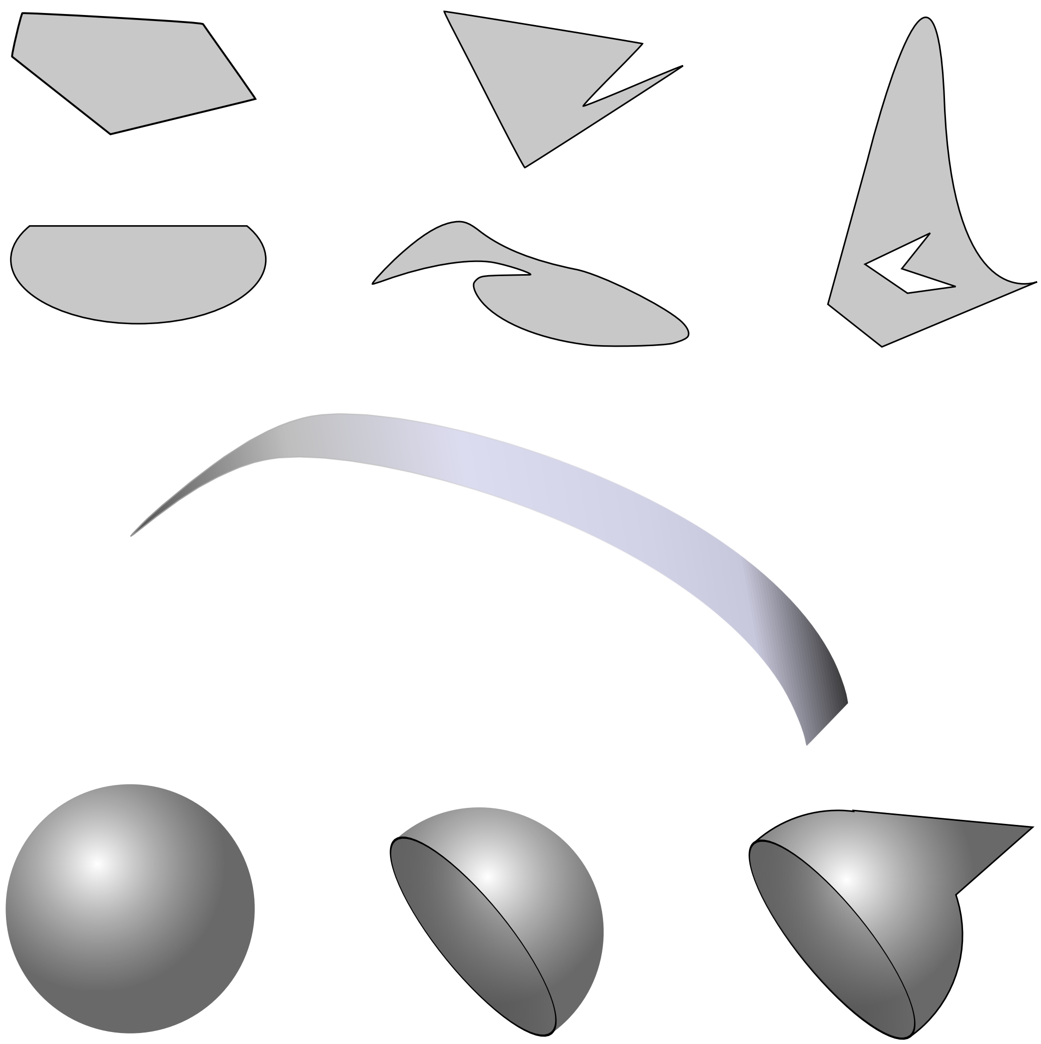


Figure 5.1: Examples of 2-manifolds. The top five are examples of planar 2-manifolds with boundaries; including convex (left) and nonconvex (middle and right) 2-manifolds, 2-manifolds with other 2-manifolds cut out (right) as holes, and with straight (top) or curved (second row) boundaries. The second row from the bottom shows a 2-manifold that is curved in 3D; any of the planar examples may be equally curved and may have internal sharp edges. The bottom row gives examples of closed 2-manifolds (surfaces; not bodies) that are either convex (left and middle) or nonconvex (right) and may have sharp edges (middle and right). Other examples of 2-manifolds include a cylinder (a rectangle glued together at two opposite edges) or a Möbius strip.

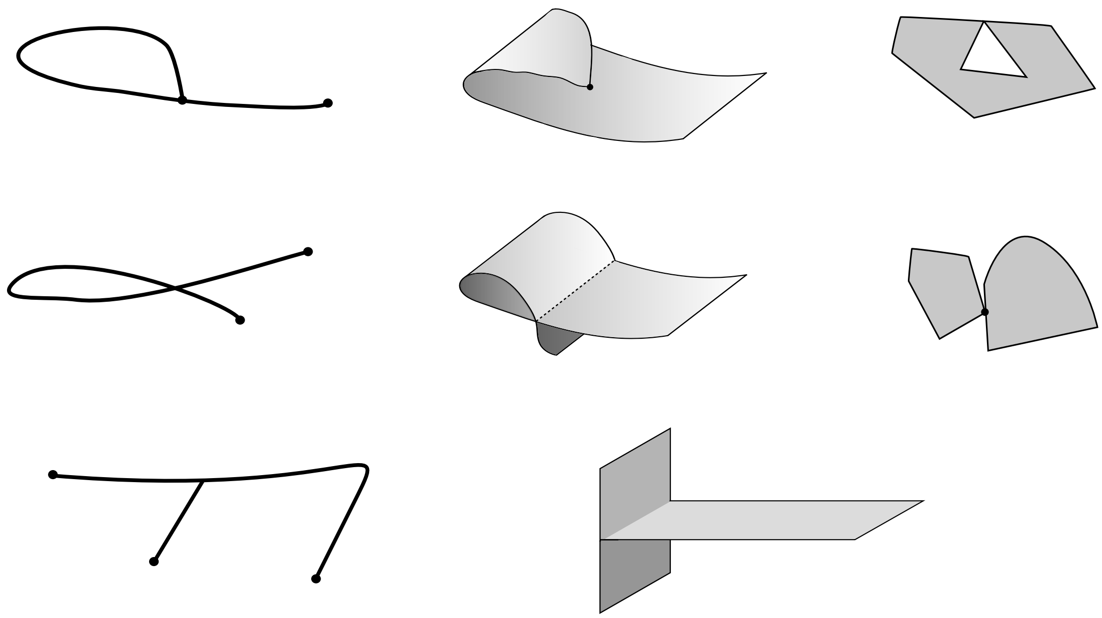


Figure 5.2: Examples of spatial entities that are not manifolds. On the left side, each of the 1D curves has some "double point": in the top example the point the curve self-connects (T-intersection) at a point, in the second row example the curve self-intersects (X-intersection) at a point, and in the bottom example the curve splits (T-intersection) at a point. The middle column contains similar examples for curved 2D areas: in the top example the area self-connects (T-intersection) in a point, in the second row example the area self-intersects (X-intersection) in a line, and in the bottom example the area splits (T-intersection) at a line. The two right-most examples have other kinds of singularities: in the top example, the inner boundary (bounding the hole) meets the outer boundary, resulting in a point that is not homeomorphic to the half space of $\mathbb{R}^2$, whereas in the right example in the second row the point where the two pieces connect is also not homeomorphic to the half space of $\mathbb{R}^2$. Note that all examples in this figure can be represented by composite manifolds.

Definition 5.5. *The closure of a set $X \subseteq \mathbb{R}^m$ is the smallest m-manifold $\overline{X}$ such that $X \subseteq \overline{X} \subseteq \mathbb{R}^m$.*

Any m-manifold—and thus any of our basic spatial entities—has, among others, the following properties:

1. Is a topologically closed point set;
2. Has no isolated or missing lower-dimensional entities ("solid entity", also known as "regular set" in topological spaces), i.e., is of a unique dimension m in the sense that it is homeomorphic to an open subset of $\mathbb{H}^m$;
3. Is a one-piece entity (self-connected);
4. Has no double points caused by self-intersection (X- or T-intersection).

While this allows singularities such as "cusps", "corners", and 'edges' (see Figure 5.1), no single point can be both a boundary and an interior point of the manifold (see Figure 5.2); otherwise the entity self-intersects and cannot be a manifold. Moreover, an m-manifold cannot be constituted of two pieces that are only "weakly connected", i.e., where the pieces are only connected by some entity of dimension $\leq m-2$ (compare Figure 5.2). Then the contact is not locally Euclidean in $\mathbb{R}^n$.

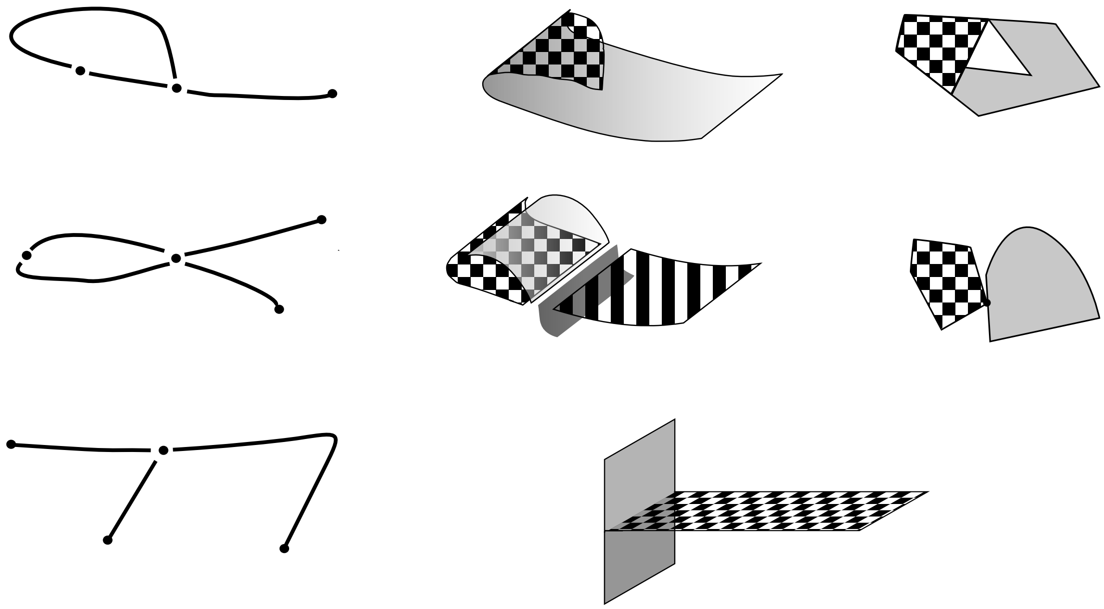


Figure 5.3: The examples from Figure 5.2 as composite manifolds. We have spaced about the individual atomic manifolds and their points of contact as necessary. Also, some atomic manifolds are displayed checkered in order to better distinguish atomic manifolds within each composite manifold. The example in the center consists of four atomic manifolds: one checkered, one striped, one dark-coloured, and one with a transparent light color.

5.3 Composite m-manifolds

Manifolds with boundaries are the most basic building blocks in our approach to characterize the class of intended structures as generalization of simplicial complexes. In our next step, we study how we can assemble those manifolds, which we will refer to as *atomic manifolds* in the sequel, to more complex spatial structures, called *composite manifolds*. While atomic manifolds are comparable to simplices, composite m-manifolds are maybe best compared to the set of simplices of the highest dimension within a simplicial complex, such as the set of all two-dimensional areas contained in a simplicial 2-complex. This gives a rough idea how we generalize simplicial complexes.

The definitions in this section are our own, they are not standard in mathematical treatises of manifolds. First, we define when a collection of atomic manifolds forms a composite manifold, and subsequently we define the interior and boundary of such a composite manifold. A composite m-manifold is a collection of m-manifolds that may touch one another only in their boundaries and that are all of equal dimension. Trivially, any atomic m-manifold is a (singleton) composite m-manifold. If two m-manifolds "touch" each other in the interior of one of the manifolds, i.e., if an interior point of one of the manifolds is also a point of the other manifold, they cannot be in common composite m-manifold. Formally, we define composite m-manifolds as follows.

Definition 5.6. *A finite collection of m-manifolds $\mathfrak{M}$ is called a* composite *m*-manifold *if and only if* $\mathrm{MF}_1 \cap \mathrm{MF}_2 \subseteq \delta\mathrm{MF}_1$ *and* $\mathrm{MF}_1 \cap \mathrm{MF}_2 \subseteq \delta\mathrm{MF}_2$ *for all* $\mathrm{MF}_1, \mathrm{MF}_2 \in \mathfrak{M}$*. We denote the composite m-manifold as* $\mathfrak{M}^m$*, m indicating the dimension* $m = \dim(\mathfrak{M}) = \dim(\mathrm{MF}_1)$ *for all* $\mathrm{MF}_1 \in \mathfrak{M}$ *of the composite m-manifold.*

Notice that the superscript in a composite m-manifold denotes its dimension. We omit the superscript when the dimension is irrelevant.

To define the boundary of a composite manifold, we first define its area and its interior boundaries. The area Σ is the set of all (interior and boundary) points that any of the atomic manifolds in a composite manifold include.

Definition 5.7. *Let $\mathfrak{M}$ be a composite manifold.*

The area of $\mathfrak{M}$ is the set of all points included in any of the manifolds in $\mathfrak{M}$:

$$\Sigma\mathfrak{M} = \{p \,:\, p \in \mathrm{MF} \text{ and } \mathrm{MF} \in \mathfrak{M}\} = \bigcup\{\mathrm{MF} : \mathrm{MF} \in \mathfrak{M}\}.$$

The interior boundary Δ_i of a composite manifold is best captured as a recursive function. Note that we can no longer rely on the existence of a neighbourhood homeomorphic to $\mathbb{R}^n$ as in the Definitions 5.2 and 5.3 to distinguish interior from boundary points because two manifolds can be glued together without their glue points becoming interior points in the traditional sense. Loosely speaking, the interior boundary comprises the topological interiors of the set of glue points between the constituent atomic manifolds in a composite manifold.

Definition 5.8. *The interior boundary $\Delta_i\mathfrak{M}$ of a composite manifold $\mathfrak{M}$ is defined as:*

$$\Delta_i\mathfrak{M} = \begin{cases} \emptyset & \text{if } |\mathfrak{M}| \leq 1, \\ \Delta_i\big(\mathfrak{M} \setminus \{\mathrm{MF}\}\big) \cup \big\{(\delta\mathrm{MF}' \cap \delta\mathrm{MF})^\circ \quad \mathrm{MF}' \in \mathfrak{M}\big\} & \text{otherwise, where } \mathrm{MF} \text{ is an arbitrary manifold in } \mathfrak{M} \end{cases}$$

Observe that we only consider the interior of the intersection of two atomic manifolds to be in its interior boundary. If, for example, two 2-manifolds in a composite manifold share a 1-manifold, only the interior of the 1-manifold is in the interior boundary of the composite manifold. A special case is where two atomic manifolds in a composite manifold share a single point, then the interior of that point will be empty (recall that all manifolds have a topology that is locally Euclidean), i.e., will not contribute to the interior boundary of the composite manifold.

Finally, we are able to define the boundary of a composite manifold as the sum of its constituent manifolds' boundaries with all interior boundaries removed.

Definition 5.9. *Let $\mathfrak{M}$ be a composite manifold. We define its boundary as*

$$\Delta\mathfrak{M} = \Big(\bigcup_{\mathrm{MF}\in\mathfrak{M}} \delta\mathrm{MF}\Big) \setminus \Delta_i\mathfrak{M}.$$

Moreover, we can define the interior of a composite manifold as the sum of its constituent manifolds' interiors together with its interior boundary.

Definition 5.10. *Let $\mathfrak{M}$ be a composite manifold. We define its interior as*

$$\Theta\mathfrak{M} = \Big(\bigcup_{\mathrm{MF}\in\mathfrak{M}} \mathrm{MF}^\circ \Big) \cup \Delta_i\mathfrak{M}.$$

Composite m-manifolds have, among others, the following properties:

1. Uniform dimension m for all manifolds in the collection;
2. No isolated or missing lower-dimensional parts ("solid entity");
3. Any two atomic m-manifolds can only share boundaries;
4. Every proper subcollection of manifolds is a composite m-manifold.

Observe that the properties 2. to 4. also apply to simplicial complexes. However, n-simplicial complexes of dimension $n \geq 2$ are not composite m-manifolds because simplicial complexes are closed under sub-complexes (compare condition (1) for simplicial complexes on page 80), i.e., every face bounding a constituent simplex must also be in the simplicial complex. On the other side, composite m-manifolds are explicitly not closed in that way (compare Definition 5.6). However, the set of all n-simplices contained in a finite simplicial complex of dimension n is a composite n-manifold.

For many practical purposes, we want to determine whether a finite set of m-manifolds is a composite m-manifold. This can be done in polynomial time complexity of $\mathcal{O}(n^2)$ using Procedure 1 with the assumption that we have an oracle that is able to determine whether the interior of a given manifold M intersects another given manifold MF, i.e., whether M° and MF share a point. This is the critical part of the procedure that depends on the specific data structure used to represent manifolds and may be nontrivial or at least not doable in polynomial time. If such an oracle or an adequate decision procedure is available, Procedure 1 simply checks that the pairwise intersections of *any* two m-manifolds in the collection do not include an interior point of either manifold.

Procedure 1 Decide whether a collection of m-manifolds is a composite m-manifold

Require: $\mathfrak{M}$ be a collection of m-manifolds.
Ensure: return true iff $\mathfrak{M}$ is a composite m-manifold.

```
while 𝔐 ≠ ∅ do
  pick a M ∈ 𝔐
  𝔐 := 𝔐 \ M
  for all MF ∈ 𝔐 do
    if M° ∩ MF ≠ ∅ or M ∩ MF° ≠ ∅ then
      return false
    end if
  end for
end while
return true
```

Similarly to Procedure 1 we can construct larger composite m-manifolds from a given composite m-manifold by adding one m-manifold at a time and ensuring that it only touches the existing m-manifolds in boundaries.

5.4 The intended structures: Complex m-manifolds

The class of structures we intend to capture are spatial configurations consisting of collections of idealized space regions, each being a composite m-manifold, arranged arbitrarily in $\mathbb{R}^n$ where n is greater or equal to the greatest dimensions of all the manifolds in the configuration. Space regions with different dimensions can coexist in a structure as long as each of them is of uniform dimension. Equally, even the composite manifolds of greatest dimension may have a codimension greater than zero. Space regions can be arranged arbitrarily, in particular in ways so that the interior of one intersects the interior or boundary of another (which we disallowed within a single composite manifold). Any space region represented by a composite manifold can also spatially contain lower-dimensional spatial regions, which are just not in the composite manifold's collection of atomic manifolds. In particular, a curve or a line (either being a 1-manifold), or an area (a 2-manifold) are different from any finite set of points (a 0-manifold) they may contain. Recall that we only consider finite collections as composite manifolds, i.e., infinite point sets are not composite manifolds of dimension zero. Equally, a line cannot completely fill a region, that is, space-filling curves are impossible because they self-intersect as we discussed before. Though a set of points and a curve may be spatial entities in one spatial arrangement, they are always of different dimensions, with the area being of higher dimension than the curve and the curve being of higher dimension than the set of points, and thereby neither of them can be identical.

Similarly to the definition of simplicial complexes, we require that the composite m-manifolds in our class of intended structures are arranged "nicely": that means the intersection of any two (atomic or composite) m-manifolds is either empty or a composite n-manifold with $n \leq m$. Earlier, we already ensured that any two atomic m-manifolds in a composite m-manifold may only intersect in their boundaries. We extend this now by the following condition:

- The intersection of any two atomic or composite manifolds in an intended structure is equivalent to some collection of composite manifolds of the structure with a distinguished composite manifold of highest dimension.

This covers three cases: (1) the intersection of two atomic m-manifolds that are in the same composite m-manifold must be a collection of composite n-manifolds with a distinguished maximal composite manifold of greatest dimension $n < m$, (2) the intersection of two atomic manifolds of dimensions n and m in different composite manifolds must be equivalent to a collection of composite manifolds with a distinguished maximal composite manifold of greatest dimension $l \leq \min(n, m)$, and (3) the intersection of two composite manifolds of dimensions m and n must be equivalent to a collection of composite manifolds with a distinguished maximal composite manifold of greatest dimension $l \leq min(n, m)$.

Why do we have to resort to the cumbersome notion of a *maximal composite manifold of greatest dimension* contained in the point intersection? This is due to the fact that two manifolds (even within the same composite manifold) may intersect in disjoint sets of various dimensions; for example, two 2-manifold (2D areas) may intersect in one or multiple 1-manifolds (curve or line segments) while also intersecting in one or multiple 0-manifolds (points) that are not contained in any of the curve or line segments. In this case, the maximal manifold of greatest dimension would be a 1-manifold containing all shared curve and line segments but not containing the 0-manifolds in the intersection that are not themselves contained in any 1-manifold of the intersection. If two 2-manifolds (not in the same composite manifold) intersect in a single or in a set of 2-manifolds, then all 1-manifolds and 0-manifolds not contained in a shared 2-manifold are ignored. The following definition formalizes the condition.

Definition 5.11. *Let $\mathfrak{M}$ be a finite collection of composite manifolds.*

$\mathfrak{M}$ is a complex manifold if and only if the following conditions are satisfied:

1. *any atomic manifold $\mathrm{MF} \in \{\mathrm{MF} :$ there exists a composite or atomic manifold $\mathrm{MF}' \in \mathfrak{M}$ such that $\mathrm{MF} \in \mathrm{MF}'\}$ is a singleton composite manifold $\{\mathrm{MF}\} \in \mathfrak{M}$;*

2. *for any $\mathrm{MF}_1^m, \mathrm{MF}_2^n \in \mathfrak{M}$ with $\Sigma\mathrm{MF}_1^m \cap \Sigma\mathrm{MF}_2^n \neq \emptyset$ there exists a nonempty collection of manifolds $\mathfrak{M}' \subseteq \mathfrak{M}$ such that*
$$\bigcup_{\mathrm{MF}' \in \mathfrak{M}'} \Sigma\mathrm{MF}' = \Sigma\mathrm{MF}_1^m \cap \Sigma\mathrm{MF}_2^n;$$
and there exists a $\mathrm{MF}_3^k \in \mathfrak{M}'$ so that for all $\mathrm{MF}_4^l \in \mathfrak{M}'$;
   (a) *$\Sigma\mathrm{MF}_4^l \subseteq \Sigma\mathrm{MF}_3^k$, or*
   (b) *$l < k$;*

3. *for any $\mathrm{MF}_1^m, \mathrm{MF}_2^n \in \mathfrak{M}$ with $\Sigma\mathrm{MF}_1^m \cap \Delta\mathrm{MF}_2^n \neq \emptyset$ and $dim(\Sigma\mathrm{MF}_1^m \cap \Delta\mathrm{MF}_2^n) < dim(\mathrm{MF}_1)$ there exists a nonempty collection of manifolds $\mathfrak{M}' \subseteq \mathfrak{M}$ such that*
$$\bigcup_{\mathrm{MF}' \in \mathfrak{M}'} \Sigma\mathrm{MF}' = \Sigma\mathrm{MF}_1^m \cap \Delta\mathrm{MF}_2^n;$$

4. *for any $\mathrm{MF}_1^m, \mathrm{MF}_2^m \in \mathfrak{M}$ with $dim(\Sigma\mathrm{MF}_1^m \setminus \Sigma\mathrm{MF}_2^m) = dim(\mathrm{MF}_1^m)$, there exists a $\mathrm{MF}_3^m \in \mathfrak{M}$ such that*
$$\Sigma\mathrm{MF}_3^m = \overline{\Sigma\mathrm{MF}_1^m \setminus \Sigma\mathrm{MF}_2^m}.$$

Let us remind the reader that our definition is totally unrelated to how the term *complex manifold* is used in differential geometry [MK71]. Our notion of a *complex manifold* will be central throughout the remainder of the thesis; some explanation of it is due. The first condition is obvious: Each composite manifold's area is the union of the areas of a collection of singleton composite manifolds in $\mathfrak{M}$. In other words, every atomic manifold in some composite manifold in $\mathfrak{M}$ is also a composite manifold in $\mathfrak{M}$.

Condition (2) requires that the point-set intersection of any two manifolds is identical to the sum of the areas of the collection $\mathfrak{M}'$ of shared manifolds. It further requires that a unique maximal manifold of maximal dimension exists in this collection $\mathfrak{M}'$. Thus, a complex manifold is closed under *point-set intersection*, though the point-set intersection may not be a single manifold; and is closed under *manifold intersection*, in which the dimension k of the maximal shared composite manifold MF_3^k is greater than or equal to the dimension l of all other composite manifolds MF_4^l in the collection $\mathfrak{M}'$. Moreover, it is easy to see that $k \leq min\{m, n\}$ must hold.

In the case where one manifold intersects another manifold in its boundary (not necessarily only in its boundary), condition (3) requires the intersection of the former manifold with the latter manifold's boundary to be covered by a collection of manifolds $\mathfrak{M}'$. However, in this collection, there does not have to exist a unique maximal manifold of maximal dimension.

Lastly, condition (4) requires a complex manifold to be closed under set differences when the set difference is of the same dimension as the minuend.

Note that $\mathfrak{M}$ is a collection with a specific structure. To make this distinction more explicit, we often write $Dom(\mathfrak{M})$ to denote the collection of composite manifolds that constitute a complex manifold $\mathfrak{M}$.

Just like every atomic or composite manifold, every complex manifold can be assigned a dimension. However, complex manifolds are—unlike atomic and composite manifolds—not of uniform dimension. Instead, a complex manifold has an embedding dimension: the largest dimension of all its constituting composite m-manifolds.

Definition 5.12. *Let $\mathfrak{M}$ be a complex manifold that is a finite collection of composite manifolds.*

We say $\mathfrak{M}$ is a finite complex manifold of dimension m (short: a finite complex m-manifold), and write $\mathfrak{M}^m$, if and only if $n \leq m$ for all $\mathrm{MF}^n \in \mathfrak{M}$ and there exists a $\mathrm{MF}^n \in \mathfrak{M}$ with $n = m$.

The class of all finite complex m-manifolds is our class of intended structures. Every complex manifold in this class is a finite set of composite manifolds, and every of its composite manifolds is composed of a finite set of atomic manifolds. We will not deal with infinite sets complex manifolds in this thesis.

Definition 5.13. *Let $\mathbb{M}$ denote the class of finite complex manifolds.*

When necessary, we restrict $\mathbb{M}$ to collections $\mathfrak{M}^m$ with a specific m; those are captured by subclasses $\mathbb{M}^m$. For example, $\mathbb{M}^3$ is the class of all finite complex manifolds $\mathfrak{M}^3$. In other words, each structure $\mathfrak{M}^3$ in $\mathbb{M}^3$ contains at least one composite 3-manifold and contains no (atomic or composite) m-manifold with $m > 3$. Equally, $\mathbb{M}^{\leq 3}$ denotes the class of all finite complex m-manifolds with $m \geq 3$, that is, the class of all finite complex manifolds that are embeddable in $\mathbb{R}^3$.

In Chapter 9, we will further restrict the class of intended structures to $\mathbb{M}_{\mathrm{dense}}$ in which every structure $\mathfrak{M}^m$ contains a manifold of *every* dimension $n \leq m$ (compare Section 9.1). This restriction will be necessary to utilize special properties of the manifolds of next-lowest dimension contained in a manifold.

Previously, we defined the interior MF° and boundary $\Delta\mathrm{MF}$ of any composite or atomic m-manifold. Within a finite complex manifold $\mathfrak{M}^m$ in the class $\mathbb{M}$, we can also define the exterior of any composite or atomic n-manifold $\mathrm{MF}^n \in \mathfrak{M}^m$ with $n \leq m$ as follows. Intuitively, the exterior of a manifold is the set of all points in the structure that are not contained in the interior or boundary of the manifold itself. Recall that $\Sigma\mathrm{MF}$ denotes the area of a manifold, which is comprised of the manifold's interior and boundary.

Definition 5.14. *Let $\mathfrak{M}$ be a structure in the class $\mathbb{M}$. Then for any $\mathrm{MF} \in \mathfrak{M}$ we define the exterior MF^- as*

$$\mathrm{MF}^- = \Big(\bigcup_{\mathrm{MF}' \in \mathfrak{M}} \Sigma\mathrm{MF}' \Big) \setminus \Sigma\mathrm{MF}.$$

While the interior and boundary of a composite manifold can be defined independently of the complex manifold it is contained in, the exterior is only definable with reference to a particular space, which is defined by the complex manifold containing the composite manifold.

More complex entities of mixed dimension are not objects of the domain, but can be captured as sets of simple entities or introduced as a separate class using a new unary relation (a sortal). We will not deal with those in any detail in the thesis.

5.5 About the structures in the class of intended structures

The class of intended structures encompasses a wide range of spatial structures if the composite manifolds are appropriately chosen. Here, we want to give a sample of the structures that are within that class.

By design all simplicial complexes are in the class of intended structures. But more general kind of structures are also in the class $\mathbb{M}$ (and in the subclass $\mathbb{M}_{\text{dense}}$ defined in Section 9.1).

One important subclass of $\mathbb{M}$ are planar geometries, these include Euclidean geometry but also, among others, elliptical and spherical geometries [Gre94]. This will become evident in Chapter 10 when we investigate the relationship between ordered incidence geometries, of which Euclidean and elliptic geometries are extensions, and our own logical theories.

The spatial arrangements we are interested in are maybe best visualized using Kandinsky's abstract paintings, such as the one shown in Figure 5.4, which is essentially a spatial arrangement of various geometric figures of dimensions zero, one, and two in a two-dimensional space. The various linear and areal features can be treated as composite 1- and 2-manifolds; the entire painting is then a complex 2-manifold. The relations we propose in this thesis can be used to qualitatively describe the painting. For example we can say that a green square partially overlaps a yellow triangle and is in superficial contact to a red square. The green square is incident but does not contain two straight line segments that start in the interior of the square. The purple disk is also incident with two line segments, but those cross the disk, i.e., they do not have an end in the interior of the disk. A notion not captured by the class $\mathbb{M}$ but relevant in Chapter 10 is that of betweenness. For example, within the large triangle in the centre of the painting, the brown straight line segment is in between the brown triangle and the disk that contains a set of dots (points). Also, the rose rectangle is in between the light green rectangle and the brown rectangle within the self-connected rectangle that is composed 16 rectangles in a diagonal row just left of the painting's centre.

One special class of arrangements of one- and zero-dimensional entities are simple graphs: any graph is a simplicial complex and thus in the class $\mathbb{M}$. We can consider the edges of a graph as 1-simplices and its vertices as 0-simplices. Schematic maps that are represented (or representable) as graphs, are thus in $\mathbb{M}$ as well. For example, public transportation networks including train, subway, streetcar, bus, and ferry routes are naturally structures in the class $\mathbb{M}$ based on their graph representation. However, a simple graph representation actually loses information about the network, namely the information which edges form a route (e.g., a bus route or a subway line). But preserving the route information is not difficult (though it may lead to duplicate edges between two vertices, resulting in a non-simple graph) while remaining within the class of structures $\mathbb{M}$. Schematic maps of public transportation networks preserve topological information while abstracting, e.g., shape (the exact geographical route) and distance information between stations [AH00; Mor96]. Other schematic topological maps that are then also in $\mathbb{M}$ include abstracted maps of hiking trails (see Figures 5.5 and 5.6), of ski trails [Fie09], of water ways, or of campgrounds.

These examples point to a more general class of structures within the intended structures: maps for various purposes. Examples of maps that are representable as complex manifolds include not just schematic maps but also topographical maps, highway maps, city maps, country maps, nautical maps, as well as small, incomplete excerpts from maps used for giving direction (as to a hotel, a restaurant, or a conference location), or even sketch maps (see Figure 5.7 for an example). Take a city map as example: its neighbourhoods, parks, and squares may be treated as two-dimensional entities, its streets and paths as one-dimensional entities, and important buildings, points of interest, or subways stops as zero-dimensional entities. A highway map may be less detailed, containing only states and cities (if large enough) as areal features, highways and other main roads as linear features and smaller towns, rest areas, and gas stations as point features. Maps used for indoor navigation, such as evacuation maps,

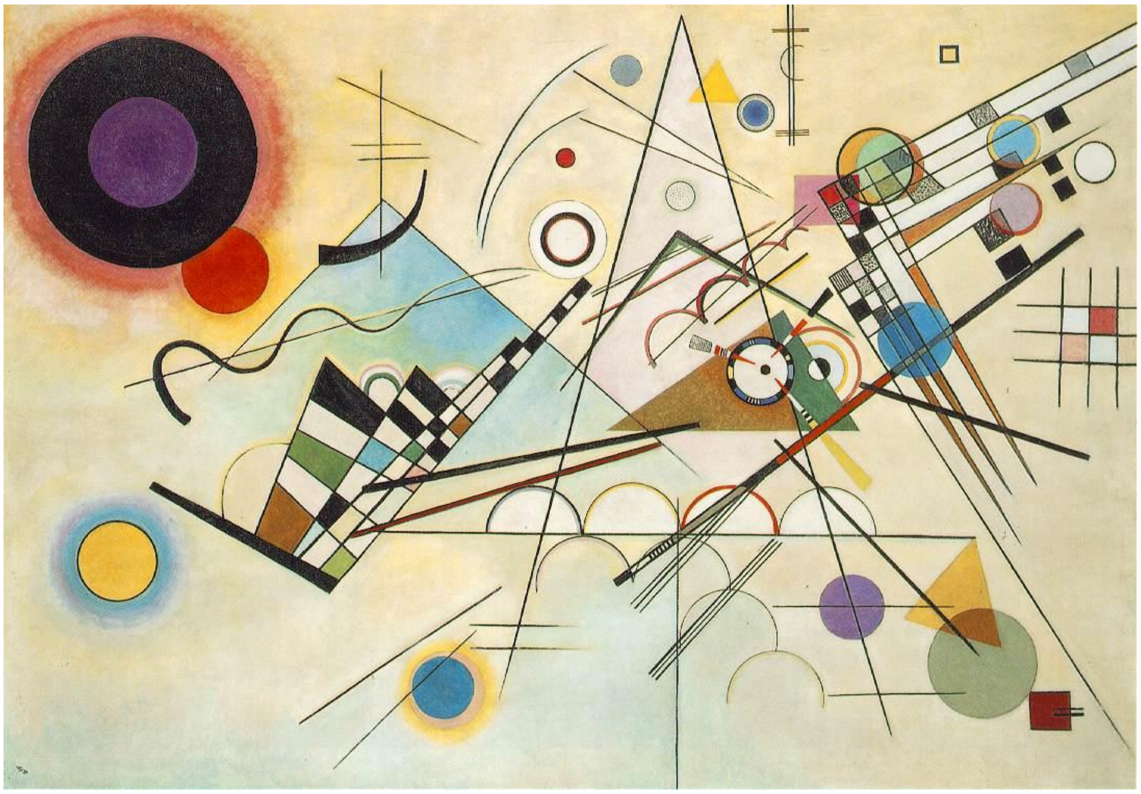


Figure 5.4: Komposition VIII (oil painting) by Wassily Kandinsky, 1923, from `http://www.wikipaintings.org/en/wassily-kandinsky/composition-viii-1923`, used under fair dealing according to *Copyright Act*, Canada, 2004.
This painting is an example of a “spatial” arrangement we aim to capture by the class $\mathbb{M}$. When viewed as a spatial arrangement, it consists of various two-, one-, and zero-dimensional spatial entities, which are composite manifolds. It contains examples of many of the spatial relations we discuss in this thesis. Examples of two-dimensional entities are areas of uniform color such as disks (filled circles), triangles, squares, and other rectangles. One-dimensional entities found in this painting are the various straight and curved line segments including half-circles or circles. Some of the areas have one-dimensional entities that may serve as their boundaries, such as the yellow disk on the left side, which is bounded by a black circle. Other areas only have implicit boundaries, for example, the yellow triangle at the top is not explicitly separated from the ground. Wherever two one-dimensional entities (which may be boundaries of two dimensional entities) meet or intersect, there is an implicit point, a zero-dimensional entity. Where a one-dimensional entity penetrates or crosses an area, their intersection is a part of the one-dimensional entity. We then say the one-dimensional entity is incident with the two-dimensional entity; for example, the blue disk at the bottom is incident with the line segment passing through it. Areas also overlap, for example in the bottom right corner the yellow triangle overlaps the green square.

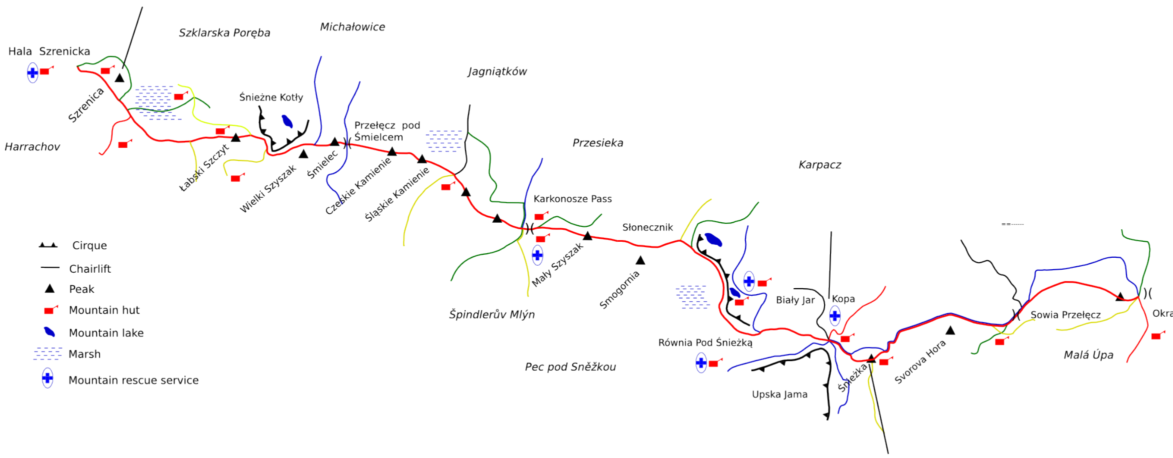


Figure 5.5: Schematic hiking map of the Polish-Czech Friendship Trail, from `http://commons.wikimedia.org/wiki/File:Friendshiptrail1.png`, used under fair dealing.
The map shows the main trail and outgoing or crossing trails (all linear features) with the trail junctions (point features). It also displays points of interest along the trail, such as peaks and mountain huts (point features). Areal features include lakes and marsh areas, with some trails passing by or through them.

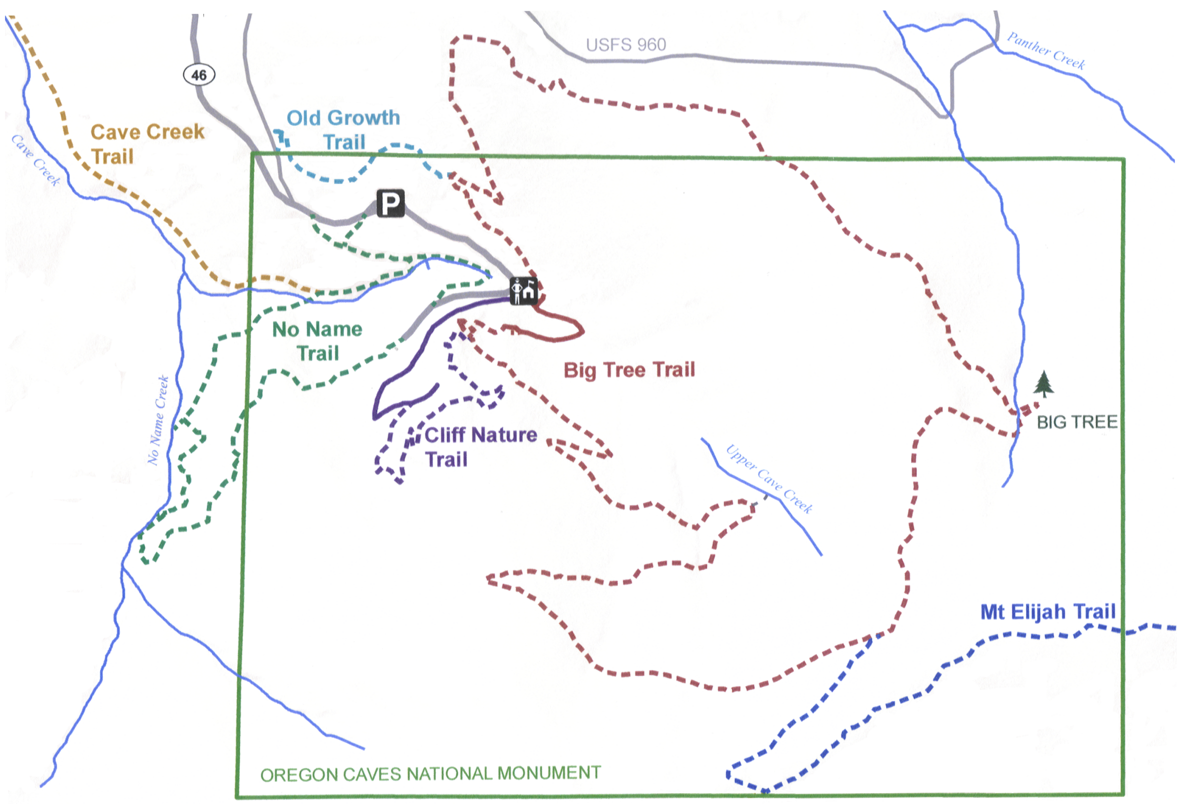


Figure 5.6: Schematic map of the hiking trails in and around Oregon Caves National Monument in the US state of Oregon, from `http://commons.wikimedia.org/wiki/File:ORCA_Hiking_trail_map_.png`, which is in the public domain as a piece of work created by a US government official.
This schematic map shows the extent of the park (an area), various trails (linear features), and trail junctions (point features). It also shows other linear features, such as creeks, main access roads as well as point features, such as the parking lot, the forest ranger hut, or the “big tree”.

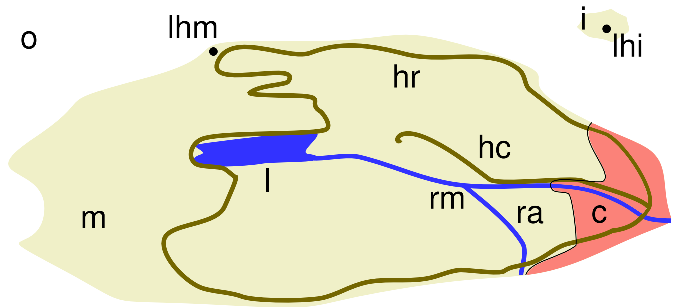


Figure 5.7: A sketch map containing entities of three different dimensions.
We have two-dimensional entities (**o**cean, **m**ain island, small **i**sland, **c**ity, **l**ake); one-dimensional entities (**r**iver **m**ain, **r**iver **a**rm, **h**ighway **r**ing, **h**ighway **c**entral); and point (zero-dimensional) entities (**l**ight**h**ouse **m**ain, **l**ight**h**ouse **i**sland).

maps showing the stores in a shopping mall, or maps of an airport, can also be modelled as complex manifolds. Likewise, idealized representations of any kind of built environment are representable as complex manifolds, see Figure 5.8 for an example. Depending on the application, buildings can be modelled as containing several multiple floors, each again composed of rooms, hallways, and separating walls, and with doors or windows as well as stairways, escalators, and elevators. We can include point features, such as the location of fire extinguishers, power outlets, or water fountains. If necessary, we can also include linear features such as wiring (electrical, communication) or utility piping (water, gas), in an abstract model of a building.

Another subclass of $\mathbb{M}$ are arrangements of arbitrary (two-dimensional) polygons in any finite-dimensional space as long as all non-decomposable polygons within any single composite manifold are only joined ("glued") along their one-dimensional boundaries. The polygons can be folded or curved in a higher dimension, or form certain kind of boxes, such as a cereal box with slots and tabs—the example discussed in the *BoxWorld* ontology in [GB11]. What kind of boxes can and cannot be modelled as composite manifolds must be investigated in more detail in the future. A special class of arrangements of polygons are 2-complexes composed of polytopes instead of simplices. In those, polygons only meet in their boundaries and all bounding straight line segments and their endpoints are in the complex as well. Such polygonal complexes are special kinds of polytopal complexes [Zie95] in which all polytopes are two-dimensional, that is, they are polygons. Generally, polytopal complexes consist of a "nice" arrangement of arbitrary polytopes, the cells, in some Euclidean space $\mathbb{R}^n$ with $n > m$ for any m-polytope in the structure. "Nice" means again that the face of every polytopes is in the structure as well and that polytopes only intersect in their faces (which are their boundaries). It is easy to see that every finite polytopal complex of finite dimension is a structure in $\mathbb{M}$. All triangulated irregular networks (TIN) [Peu+78], a spatial representation popular in geographic information systems, and regular grids as used for raster representations are special cases of polytopal complexes. Other special classes of polytopal complexes include polyhedra assembled along their boundaries to more complex objects in three-dimensional space. For example, the spatial structure of the *BlocksWorld* domain [Win72] frequently used as toy example in AI planning could be model as two- or three-dimensional spatial arrangement; each block is a polyhedron and no two blocks overlap. In CAD, CAM, and computer graphics so-called *meshes* of polygons (usually simplices) are used to represent polyhedra. Such representations of solid objects are consequently also in the class $\mathbb{M}$.

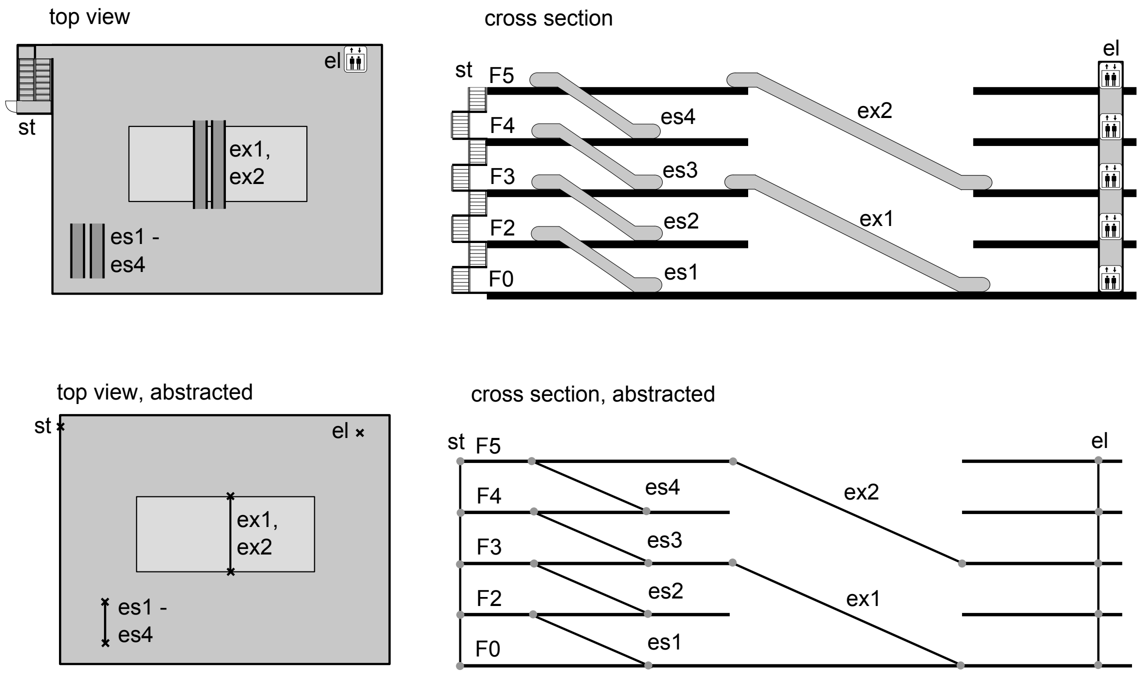


Figure 5.8: Top view and cross section of a building with escalators and an elevator. While the building would intuitively be considered a 3D object, each floor can be treated as a 2D area, if we are, e.g., only interested in navigating the building for which room heights are usually irrelevant (as long as they are assumed to be of sufficient height). Escalators ($es1 - es4$ and $ex1$, $ex2$), elevators (el), and staircases (st) could be treated as line segments as long as it is clear that, e.g., a person can navigate those lines (as we usually assume a person or a car can navigate “lines” denoting streets on a map). Each floor would then, for example, have various points from which it is possible to access other floors. For many tasks, in particular navigation, such a simplified view would be sufficient. The bottom row shows the same views in a abstract way. While the building may be considered a 3D entity (not visible), the floors are 2D areas, connected by one-dimensional line segments, which are abstractions of staircases, escalators, and elevators. They connect floors and have access points (the point where they meet the floor). This abstract representation is perfectly fine to solve any coarse navigational tasks, e.g., how to get from one point to another point in the building or how to evacuate the building in the case of an emergency.

Polytopal complexes further generalize to so-called CW-complexes [LW69]. But the exact relationship between CW-complexes and the class $\mathbb{M}$ remains to be investigated (**Question 9**).

Chapter 6

A basic first-order theory of multidimensional mereotopological space[1]

One way to overcome the restrictions in the expressivity of equidimensional mereotopologies is by allowing spatial configurations that consist of entities of multiple dimensions. While a wealth of theories capturing the mereotopological relations between spatial entities of equal dimension are known, only few theories define relations between entities of different dimensions. But Freeman [Fre75] noted early on that many relations used in everyday language have intrinsic dimensionality constraints in that they, for example, neglect the width and/or thickness of some physical objects. In that sense, humans often use idealizations of physical space and, since we are primarily concerned with abstract spatial entities, such idealizations are appropriately captured by entities of different dimensions. Moreover, there are algorithms available to effectively determine the minimal embedding dimension of a set of points in space [Dey+02], thereby assigning a dimension to entities for which only some points are given.

Most work on non-equidimensional mereotopological relations, in particular in the field of geographic space, focus on area-area, area-line, area-point, line-line, line-point, and point-point relations in two-dimensional space, e.g., [BH11; CDFO93; EH91; McK+05]. This results in a large set of differing relations of which many only differ in the dimensions of the involved entities. For example, McKenney et al. [McK+05] distinguish 61 line-line relations alone, a set way too large for interaction with humans as well as for automated reasoning. Moreover, only McKenney et al. explicitly define the dimension of entities in their framework, but they rely on the topological definition of Lebesgue covering dimension. TO provide such a topological definition of dimension in a logical theory requires us to first axiomatize a large mathematical apparatus.

Truly multidimensional mereotopological theories where the absolute (or numeric) dimensions do not matter have only been proposed by Galton [Gal04] and Gotts [Got96]. Both never axiomatize dimension explicitly; instead Galton relies on boundaries to construct entities of increasingly lower dimensions, while Gotts uses a single primitive relation of 'including a chunk', *INCH*, a rather unintuitive multidimensional mereotopological relation, to topologically relate entities of potentially different dimensions.

[1] The work in this chapter is an extension and – to some extent – a simplification of [HG11a]. Some of the simplifications have been previously used in [HG11b].

We follow in the footsteps of Gotts by axiomatizing the mereotopological relations between two entities independent of their dimensions and independent of boundaries, but using a notion of spatial containment, *Cont*, that is more intuitive than Gott's relation *INCH*. Together with a notion of relative dimension, we can distinguish a small set of specialized topological relations which are not restricted to specific absolute dimensions. In this dimension-independent approach to mereotopology we favour neither the bottom-up approach (defining higher-dimensional entities in terms of points) employed in classical geometry nor the top-down approach (taking higher-dimensional regions as foundational and reconstructing dependent lower-dimensional entities) employed in equidimensional mereotopology. Instead, the objective here is to axiomatize topological and mereological relations in an unsorted theory where entities of various dimensions co-exist as first-class domain objects, similar to Hilbert's axiomatization [Hil71] of Euclidean geometry. The challenge is to separate the relations that can hold between entities regardless of their dimension from the relations that constrain the dimensions of the involved entities. Addressing this challenge is the main purpose of this chapter, in which we develop a basic theory of multidimensional mereotopology from first principles.

6.1 A naïve theory of relative spatial dimensions

Various notions of dimension have been employed within theories of qualitative space. We want to axiomatize dimension in the weakest possible way which is still suitable for defining spatial relations that are limited to entities of certain (relative) dimensions. For example we want to be able to express that region A has a higher dimension than region B or the intersection of regions A and B has a lower dimension than either one. Thereby it is unnecessarily restrictive to e.g., require that dimensions can be added or subtracted or restrict the total number of distinct dimensions. In other words, the sought axiomatization should be just strong enough to allow us to compare the dimensions of spatial entities.

A brief look at the various definitions of dimension in topology can be of help. There we find the small and large inductive dimensions, the Lebesgue covering dimension [compare McK+05], the Hausdorff dimension, and the notion of dimension in the theory of manifolds. [Eng95] gives a good overview of dimension from the topological perspective. Other notions of dimensions, e.g., those used for vector spaces or Hilbert spaces, are difficult to include in a qualitative theory of space.

A theory of dimension that suits our needs can be constructed reusing core ideas from inductive definitions of dimension. However, the relevant topological definitions are either still overly restrictive or rely on a heavy topological apparatus of which we would like to rid ourselves. In [HG11a] we presented our most basic theory of (relative) dimension, called DI_{basic}, which is based on three primitive relations: two binary relations of relative dimension, $<_{\text{dim}}$ and $=_{\text{dim}}$, and a unary relation denoting a zero region, *ZEX*. The intended interpretations of $x <_{\text{dim}} y$ and $x =_{\text{dim}} y$ are 'x has a lower dimension than y' and 'x and y have the same dimension', respectively. This basic theory could be axiomatized equivalently using $\leq_{\text{dim}}$ as the only primitive relation, which is a preorder. DI_{basic} allows models with incomplete information about the relative dimension between pairs of entities, i.e., some pairs of entities may not be dimensionally comparable at all. This basic theory is already sufficiently strong to distinguish mereotopological relations that depend on the relative dimension between two entities. Here, the theory DI_{basic} will not be discussed in more detail, we refer the interested reader to [HG11a].

Instead we will work with a stronger theory (that we also used in [HG11b]), in which all entities are dimensionally comparable. In this stronger theory denoted as

$$DI_{\mathrm{linear-unbounded}} = \{\text{D-A1} - \text{D-A5, D-D1} - \text{D-D7}\}$$

the classes of equidimensional entities are linearly ordered by $<_{\mathrm{dim}}$. More precisely, the relation $<_{\mathrm{dim}}$ is a strict weak order, whereas $\leq_{\mathrm{dim}}$ is a total order. For these reasons, we call this stronger theory the theory of linear (relative) dimension. It will be the focus of our discussion here. This theory can be defined using $<_{\mathrm{dim}}$ as the only primitive relation of relative dimension together with ZEX as primitive relation denoting a zero region.

(D-D1) $x >_{\mathrm{dim}} y \leftrightarrow x <_{\mathrm{dim}} y$ (greater dimension)

(D-D2) $x =_{\mathrm{dim}} y \leftrightarrow x \not<_{\mathrm{dim}} y \wedge y \not<_{\mathrm{dim}} x$ (equal dimension)

(D-D3) $x \leq_{\mathrm{dim}} y \leftrightarrow x <_{\mathrm{dim}} y \vee x =_{\mathrm{dim}} y$ (lesser or equal dimension)

(D-D4) $x \geq_{\mathrm{dim}} y \leftrightarrow x >_{\mathrm{dim}} y \vee x =_{\mathrm{dim}} y$ (greater or equal dimension)

(D-D5) $MaxDim(x) \leftrightarrow \forall y\, [y \leq_{\mathrm{dim}} x]$ (maximal-dimensional entity)

(D-D6) $MinDim(x) \leftrightarrow \neg ZEX(x) \wedge \forall y\, [y <_{\mathrm{dim}} x \rightarrow ZEX(y)]$ (minimal nonzero dimension)

(D-D7) $x \prec_{\mathrm{dim}} y \leftrightarrow x <_{\mathrm{dim}} y \wedge \forall z\, [z \leq_{\mathrm{dim}} x \vee y \leq_{\mathrm{dim}} z]$ (next highest dimension)

(D-A1) $x \not<_{\mathrm{dim}} x$ ($<$ irreflexive)

(D-A2) $x <_{\mathrm{dim}} y \rightarrow y \not<_{\mathrm{dim}} x$ ($<$ asymmetric)

(D-A3) $x <_{\mathrm{dim}} y \wedge y \leq_{\mathrm{dim}} z \rightarrow x <_{\mathrm{dim}} z$ ($<_{\mathrm{dim}}$ transitive[2])

(D-A4) $ZEX(x) \wedge ZEX(y) \rightarrow x = y$ (unique ZEX)

(D-A5) $ZEX(x) \wedge \neg ZEX(y) \rightarrow x <_{\mathrm{dim}} y$ (ZEX has minimal dimension)

Axiom Set 6.1: Axioms D-A1 – D-A5 and definitions D-D1 – D-D6 of $DI_{\mathrm{linear-unbounded}}$.

In addition to the standard definitions of the other ordering relations: $>_{\mathrm{dim}}$, $=_{\mathrm{dim}}$, $\leq_{\mathrm{dim}}$, and $\geq_{\mathrm{dim}}$ (D-D1 – D-D4), we define what it means for an entity to be of maximal or of minimal nonzero dimension (D-D5 and D-D6). Finally, D-D7 defines the notion of the next-highest dimension, something like the successor function used for the natural numbers. The definition will play a prominent role in Chapter 9.

The relation $<_{\mathrm{dim}}$ is irreflexive, asymmetric, and transitive (a strict partial order; D-A1 – D-A3). D-A4 and D-A5 ensure that a potential zero region (we reuse the term ZEX from [Got96]) is unique and of lowest dimension. We do not claim that the zero region is ontologically meaningful. D-A10 demands a lowest-dimensional entity (apart from ZEX) without preventing infinite-dimensional models. As this paper will show, this theory is sufficiently strong to distinguish mereotopological relations that depend on dimensions.

The theorems D-T1 – D-T3 verify that the defined relation $=_{\mathrm{dim}}$ is indeed an equivalence relation, i.e., that $=_{\mathrm{dim}}$ is reflexive, symmetric, and transitive. We also verify that two entities $x =_{\mathrm{dim}} y$ behave equivalently with respect to other entities of differing dimension (D-T4, D-T5).

(D-T1) $x =_{\mathrm{dim}} x$ ($=_{\mathrm{dim}}$ reflexive)

(D-T2) $x =_{\mathrm{dim}} y \rightarrow y =_{\mathrm{dim}} x$ ($=_{\mathrm{dim}}$ symmetric)

[2] Notice that the axiom D-A3 in [HG11b] contains a typo corrected here.

(D-T3) $x =_{\text{dim}} y \wedge y =_{\text{dim}} z \to x =_{\text{dim}} z$ ($=_{\text{dim}}$ transitive)

(D-T4) $x =_{\text{dim}} y \wedge z <_{\text{dim}} x \to z <_{\text{dim}} y$ ($=_{\text{dim}}$ renders $<_{\text{dim}}$ transitive)

(D-T5) $x =_{\text{dim}} y \wedge x <_{\text{dim}} z \to y <_{\text{dim}} z$ ($=_{\text{dim}}$ renders $<_{\text{dim}}$ transitive)

Lemma 6.1. $DI_{\text{linear}-\text{unbounded}} \models$ *{D-T1 – D-T5}*

While $DI_{\text{linear}-\text{unbounded}}$ and all other theories of relative dimension considered here are agnostic about the existence of a zero region to accommodate extensions in which such region is either desirable or convenient, an extensions by Z-A1 to $DI^{0}_{\text{linear}-\text{unbounded}}$ or by NZ-A1 to $DI^{\neg 0}_{\text{linear}-\text{unbounded}}$ force/prevent a zero region. Obviously, Z-A1 and NZ-A1 cannot be used together, a inconsistent theory would result. In general, we denote the inclusion of Z-A1 or NZ-A1 by a superscript ^{0} or $^{\neg 0}$ in any theory.

(Z-A1) $\exists x ZEX(x)$ (existence of a *ZEX*)

(NZ-A1) $\neg ZEX(x)$ (no *ZEX* exists)

Axiom Set 6.2: Axioms Z-A1 and NZ-A1.

(D-A6) $\exists x\,[MinDim(x)]$ (a lowest nonzero dimensional entity exists)

(D-A7) $\exists x\,[MaxDim(x)]$ (bounded ≡ a maximal dimension exists)

(D-A8) $\neg MaxDim(x) \to \exists y\,[x \prec_{\text{dim}} y]$ (discrete set of dimensions: next highest dimension)

(D-A9) $\neg ZEX(x) \wedge \neg MinDim(x) \to \exists y\,[y \prec_{\text{dim}} x]$ (discrete set of dimensions: next lowest dimension)

Axiom Set 6.3: Axioms D-A6 – D-A9 of the linear theory of relative dimension $DI_{\text{linear}-\text{bounded}-\text{discrete}}$.

Neither of D-D5 – D-D7 force entities of lowest, highest, or next-highest dimension to exist. If we want to enforce that, we need to extend the theory $DI_{\text{linear}-\text{unbounded}}$ to more restrictive theories of bounded linear dimension and discrete bounded linear dimension. For most practical applications, it makes sense to at least assume that a lowest dimension exists. This is enforced in

$$DI_{\text{linear}} = DI_{\text{linear}-\text{unbounded}} \cup \text{ D-A6.}$$

Other extensions of practical importance are the theory of bounded dimension $DI_{\text{linear}-\text{bounded}}$, which non-conservatively extends DI_{linear} by D-A7, and $DI_{\text{linear}-\text{discrete}}$, which non-conservatively extends DI_{linear} by D-A8 and D-A9. Except for some models that are only of theoretical topological relevance, we would expect the set of different dimensions to be always discrete as in $DI_{\text{linear}-\text{discrete}}$. The strongest theory is

$$DI_{\text{linear}-\text{bounded}-\text{discrete}} = DI_{\text{linear}-\text{bounded}} \cup DI_{\text{linear}-\text{discrete}}.$$

$DI_{\text{linear}-\text{bounded}-\text{discrete}}$ is the most restrictive theory of relative dimension of interest to our work. The complete family of theories of relative dimension is depicted in Figure 6.1.

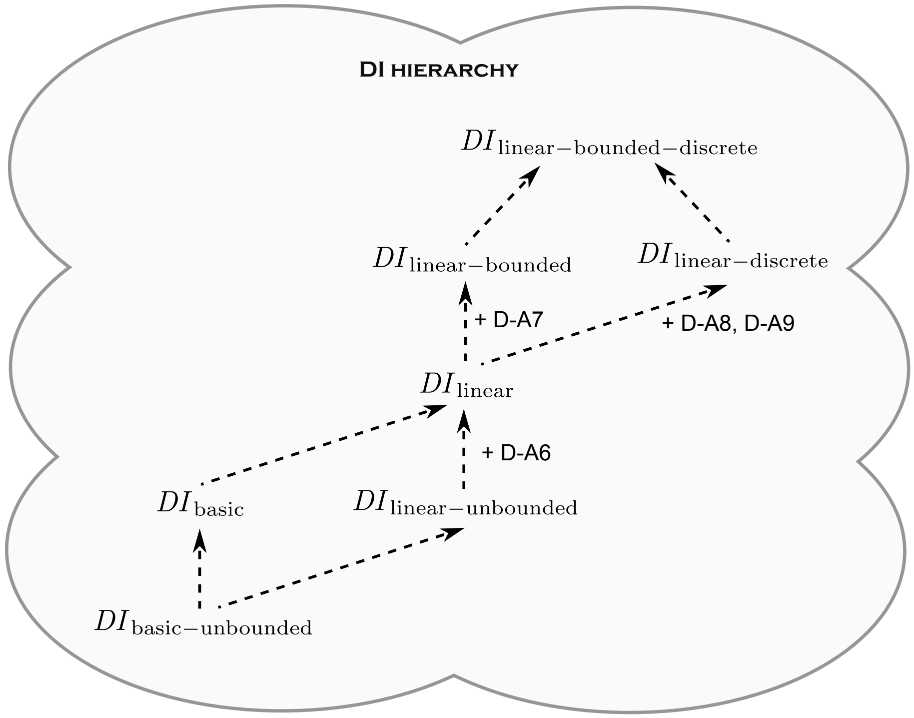


Figure 6.1: The hierarchy *DI* of theories of relative dimension.

6.2 Dimension-independent spatial relations

We proceed by examining the mereological and topological relations that can hold between spatial entities independent of their dimension. On the mereological side this is spatial containment, denoted by $Cont(x, y)$, and on the topological side it is contact, denoted by $C(x, y)$. Though we choose *Cont* as spatial primitive, C would serve equally well as primitive which can define *Cont*. We first observed and studied this interchangeability of a topological and mereological primitive in [HWG09] for the equi-dimensional mereotopology of Asher & Vieu [AV95].

6.2.1 Containment as mereological relation

What parthood is to equi-dimensional mereotopology, containment is to dimension-independent mereotopology. In its point-set interpretation, we say 'y contains x', i.e., $Cont(x, y)$, if x represents a nonempty set of points and every point in space occupied by x is also occupied by y. A region can contain not only a (smaller) region of the same dimension (equi-dimensional parthood), but also a lower-dimensional entity. e.g., a 2D-surface can contain another 2D-surface, a line, or a point. Containment is a nonstrict partial order. We again use *ZEX* to denote a zero region which neither contains nor is contained in any other region[3]. For the basic theory of containment,

$$CO_{\text{basic}} = \{\text{C-A1} - \text{C-A4, D-A4}\},$$

[3]This is a somewhat arbitrary choice and justified primarily by its cognitive coherence. Mathematically, it would be fine to assume that the zero region is contained in any other region, but using containment in this way seems counterintuitive.

we make no assumption about the (non-)existence of a zero region, but we include again D-A4 so that the zero region is unique if it exists at all. Two extensions are feasible: $CO^{0}_{\text{basic}} = CO_{\text{basic}} \cup \{\text{Z-A1}\}$ and $CO^{\neg 0}_{\text{basic}} = CO_{\text{basic}} \cup \{\text{NZ-A1}\}$.

(C-A1) $\neg ZEX(x) \leftrightarrow Cont(x,x)$ (*Cont* reflexive and definition of *ZEX*)

(C-A2) $Cont(x,y) \wedge Cont(y,x) \rightarrow x = y$ (*Cont* antisymmetric)

(C-A3) $Cont(x,y) \wedge Cont(y,z) \rightarrow Cont(x,z)$ (*Cont* transitive)

(C-A4) $ZEX(x) \rightarrow \forall y\,[\neg Cont(x,y) \wedge \neg Cont(y,x)]$ (nothing contains or is contained in the zero entity)

Axiom Set 6.4: Axioms C-A1 – C-A4 of the basic theory of containment CO_{basic}.

Recall the characterization of the structures we intend to capture: compositions of manifolds with boundaries. In particular, we perceive each entity as including its boundary (if there is one)—we do not distinguish an entity from its interior. We will get more precise in the treatment of boundaries in Chapter 9.

It is natural to assume that containment is extensional, that is, if two entities contain exactly the same set of entities, then they are equal. If only a single entity satisfying *ZEX* exists, as posited by D-A4, we can actually prove this weak kind of extensionality.

(C-T1) $\forall z\,[Cont(z,x) \leftrightarrow Cont(z,y)] \rightarrow x = y$ (*Cont* extensional)

Lemma 6.2. $CO_{\text{basic}} \models$ *C-T1*

If one entity x is contained in another entity y, this implies that they could only be equal if y in turn contains x. See Figure 6.2 for an example. Subsequently, we include C-T1 as an axiom, it will simplify some of the automated proofs. A stronger version of extensionality requires that no two entities contain exactly the same entities apart from themselves (known as *strong supplementation* from [CV99a], compare axiom EP-E2).

6.2.2 Contact as definable topological relation

Now contact C is definable in terms of containment (C-D), resembling the definition for overlap, $\forall x,y[O(x,y) \leftrightarrow \exists z(P(z,x) \wedge P(z,y))]$, in many equi-dimensional mereotopologies.

(C-D) $C(x,y) \leftrightarrow \exists z\,[Cont(z,x) \wedge Cont(z,y)]$ (contact)

Axiom Set 6.5: Definition C-D of the theory CO_{C}.

In $CO_{\text{basic}} \cup$ C-D (and all extensions thereof) the contact relation is provably reflexive, symmetric and monotone with respect to containment (C-T2 to C-T5).

(C-T2) $\neg ZEX(x) \rightarrow C(x,x)$ (C reflexive)

(C-T3) $C(x,y) \rightarrow C(y,x)$ (C symmetric)

(C-T4) $ZEX(x) \rightarrow \neg C(x,y)$ (nothing in contact with zero entity)

(C-T5) $Cont(x,y) \rightarrow \forall z\,[C(z,x) \rightarrow C(z,y)]$ (*Cont* implies C monotone)

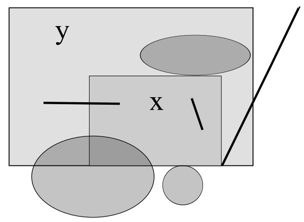


Figure 6.2: A model of CO_{C} in which $x \neq y$ is intended even though $x, y \in \mathbf{M}$ are in contact to all entities (assuming the displayed entities are the only existing ones). Hence extensionality of C (C-E2) is violated, but extensionality of *Cont* (C-T1) is satisfied because $\mathbf{Cont}(x, y)$ but $\neg\mathbf{Cont}(y, x)$.

Lemma 6.3. $CO_{\mathrm{basic}} \cup$ *C-D* $\vDash$ *{C-T2 – C-T5}*

The converse of C-T5 is not entailed and posited in a slightly weaker form as C-E1. The way C-E1 is expressed here is due to the fact that we do not want to force extensionality of C.

(C-E1) $\neg ZEX(x) \wedge \forall z\,[C(z, x) \rightarrow C(z, y)] \wedge \exists z\,[C(z, y) \wedge \neg C(z, x)] \rightarrow Cont(x, y) \wedge x \neq y$ (C strictly monotone implies *Cont*)

(C-E2) $\forall z\,[C(z, x) \leftrightarrow C(z, y)] \rightarrow x = y$ (C extensional)

Axiom Set 6.6: Extension axioms C-E1 and C-E2 of the theory CO_{basic}.

We obtain

$$CO_{\mathrm{C}} = \{\text{C-A1 – C-A4, D-A4, C-E1, C-D}\}.$$

It will turn that the axiom C-A5 is not needed throughout the thesis. In other words, no subsequent results rely on this axiom.

As side-effect, C-D mereologically closes the set of all entities in a very crude way. Contact between two entities requires the existence of a common contained entity—interpretable as intersection.

A useful and common assumption is extensionality of C (C-E2). It goes far beyond extensionality of *Cont* as already required in CO_{basic}: An entity x contained in y that is in contact to all entities in contact to y, must be identical to y even if y is not contained in x, compare Figure 6.2.

6.3 Interaction of dimension and containment

The theories of containment, such as CO_{basic}, can be combined with any extension of DI_{linear} by axiomatizing the direct relationship between containment and relative dimension: if x is contained in y, then x must have a dimension that is the same as or lower than that of y. We are particularly interested in the theory

$$CODI_{\mathrm{linear}} = CO_{\mathrm{basic}} \cup DI_{\mathrm{linear}} \cup \{\text{C-D, CD-A1}\}$$

as the weakest such combination.

Another common assumption is that the nonzero entities of lowest dimension are indivisible (CD-E1). Indivisibility is justified as long as the nonzero entities of lowest dimension represent points. If the entities of the lowest dimension represent lines or surfaces it is too strong an assumption. Though a necessary

(CD-A1) $\mathit{Cont}(x,y) \rightarrow x \leq_{\text{dim}} y$
(a contained entity has a dimension no greater than that of the entity it is contained in)

Axiom Set 6.7: Axiom CD-A1 for the basic theory of containment and linear dimension, $\mathit{CODI}_{\text{linear}}$.

extension e.g., for incidence geometry, we do not include CD-E1 in our general theory of containment and dimension.

(CD-E1) $\mathit{MinDim}(x) \rightarrow \forall y\,[\mathit{Cont}(y,x) \rightarrow x = y]$ (entities of lowest dimension are indivisible)

Axiom Set 6.8: Extension Axiom CD-E1 of the *CODI* hierarchy.

We use $\mathit{CODI}_{\text{linear}}$ to define three types of contact depending on the dimension of the entities and their common entity. We distinguish two types of strong contact and one type of weak contact. This does not depend on further restrictions of the theory of relative dimension. But first we define the useful notion of equi-dimensional parthood, i.e., containment between two entities of equal dimension (EP-D). For convenience, we will often use the definable relation of proper parthood, *PP* (EPP-D).

(EP-D) $P(x,y) \leftrightarrow \mathit{Cont}(x,y) \wedge x =_{\text{dim}} y$ (equidimensional parthood)
(EPP-D) $PP(x,y) \leftrightarrow P(x,y) \wedge x \neq y$ (equidimensional proper parthood)

Axiom Set 6.9: Definitions EP-D and EPP-D of the *CODI* hierarchy.

We verify that even in the relatively weak theory $\mathit{CODI}_{\text{linear}}$ parthood is a nonstrict partial order, that is, reflexive, antisymmetric, and transitive (EP-T1 to EP-T3). Parthood further implies contact (EP-T8). Other simple transitivity properties in interaction with dimension constraints (EP-T4 to EP-T7) also hold in $\mathit{CODI}_{\text{linear}}$.

(EP-T1) $\neg \mathit{ZEX}(x) \rightarrow P(x,x)$ (P reflexive)

(EP-T2) $P(x,y) \wedge P(y,x) \rightarrow x = y$ (P antisymmetric)

(EP-T3) $P(x,y) \wedge P(y,z) \rightarrow P(x,z)$ (P transitive)

(EP-T4) $P(x,y) \wedge z <_{\text{dim}} x \rightarrow z <_{\text{dim}} y$ (parthood preserves relative dimension)

(EP-T5) $P(x,y) \wedge y <_{\text{dim}} z \rightarrow x <_{\text{dim}} z$ (parthood preserves relative dimension)

(EP-T6) $P(x,y) \wedge z =_{\text{dim}} x \rightarrow z =_{\text{dim}} y$ (parthood preserves relative dimension)

(EP-T7) $P(x,y) \wedge z =_{\text{dim}} y \rightarrow x =_{\text{dim}} z$ (parthood preserves relative dimension)

(EP-T8) $P(x,y) \rightarrow C(x,y)$ (parthood requires contact)

(EP-T9) $\forall z\,[P(z,x) \leftrightarrow P(z,y)] \rightarrow x = y$ (P extensional)

Lemma 6.4. $\mathit{CODI}_{\text{linear}} \cup$ *{EP-D, EPP-D}* $\models$ *{EP-T1 – EP-T9}*

Analogously to the extensionality of *Cont* (C-T1), EP-T9 does not mean that two entities which have identical parts (apart from themselves) are necessarily identical. In other words, the notion of weak supplementation (EP-E1, compare page 109) from [CV99a] may still fail.

Now we can characterize the relationship between containment and dimension using the parthood relation. Recall that the axioms of relative dimension force $<_{\text{dim}}$ to be a strict partial order, while the relations *Cont* and *P* both define nonstrict partial orders. We can then formally characterize the relationship between these three relations as follows.

Theorem 6.1. *In a model $\mathcal{M}$ of $CODI_{\text{linear}} \cup \{EP\text{-}D\}$, $\mathbf{P}_{\mathcal{M}}$ and $(<_{\mathbf{dim}})_{\mathcal{M}}$ form a partition of $\mathbf{Cont}_{\mathcal{M}}$.*

Proof. It suffices to prove:

$$CODI_{\text{linear}} \cup \{\text{EP-D}\} \models Cont(x, y) \rightarrow P(x, y) \vee x <_{\text{dim}} y$$

and

$$CODI_{\text{linear}} \cup \{\text{EP-D}\} \models \neg P(x, y) \vee \neg x <_{\text{dim}} y.$$

The first sentence follows immediately from CD-A1, EP-D, and D-D4 and the second sentence follows from EP-D and D-D2. ☐

In other words, the containment relation *Cont* can be broken down into two disjoint and exhaustive subrelations *P*, which only holds for equidimensional entities, and $<_{\text{dim}}$, which only holds for non-equidimensional entities.

We also want to define a set of jointly exhaustive, pairwise disjoint subrelations of contact based on the relative dimensions among the two entities and their shared entity. To do that, we first introduce a few more definitions using parthood.

6.3.1 Maximal and minimal entities within a dimension

A special role in representations of space have entities of a given dimension that are not properly contained in any other entity of the same dimensions. In classical geometries, these are, e.g., lines or planes. ME-D1 defines the maximal entities within a dimension. Analogously, ME-D2 defines the minimal entities of a dimension, that is, the entities that have no proper parts.

(ME-D1)	$Max(x) \leftrightarrow \neg ZEX(x) \wedge \forall y\, [\neg PP(x, y)]$	(maximal in a dimension)
(ME-D2)	$Min(x) \leftrightarrow \neg ZEX(x) \wedge \forall y\, [\neg PP(y, x)]$	(minimal in a dimension)

Axiom Set 6.10: Definitions ME-D1 and ME-D2 of the *CODI* hierarchy.

These notions of ‘maximal in a dimension’ and ’minimal in a dimension’, $Max(x)$ and $Min(x)$, should not be confused with the notions of maximal and minimal dimension, $MaxDim(x)$ and $MinDim(x)$ as defined earlier. $Max(x)$ singles out the maximal entities within a single dimension, that is, the maximal entities with respect to the order defined by parthood, while $MaxDim(x)$ singles out the entities of greatest dimension, i.e., the entities that are maximal with respect to the order defined by relative dimension. In the intended structures, all atomic manifolds are minimal entities. Because each intended structures has a finite domain, i.e., has only a finite number of atomic and composite manifolds, maximal entities are guaranteed to exist.

In our work, the minimal and maximal entities will play key roles for many properties we intend to prove and for proving satisfiability of select theories. Because our intended models as described in

Chapter 5 are finite, we will often extend $CODI_{\text{linear}} \cup \{$EP-D, EPP-D$\}$ by ME-E1. This axiom requires models to be atomic, that is, every entity in the domain of a model must contain some atomic, i.e., minimal entity of the same dimension.

(ME-E1) $\forall x\big[\neg ZEX(x) \rightarrow \exists y[P(y,x) \wedge Min(y)]\big]$ (atomic: every nonzero entity has a minimal part)

Axiom Set 6.11: Extension axiom ME-E1 of the *CODI* hierarchy.

6.3.2 Relative dimension distinguishes three types of contact

In this subsection we will define three kinds of contact relations that are definable in $CODI_{\text{linear}}$. Together those three relations are jointly exhaustive and pairwise disjunct subrelations of contact. They are very natural in that they are often distinguished intuitively by humans. Unsurprisingly, they have occurred in more specialized forms and under different names throughout the literature on topological spatial relations, e.g., in [AV95; Ege89; RCC92].

Equidimensional strong contact: Partial overlap

Partial overlap (PO-D) is the strongest of our contact relations, it holds when two regions share a part, that is, they *overlap in a part*[4]. Partial overlap is in $CODI_{\text{linear}}$ a reflexive and symmetric relation requiring equi-dimensionality (PO-T1 to PO-T3). It is effectively what is known as overlap, O, in most equidimensional mereotopologies.

(PO-D) $PO(x,y) \leftrightarrow \exists z\,[P(z,x) \wedge P(z,y)]$ (partial overlap)

Axiom Set 6.12: Definition PO-D of the *CODI* hierarchy.

(PO-T1) $\neg ZEX(x) \rightarrow PO(x,x)$ (PO reflexive)

(PO-T2) $PO(x,y) \rightarrow PO(x,y)$ (PO symmetric)

(PO-T3) $PO(x,y) \rightarrow x =_{\text{dim}} y$ (PO requires equidimensionality)

Lemma 6.5. $CODI_{\text{linear}} \cup$ *{EP-D, PO-D}* $\models$ *{PO-T1 – PO-T3}*

Non-equidimensional strong contact: Incidence

Entities of different dimension can also be in strong contact. We generalize partial overlap to incidence (Inc-D) by requiring that the common element is an equi-dimensional part of exactly one of them. Even in the weak theory $CODI_{\text{linear}}$ incidence is irreflexive, symmetric, and prevents equi-dimensionality (Inc-T1 to Inc-T3). Inc-T4 to Inc-T6 show its dimension constraint.

[4] The name 'partial overlap' for this relation here may be a little confusing because it differs from its use as base relation in the RCC [RCC92] in which it is defined as $PO(x,y) \leftrightarrow O(x,y) \wedge \neg P(x,y) \wedge \neg P(y,x)$. The term 'partial' here refers to the fact that the entities overlap in an equidimensional part, not just in some lower-dimensional entity. The relation PO here allows one entity to be a part of the other.

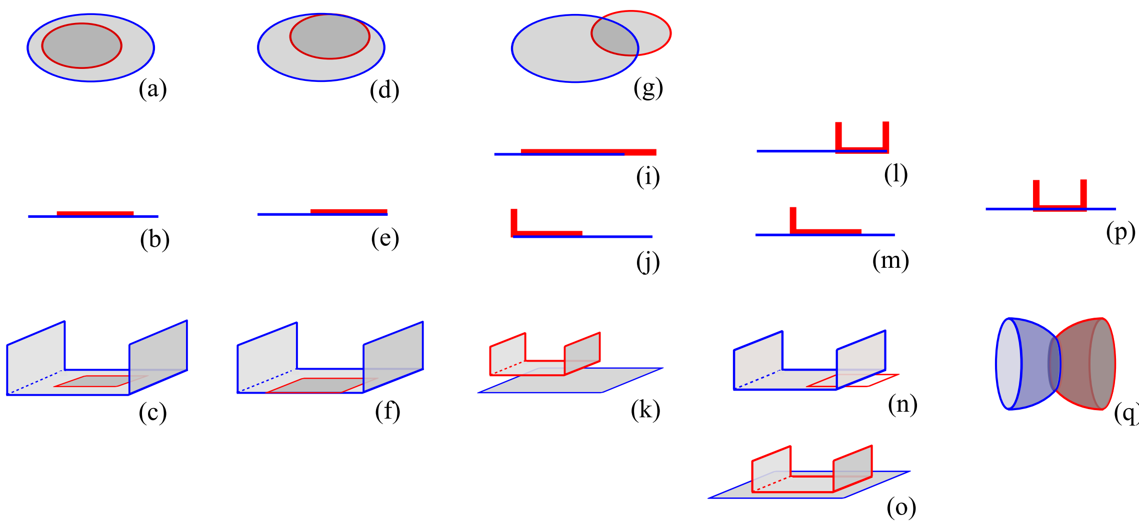


Figure 6.3: Examples of two partially overlapping entities.

(Inc-D) $Inc(x, y) \leftrightarrow \exists z[(Cont(z, x) \land P(z, y) \land z <_{\text{dim}} x) \lor (P(z, x) \land Cont(z, y) \land z <_{\text{dim}} y)]$ (incidence)

Axiom Set 6.13: Definition Inc-D of the *CODI* hierarchy.

(Inc-T1) $\neg Inc(x, x)$ (Inc irreflexive)

(Inc-T2) $Inc(x, y) \rightarrow Inc(y, x)$ (Inc symmetric)

(Inc-T3) $x =_{\text{dim}} y \rightarrow \neg Inc(x, y)$ (equidimensionality prevents Inc)

(Inc-T4) $Inc(x, y) \rightarrow x <_{\text{dim}} y \lor y <_{\text{dim}} x$ (Inc requires comparability of entities)

(Inc-T5) $Cont(x, y) \land x <_{\text{dim}} y \rightarrow Inc(x, y)$ (containment of a lower-dimensional entity requires Inc)

(Inc-T6) $Inc(x, y) \land P(y, z) \rightarrow Inc(x, z)$ (Inc transitive with respect to parthood)

Lemma 6.6. $CODI_{\text{linear}} \cup$ *{EP-D, Inc-D}* $\models$ *{Inc-T1 – Inc-T6}*

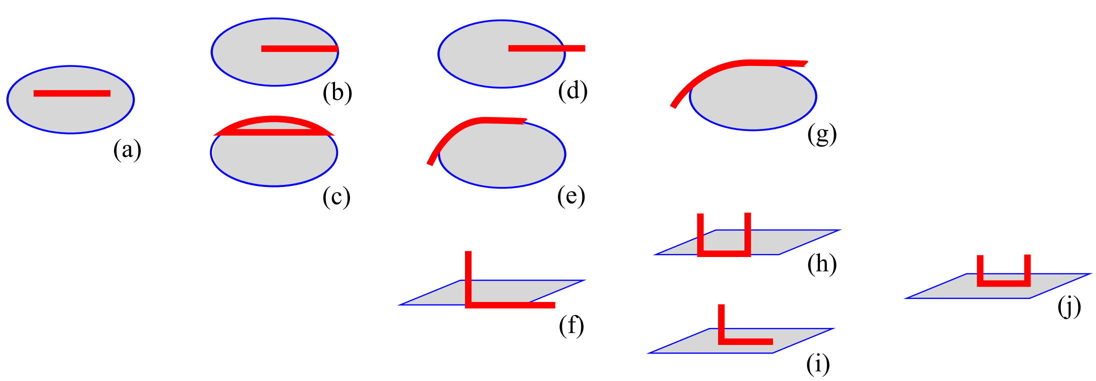


Figure 6.4: Examples of two incident entities.

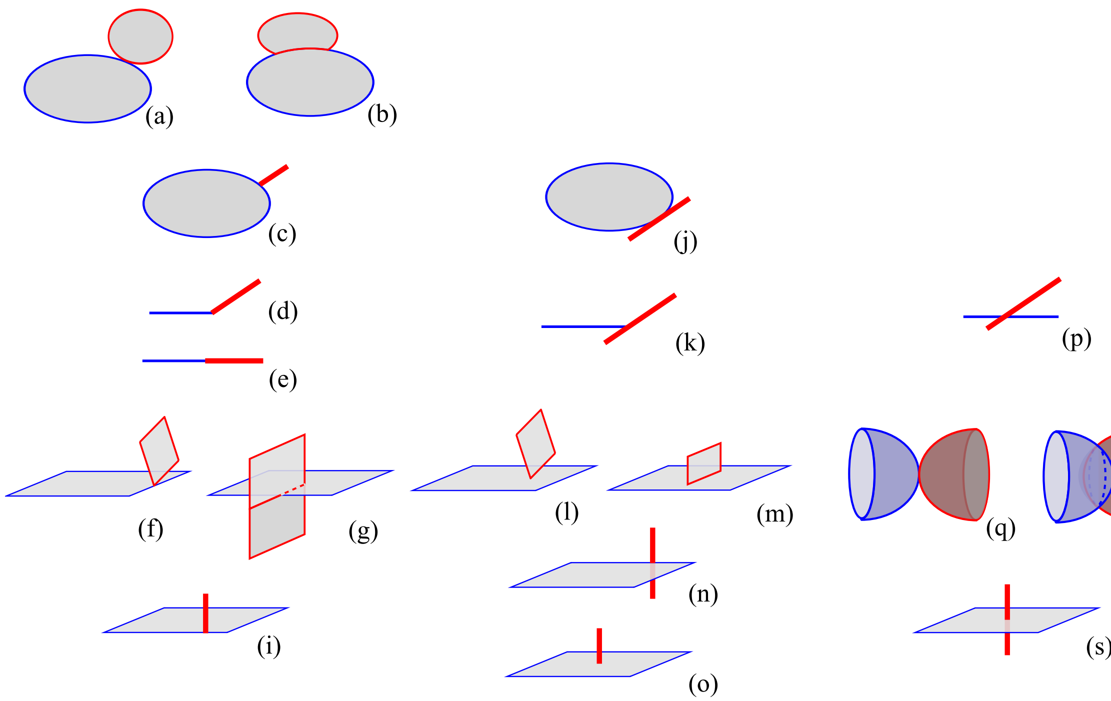


Figure 6.5: Examples of two entities in superficial contact.

Weak contact: Superficial contact

In contrast to partial overlap and incidence, superficial contact *SC* (SC-D) is a weak contact in the following sense: the shared entity must be of a lower dimension than both of the entities in contact (SC-T4). SC-T1 provides an alternative definition more closely linked to contact. *SC* is provably irreflexive and symmetric (SC-T2, SC-T3).

(SC-D) $SC(x,y) \leftrightarrow \exists z[Cont(z,x) \wedge Cont(z,y)] \wedge \forall z[Cont(z,x) \wedge Cont(z,y) \rightarrow z <_{\text{dim}} x \wedge z <_{\text{dim}} y]$ (superficial contact)

Axiom Set 6.14: Definition SC-D of the *CODI* hierarchy.

(SC-T1) $SC(x,y) \leftrightarrow C(x,y) \wedge \neg\exists z[Cont(z,x) \wedge P(z,y)] \wedge \neg\exists z[P(z,x) \wedge Cont(z,y)]$

(alternative definition of *SC*)

(SC-T2) $\neg SC(x,x)$ (*SC* irreflexive)

(SC-T3) $SC(x,y) \rightarrow SC(y,x)$ (*SC* symmetric)

(SC-T4) $SC(x,y) \rightarrow \exists z\,[z <_{\text{dim}} x \wedge z <_{\text{dim}} y \wedge Cont(z,x) \wedge Cont(z,y)]$

(*SC* requires a shared entity of a lower dimension)

Lemma 6.7. $CODI_{\text{linear}} \cup$ *{EP-D, SC-D}* $\vDash$ *{SC-T1 – SC-T4}*

The relation of 'external contact', *EC*, as used in traditional equi-dimensional mereotopology [see e.g. AV95; Coh+97b], where all regions are of the same dimension, is a special case of *SC*. The intended interpretation of $EC(x, y)$ is that 'x and y are connected in their boundaries only'. When x and y are of maximal dimension, then $SC(x, y)$ is equivalent to $EC(x, y)$. However, entities of nonmaximal dimension that are in superficial contact may or may not be in external contact. We will only be able to make such a distinction in Chapter 9.

Exhaustiveness and disjointness

The three defined relations then form a set jointly exhaustive, pairwise disjoint (JEPD) subrelations of contact in

$$CODI = CODI_{\text{linear}} \cup \{\text{C-D, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2}\}$$

which is merely a definitional extension of $CODI_{\text{linear}}$. The same applies to $CODI_{\text{unbounded}}$ (which will be of more interest in Section 8.2), to *CODI* with D-A6 omitted, and to all extensions of *CODI* like the ones we will introduce in Chapter 7.

Theorem 6.2. *In a model $\mathcal{M}$ of CODI, $\mathbf{PO}_{\mathcal{M}}$, $\mathbf{Inc}_{\mathcal{M}}$, and $\mathbf{SC}_{\mathcal{M}}$ form a partition of $\mathbf{C}_{\mathcal{M}}$.*

Proof. It suffices to show that in *CODI*, *PO*, *SC*, and *Inc* are an exhaustive set of subrelations of contact (CD-T1 to CD-T4) which are pairwise disjoint (CD-T5 to CD-T10).

(CD-T1) $PO(x, y) \rightarrow C(x, y)$

(CD-T2) $SC(x, y) \rightarrow C(x, y)$

(CD-T3) $Inc(x, y) \rightarrow C(x, y)$

(CD-T4) $C(x, y) \rightarrow PO(x, y) \vee SC(x, y) \vee Inc(x, y)$

(CD-T5) $PO(x, y) \rightarrow \neg SC(x, y)$

(CD-T6) $PO(x, y) \rightarrow \neg Inc(x, y)$

(CD-T7) $SC(x, y) \rightarrow \neg PO(x, y)$

(CD-T8) $SC(x, y) \rightarrow \neg Inc(x, y)$

(CD-T9) $Inc(x, y) \rightarrow \neg PO(x, y)$

(CD-T10) $Inc(x, y) \rightarrow \neg SC(x, y)$

CD-T1 – CD-T10 are automatically provable. This proves that *PO*, *Inc*, and *SC* are indeed jointly exhaustive and pairwise disjoint subrelations of C in *CODI* and all extensions thereof. □

Supplementation

An important principle in mereologies and in mereotopologies is supplementation [CV99a; Sim87]. Essentially, supplementation requires that any x that has some proper part, has at least two proper parts, namely *complementary proper parts*, that do not overlap. This weak definition of supplementation is captured by EP-E1. A stronger notion of supplementation is captured by EP-E2, it says that any two entities x and y that are not in parthood relation to each other must differ in at least one part, that is, there is a part z of y that does not partially overlap x. While EP-E1 and EP-E2 are tailored to equidimensional entities, we can generalize EP-E2 to the general multidimensional case. This results in

EP-E3, formalizing the idea that any entity y that is not contained in x has a part z that is neither incident with x nor overlaps x.

(EP-E1) $PP(y,x) \rightarrow \exists z\,[P(z,x) \wedge \neg PO(z,y)]$ (weak supplementation)

(EP-E2) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg P(y,x) \rightarrow \exists z\big[P(z,y) \wedge \neg PO(z,x)\big]$ (strong supplementation)

(EP-E3) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg Cont(y,x) \rightarrow \exists z\,[P(z,y) \wedge \neg Inc(z,x) \wedge \neg PO(z,x)]$
(strong supplementation of containment)

Axiom Set 6.15: Extension axioms EP-E1 – EP-E3 of the *CODI* hierarchy.

EP-E1 – EP-E3 are not theorems of *CODI*. One model $\mathcal{M}$ of *CODI* that is a counterexample to EP-E1 contains two elements $a, b \in \mathbf{M}$ such that $\langle a,b\rangle \in \mathbf{P}_{\mathcal{M}}$, $\langle b,a\rangle \notin \mathbf{P}_{\mathcal{M}}$ and for any $c \in \mathbf{M}$ with $c \neq b$ we have

$$\langle c,a\rangle \in \mathbf{P}_{\mathcal{M}} \iff \langle c,b\rangle \in \mathbf{P}_{\mathcal{M}}.$$

In this model a and b are distinct even though all proper parts of b are also parts of a. This means that the topological interior and the topological closure of a region may not be identical. Then the stronger version thereof, strong supplementation [CV99a], as captured by EP-E2 may also fail.

Equally, strong supplementation for containment, captured by EP-E3, is not provable. For a specific model of *CODI* let $a, b \in \mathbf{M}$ be two arbitrary domain entities. Let c denote the domain entity that satisfies the existentially quantified formula

$$\exists z[P(z,y) \wedge \neg Inc(z,x) \wedge \neg PO(z,x)]$$

for the variable assignments $x := a$ and $y := b$. Observe that any domain entity $d \in \mathbf{M}$ that is contained in c, i.e., with $\langle d,c\rangle \in \mathbf{Cont}_{\mathcal{M}}$, may also be contained in a as well. In other words, the formula

$$P(z,y) \wedge \neg Inc(z,x) \wedge \neg PO(z,x) \rightarrow \forall v[Cont(v,z) \rightarrow \neg Cont(v,x)]$$

may not be valid. This is intentional because a and b (or for that matter, any x, y that satisfy the antecedent of EP-E3) may share entities of lower dimension, but no part of y can be contained in x. This is adequately captured by $\neg Inc(z,x)$ and $\neg PO(z,x)$ in the antecedent of the above formula. Later (on page 121), after we have introduced an intersection operation, we will be able to simplify EP-E3. In the extensions of *CODI* by axioms that close all models under intersections and differences, we will be able to prove EP-E1 – EP-E3.

6.4 Summary

In this chapter we have introduced a linear theory of relative dimension, DI_{linear}, that makes the following ontological assumptions:

- All entities are dimensionally comparable.
- The order over dimensions is linear.
- There is a lowest nonzero dimension.

Weaker theories of dimension are feasible, such as $DI_{basic-unbounded}$ [HG11a], but are not of relevance here. We have also considered extensions of DI_{linear} by the following additional ontological assumptions:

- The order over dimensions is discrete (*discrete* dimensions).
- There is a highest dimension (*bounded* set of dimensions).

Likewise, we introduced a basic theory of spatial containment that works independently of the relative dimension of the involved spatial entities. Though this theory is based on a mereological primitive relation, it is also capable of defining the topological relation of contact.

By combining the basic theory of containment, CO_{basic} *cup* C-D, with the linear theory of relative dimension, DI_{linear}, and adding CD-A1 to ensure their proper interaction, we defined a basic theory of containment and linear dimensions, $CODI_{linear}$. This theory can again be extended by the various restrictions of the relative dimension relation from the *DI* hierarchy. The three resulting hierarchies and their relationships are illustrated in Figure 6.6.

In this combined theory, we can further refine the topological relation of contact into three types of contact: (partial) overlap, incidence, and superficial contact. Independent of the relative dimension of the involved entities, these exhaustively classify contact. Moreover, these relations are disjoint kinds of contact. The definitions of these specialized contact relations together with a few other definitions form a definitional extension of $CODI_{\text{linear}}$ that we refer to as *CODI*. This theory will play a central role throughout the thesis. That the three defined kinds of contact are jointly exhaustive and pairwise disjunct (JEPD) has an important practical consequence: it allows us to define a spatial calculus from these mereotopological relations, which is the multidimensional equivalent of the Region Connection

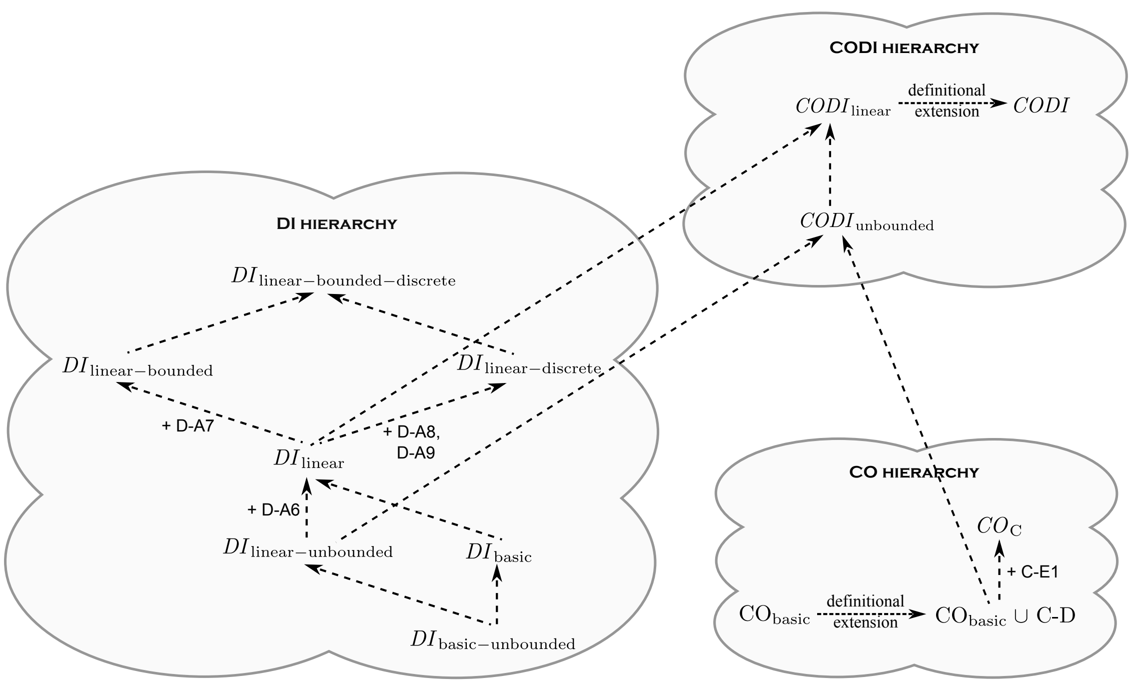


Figure 6.6: The *CODI* hierarchy and its relationship to the *DI* and *CO* hierarchies. Theories of the *CODI* hierarchy can be obtained as combination of theories of the dimension hierarchy *DI* and the small hierarchy of containment, *CO*.

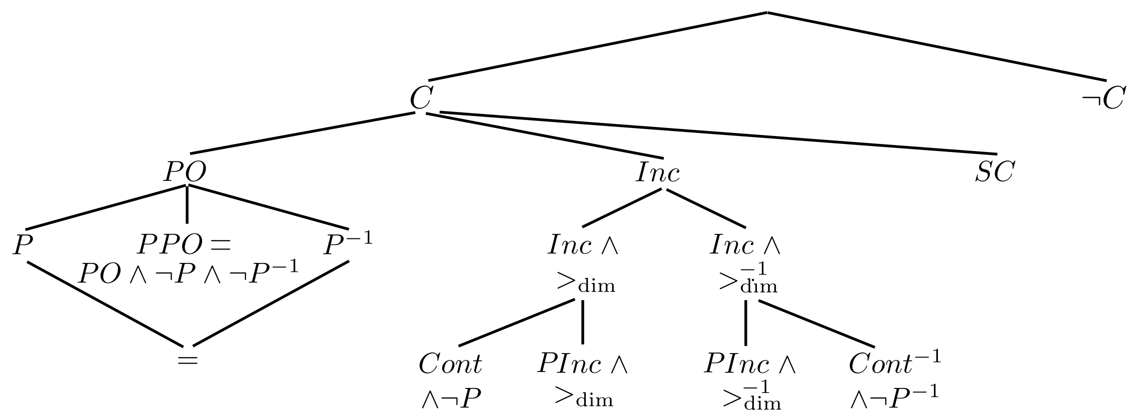


Figure 6.7: The lattice of jointly exhaustive, pairwise disjoint binary relations resulting from *CODI*. On the coarsest level it distinguishes the four symmetric relations *PO*, *Inc*, *SC*, and $\neg C$. On the finest level, it distinguishes ten relations in total (including identity =, of which four are symmetric (*PPO*, *SC*, $\neg C$, and =) while the three nonsymmetric relations *P*, *Cont* $\wedge \neg P$, and *PInc*$\wedge >_{\text{dim}}$ come also as inverses, denoted by superscript $^{-1}$.

Calculus and related calculi (in particular the 9-intersection model by Egenhofer et al. [Ege89; Ege91; EF91; EH91]).

The resulting lattice of binary relations with refinements between relations is depicted in Figure 6.7. Note that the right-hand side of the lattice (the refinements of *PO*) together with $\neg C$ are identical with the five binary relations defined in the spatial calculus RCC-5 [Ren02].

Not all relations in this lattice have a explicit name; some are expressed by a sentence (such as *proper partial overlap*, *PPO* which is $PO \wedge \neg P \wedge \neg P^{-1}$ or $PInc \wedge x >_{\text{dim}} y$ which is proper incidence where the first participating entity is of greater dimension than the second participating entity). On the coarsest level, we have four symmetric relations: *PO*, *Inc*, *SC*, and $\neg C$. By Theorem 6.2 those four relations are jointly exhaustive and pairwise disjoint, that is, between any pair of entities in a model of *CODI* (with the necessary definitions) exactly one of those relations holds.

The theory *CODI* with its various definitions will form the base theory for the remaining chapters. We will extend this theory by additional axioms and new primitive, i.e., undefinable, relations to show how to reconstruct other spatial theories from this weak theory. First, we will investigate how to close this theory under standard mereological closure operations (intersection, difference, sum) in Chapter 7. Using some of those closure axioms, we can then in Chapter 8 show how to extend *CODI* to reconstruct other mereotopologies, such as the Region Connection Calculus and the INCH Calculus. By the nature of the necessary axioms, this will confirm that *CODI* is indeed a weak multidimensional mereotopology.

Chapter 7

Closure operations in multidimensional mereotopological space

So far, we have not postulated the existence of any entities beyond the zero entity and entities of minimal and maximal dimensions. Often, in mereotopology we want to ensure that certain other entities exist as well, such as the intersection between two entities in an entity by itself. In this section we will study ways to close the basic multidimensional mereotopology by introducing mereological closure operations of intersection, difference, sum, and universal. For this task, we heavily rely on the insights gained about mereological and topological closure operations in equidimensional mereotopologies from Chapter 4 and from Casati & Varzi's exposition [CV99a]. Note that we are pragmatic about the existence of additional entities forced by mereological closures (under one or multiple of the mereological closure operations intersection, difference, sum, and universal)—we make no ontological assumptions whether the introduced entities are ontologically meaningful; instead, we provide an various strength of the theories that one can adopt on a case-by-case basis. Since it will be useful in some domains or applications, we provide the axioms for the closures. They are particularly useful for modelling abstract space, just like we will use this mereologically closed variant of *CODI* in Chapter 11, as opposed to modelling a space of physical objects. Moreover, the closure operations allow us to define mereologically closed spatial theories as extensions of the *CODI* hierarchy.

Since multidimensional mereotopologies have received much less attention than equidimensional mereotopologies, mereological closure operations among entities of different dimensions are not well understood. Only [Gal96; Gal04; Got96] studied mereological closures in multidimensional mereotopologies to some extent, but restricted their attention to mereological closures of entities of equal dimension with axioms almost identical to those found in equidimensional mereotopology. Thereby they did not consider the interesting cases of intersections, differences, or sums of entities of different dimension, altogether. The lack of considering intersections of entities of different dimension particularly limits the usefulness of those theories, while the lack of differences and sums can be justified because their adequacy is controversial [CV99a]. But most problematic from an implementation point of view is that their closure operations are not total functions, i.e., they do not apply to arbitrary pairs of entities.

Here, we propose one way of closing our multidimensional mereotopology under intersections, differences, and sums. There are three properties that guide our axiomatization. First, all closure operations shall be total functions, i.e., defined for any pair of entities—independent of their relative dimension—in terms of the primitive and defined relations we introduced in the previous chapter. Secondly, all closure operations shall behave as in equidimensional mereotopologies for entities of equal dimension. The second condition ensures that equidimensional mereotopology can be defined as an extension of our theory in the subsequent chapter. In particular, this implies that the entities of equal dimension in each model of our theory form a Boolean algebra, a claim which Theorem 7.6 will verify. Thirdly, the resulting entities must be again of uniform dimension in order to be representable by composite manifolds as the domain entities of the intended class of structures.

There are drawbacks associated to our approach. In particular, the closure operations are kind of 'coarse', they lose entities of smaller dimensions—something we cannot avoid if we want to ensure that the entities resulting from the mereological closures are again of uniform dimension. For the same reason, the definitions of the closure operations are not particularly elegant. For example, associativity cannot be guaranteed for intersections, and distributivity cannot be guaranteed for entities of different dimensions. We will study which of the standard properties of intersections, differences, and sums still hold and which may fail.

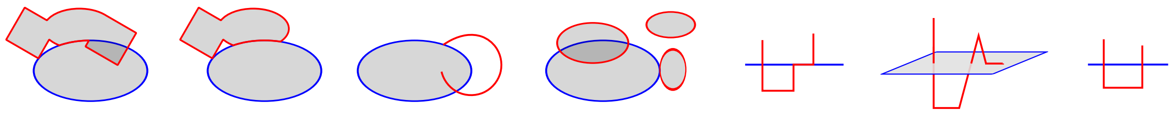
We proceed as following: first we introduce the downward mereological closure operations intersection and difference in Section 7.1 and 7.2, respectively, resulting in the theory $CODI_{\downarrow}$. As part of Section 7.2, we generalize the so-called supplementation principles of equidimensional mereotopologies to the multidimensional case and show that they become provable in $CODI_{\downarrow}$ (Section 7.2.1) and introduce the definable concept of self-connectedness (Section 7.2.3). At the end, we show that in atomic models of $CODI_{\downarrow}$, every entity is the sum of a set of minimal entities. This helps us that every structure in the class of intended structures, $\mathbb{M}$, as defined in Chapter 5, yields a model of $CODI_{\downarrow}$ (Theorem 7.4). In other words, the intended structures satisfy all axioms of $CODI_{\downarrow}$.

Subsequently, we extend the theory by the upward mereological closure operations sum and universal in Sections 7.3 and 7.4, resulting in the theory $CODI_{\updownarrow}$. The second core result of this chapter is Theorem 7.6: it uses the partitioning of a model $CODI_{\downarrow}$ into maximal sets of entities of equal dimension from Theorem 6.1 and shows that each set in the partition forms a Boolean lattice with parthood P defining its underlying partial order.

Along the way, we prove that each of the three closure operations are total function, i.e., are uniquely defined for arbitrary pairs of domain entities (Theorems 7.1, 7.2, and 7.5) and that the universal is uniquely defined (Theorem 7.7). In particular, the operations are defined functions, they are not primitive. Hence, all theories we construct in this chapter will be extensions of the 'containment-dimension mereotopology' *CODI* defined in the previous chapter. We also prove important intuitive properties about the operations and their interaction. At the same time, we discuss properties that are valid for equidimensional closures, but not generally valid in the multidimensional case. If possible, we give relative dimension constraints of when such properties hold.

7.1 Intersections

The definition of contact from Chapter 6 already implies that two entities are only in contact if there exists a common entity that is contained in both. We can capture all entities satisfying this condition



(a) (b) (c) (d) (e) (f) (g)

Figure 7.1: The spatial configurations (a) to (f) are examples in which the entities shared by two entities are not all of the same dimension. In (a), the entities share something that consists of a 2D region, a 1D line segment, and a point. This is not an entity of uniform dimension itself; the intersection consists only of the 2D region. Similarly for the examples (b) to (f). In (g) the intersection is not connected, but of uniform dimension. Thus both points are contained in the intersection of the two entities.

by a ternary intersection relation $IntCont(z, x, y)$ meaning that 'z is contained in both x and y' (Int-D). This naïve definition of the intersection of two entities can be of mixed dimension and is therefore not always a domain entity, which are all of uniform dimension. For example, two two-dimensional areas can intersect in a linear (one-dimensional) entity and in a separate point not contained in the linear entity (compare Figure 7.1). Then, the intersection cannot be an element in the domain.

(Int-D) $IntCont(z, x, y) \leftrightarrow Cont(z, x) \wedge Cont(z, y)$ (intersection containment)

Axiom Set 7.1: Definition Int-D of the *CODI* hierarchy.

In *CODI* we can prove the following properties about *IntCont*.

(Int-T1) $\neg ZEX(x) \rightarrow IntCont(x, x, x)$ (*IntCont* reflexive)

(Int-T2) $IntCont(z, x, y) \rightarrow IntCont(z, y, x)$ (*IntCont* symmetric in the last two places)

(Int-T3) $IntCont(z, x, y) \rightarrow IntCont(z, x, x)$ (*IntCont* reflexive in the last two places)

(Int-T4) $\neg C(x, y) \leftrightarrow \forall z\, [\neg IntCont(z, x, y)]$ (empty intersection iff $\neg C(x, y)$)

Lemma 7.1. *CODI* $\cup$ *Int-D* $\models$ *{Int-T1 – Int-T4}*

For two entities that partially overlap, that is, for two entities of identical dimension that share a part of the same dimension, a stronger notion of closure under mereological intersection requires a single, unique shared entity to exist (Int-E1), though this shared entity may consist of several scatted, i.e., disconnected, pieces. This is essentially the closure operation in equidimensional mereotopologies.

(Int-E1) $PO(x, y) \rightarrow \forall v\, [P(v, x) \wedge P(v, y) \rightarrow P(v, x \cdot y)]$
(intersection closure $x \cdot y$ for partially overlapping entities x, y)

Axiom Set 7.2: Extension axiom Int-E1 of the *CODI* hierarchy.

Int-E1 is not entailed by *CODI* $\cup$ Int-D. But instead of simply adding Int-E1 as an axiom, we want to generalize this idea to the multidimensional case by defining the *maximal intersection of highest dimension* among all entities shared by x and y. We define this maximal intersection of highest dimension as the function $x \cdot y$ (Int-A1 – Int-A4). This intersection could still consist of two or more disconnected parts, but now those must be at least of equal dimension. All other, lower-dimensional entities captured

(Int-A1) $\neg C(x, y) \rightarrow ZEX(x \cdot y)$ (empty intersection)

(Int-A2) $\neg ZEX(x \cdot y) \rightarrow Cont(x \cdot y, x)$ ($x \cdot y$ is contained in the intersecting entities)

(Int-A3) $Cont(z, x) \wedge Cont(z, y) \rightarrow z \leq_{\text{dim}} x \cdot y$
($x \cdot y$ has a dimension greater or equal to all entities contained in both x and y)

(Int-A4) $Cont(z, x) \wedge Cont(z, y) \wedge z =_{\text{dim}} x \cdot y \leftrightarrow P(z, x \cdot y)$ (every entity of the dimension of $x \cdot y$ that is contained in both x and y is a part of $x \cdot y$ and vice versa)

Axiom Set 7.3: Axioms Int-A1 – Int-A4 of $CODI_{\downarrow}$.

by *IntCont* are lost in this functional intersection unless they are contained in a part of the intersection of highest dimension. Then they are still contained in the intersection by transitivity of containment.

First, we want to verify that the intersection operation $\cdot$ as restricted by $CODI \cup$ {Int-A1 – Int-A4} is indeed a total function.

Theorem 7.1. *The operation $\cdot$ is a total function in $CODI \cup$ {Int-A1 – Int-A4}.*

Proof. In order to proof that $\cdot$ is a total function, i.e., that $x \cdot y$ is uniquely defined for every pair x, y, we must show that $x \cdot y$ is defined for every pair x, y and that it is unique.

Let x, y be arbitrary entities. We split the proof into two cases.

Case (I): Assume $\neg\mathbf{C}(x, y)$.

Then $(x \cdot y) \notin \mathbf{ZEX}_{\mathcal{M}}$ satisfies the constraints and by D-A4 $x \cdot y$ must also be unique (since there is a unique element $z \in \mathbf{ZEX}_{\mathcal{M}}$ by D-A4).

Case (II): Assume $\mathbf{C}(x, y)$.

Then by C-D, there must exist some $z \in \mathbf{M}$ such that $\mathbf{Cont}(z, x)$ and $\mathbf{Cont}(z, y)$. We now claim that every such entity z of highest dimension is contained in $x \cdot y$. By Int-A3, we have $z \leq_{\mathbf{dim}} x \cdot y$.

Subcase (II.i): Assume $z <_{\mathbf{dim}} x \cdot y$.

Then some other entity $v \in \mathbf{M}$ with $z <_{\mathbf{dim}} v$ must exist such that $\mathbf{P}_{\mathcal{M}} v, x \cdot y)$ and thus by Int-A2, $\mathbf{Cont}(v, x)$ and $\mathbf{Cont}(v, y)$. Hence z cannot be an entity of highest dimension contained in both x and y and hence $\neg\mathbf{P}_{\mathcal{M}} z, x \cdot y)$.

Subcase (II.ii): Assume $z =_{\mathbf{dim}} x \cdot y$.

Then by Int-A4, we have $\mathbf{P}_{\mathcal{M}}(z, x \cdot y)$. Hence, we can determine uniquely the set of parts of $x \cdot y$ and by EP-T9 this uniquely identifies $x \cdot y$.

Since all entities $x, y \in \mathbf{M}$ are either in contact or not, no other cases are possible. Therefore, the intersection $x \cdot y$ is uniquely determined for arbitrary entities $x, y \in \mathbf{M}$. □

Next we prove some essential properties of the intersection operation. We start by verifying that the intersection operation is commutative (Int-T5) and behaves as expected for the three types of contact, *PO*, *Inc*, and *SC* (Int-T7 to Int-T9). Moreover, Int-E1 is now a theorem in $CODI \cup$ {Int-A1 – Int-A4}.

(Int-T5) $x \cdot y = y \cdot x$ (intersection commutative)

(Int-T6) $ZEX(x \cdot y) \leftrightarrow \neg C(x, y)$ (zero intersection only for disconnected entities)

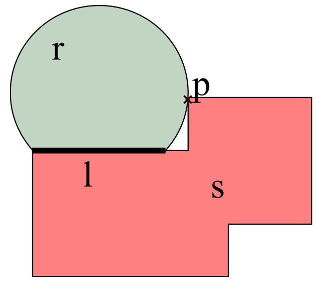


Figure 7.2: Intersection $\cdot$ is not associative in this model of *CODI* $\cup$ {Int-A1 – Int-A4}. Consider $r \cdot (s \cdot p) = r \cdot p = p \neq \emptyset = l \cdot p = (r \cdot s) \cdot p$.

(Int-T7) $PO(x, y) \rightarrow x =_{\text{dim}} x \cdot y =_{\text{dim}} y$

(PO: the max. intersection is of the same dimension as both intersecting entities)

(Int-T8) $Inc(x, y) \wedge x <_{\text{dim}} y \rightarrow x \cdot y =_{\text{dim}} x \wedge x \cdot y <_{\text{dim}} y$

(Inc: the max. intersection is of a lower dimension than one of the intersecting entities)

(Int-T9) $SC(x, y) \rightarrow x \cdot y <_{\text{dim}} x \wedge x \cdot y <_{\text{dim}} y$

(SC: the max. intersection is of a lower dimension than both intersecting entities)

Lemma 7.2. *CODI* $\cup$ *{Int-A1 – Int-A4}* $\vDash$ *{Int-T5 – Int-T9, Int-E1}*

In equidimensional mereotopology, the intersection operation is always associative, i.e., it satisfies

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z.$$

But unlike the equidimensional case, the intersection operation in the multidimensional case is not always associative as illustrated by Figure 7.2. Non-associativity only occurs for entities whose intersection is nonuniform, i.e., whose intersection contains disconnected entities of differing dimensions. However, the intersection operation satisfies a weaker form of associativity known as the left- and right-alternative laws (Int-T11 and Int-T12). But first we prove that the intersection operation is idempotent (Int-T10), a property that simplifies the proofs of Int-T11 and Int-T12.

(Int-T10) $x \cdot x = x$ ($\cdot$ idempotent)

(Int-T11) $(x \cdot x) \cdot y = x \cdot (x \cdot y)$ ($\cdot$ left-alternative)

(Int-T12) $y \cdot (x \cdot x) = (y \cdot x) \cdot x$ ($\cdot$ right-alternative)

Lemma 7.3. *CODI* $\cup$ *{Int-A1 – Int-A4}* $\vDash$ *{Int-T10 – Int-T12}*

Proof. First, idempotence (Int-T10) of the intersection operation follows directly from Int-A4 as the automatic proof shows.

By commutativity (Int-T5) of $\cdot$ both left- and right-alternative (Int-T11 and Int-T12) are equivalent. To prove either, it suffices to prove a simpler equivalent property according to the following logical derivation:

$$\begin{aligned} & y \cdot (x \cdot x) &= (y \cdot x) \cdot x & \\ \Leftrightarrow\quad & (x \cdot x) \cdot y &= x \cdot (x \cdot y) & \quad \text{(Int-T5)} \\ \Leftrightarrow\quad & x \cdot y &= x \cdot (x \cdot y) & \quad \text{(Int-T10)} \end{aligned}$$

To prove Int-T11 and Int-T12, it thereby suffices to prove

(Int-T11′) $x \cdot y = x \cdot (x \cdot y)$ (· left- and right-alternative)

By EP-T2 it then suffices to prove the following two sentences about the parthood relations:

$$\neg ZEX(x \cdot y) \rightarrow P(x \cdot y, x \cdot (x \cdot y))$$

$$\neg ZEX(x \cdot y) \wedge \neg ZEX(x \cdot (x \cdot y)) \rightarrow P(x \cdot (x \cdot y), x \cdot y)$$

which are again proved automatically. □

As a result, we get the following weak characterization of the intersection operation in *CODI* ∪ {Int-A1 – Int-A4}.

Corollary 7.1. *Let $\mathcal{M}$ be a model of CODI ∪ {Int-A1 – Int-A4} with domain M.*
Then $(M, \cdot)$ is a commutative alternative magma (groupoid).

Proof. $\mathcal{M}$ is a magma since the intersection operation · is closed in M, that is, for all $x, y \in M$, $(x \cdot y) \in M$. By Int-T5 $\mathcal{M}$ is commutative, while by Lemma 7.3 it is alternative. □

General associativity (Int-E2) nonconservatively extends *CODI* ∪ {Int-A1 – Int-A4}.

(Int-E2) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (· associative)

Axiom Set 7.4: Extension axiom Int-E2 of $CODI_{\downarrow}$.

7.2 Differences

Models of equidimensional mereotopologies are usually closed under complementation. That is, in the presence of a universal entity U (something that everything else is a part of), for every nonzero entity y a complement y' exists if $y \neq U$. Even if no universal entity exists, it is reasonable to require that relative complements exists: an entity y with $y \notin ZEX$ that is a proper part of x has a relative complement $x - y$ if $y \neq x$. Relative complementation is better known as the difference, e.g., in set theory.

In order to ensure that the difference between two entities is always of a uniform dimension, we follow an approach analogue to how we defined the intersection operation · by always taking the difference of the largest dimension only. In particular, the difference whose minuend is of a greater dimension than the subtrahend must always be the minuend itself (Dif-A2): we ignore lower-dimensional artefacts for differences so that all entities are still of uniform dimension and are regular in their topological interpretation. But for two entities of equal dimension, we define the difference so that it behaves as in

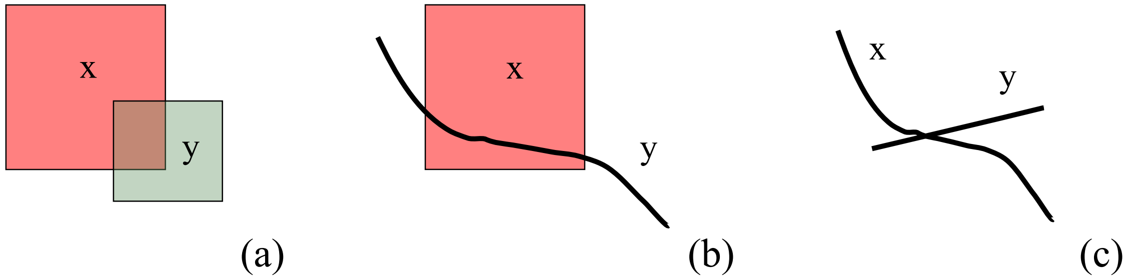


Figure 7.3: Two examples in which nontrivial differences exist. In (a) both differences $x - y$ and $y - x$ are defined as expected, while in (b) only the difference $y - x$ is meaningful, while $x - y = x$. This is because in (a) the intersection $x \cdot y$ is a proper part of both x and y and thus the differences $x - (x \cdot y)$ and $y - (x \cdot y)$ must exist. In (b), the intersection $x \cdot y$ is a proper part of y but not a proper part of x (even though it is contained in x), thus only the difference $y - (x \cdot y)$ differs from y. In (c) we have both $x - y = x$ and $y - x = y$ since $x \cdot y <_{\mathbf{dim}} x, y$.

equidimensional mereotopology [compare Gal96; Got96]. Then, for example, a line segment contained in a line or a greater line segment requires that a difference exists—which is also a line segment, though possibly disconnected. Equally, the difference between, e.g., two 2D areas is well-defined, compare Figure 7.3(a). For two entities in contact, this idea extends to differences whose subtrahend is of a greater dimension than the minuend (Dif-A3). For example, there exists a difference of uniform dimension between a line and a two-dimensional area incident with the line; it is the line minus the intersection between the line and the area, which is a (possibly disconnected) line segment which is a part of the original line, compare Figure 7.3(b). This works in general because the intersection is always of no greater dimension than either of the intersecting entities as captured by Dif-A3. Later, in Dif-T7, we verify that the difference between x and y is indeed the difference between x and the intersection of x and y.

Dif-A1 establishes the dimension of the difference between any two entities x and y, which is either of the same dimension as x or is empty. While Dif-A2 captures the easy case of the difference between a higher- and a lower-dimensional entity, Dif-A3a to Dif-A3c axiomatize the constitution of the difference between two entities of which the minuend is of equal or lower dimension than the subtrahend. Dif-A4 captures the exact conditions when the difference may be empty: either the minuend is zero or the minuend is contained in the subtrahend.

(Dif-A1) $\neg ZEX(x - y) \rightarrow x - y =_{\text{dim}} x$ (dimension of the difference $x - y$)

(Dif-A2) $y <_{\text{dim}} x \rightarrow x - y = x$ (difference $x - y$ for a lower-dimensional y)

(Dif-A3a) $x \leq_{\text{dim}} y \rightarrow [Cont(z, x) \wedge z \cdot y <_{\text{dim}} z \rightarrow Cont(z, x - y)]$

(Dif-A3b) $x \leq_{\text{dim}} y \rightarrow [Cont(z, x - y) \rightarrow Cont(z, x)]$

(Dif-A3c) $x \leq_{\text{dim}} y \rightarrow [P(z, x - y) \rightarrow z \cdot y <_{\text{dim}} z]$

(Dif-A3a – Dif-A3c: constitution of $x - y$ when y has equal or greater dimension than x)

(Dif-A4) $ZEX(x - y) \leftrightarrow ZEX(x) \vee Cont(x, y)$

(zero difference $x - y$ only when x is contained in y or x is a zero entity)

Axiom Set 7.5: Axioms Dif-A1 – Dif-A4 of $CODI_{\downarrow}$.

Note that because differences must exist for all pairs of entities, we always have $ZEX(x-x)$. Hence an entity $x \in \mathbf{ZEX}_{\mathcal{M}}$ must exist, that is, Z-A1 is a theorem of $CODI \cup \{$Dif-A1 – Dif-A4$\}$.

Lemma 7.4. *CODI* $\cup$ *{Dif-A1 – Dif-A4}* $\models$ *Z-A1*

We define the 'containment-dimension mereotopology with downwards closure' as the theory

$$CODI_{\downarrow} = CODI \cup \{\text{Int-A1} - \text{Int-A4}, \text{Dif-A1} - \text{Dif-A4}\}.$$

Note that $CODI_{\downarrow} = CODI_{\downarrow}^{0}$ because of Lemma 7.4.

Again, we want to verify that the difference operation as axiomatized in $CODI_{\downarrow}$ is a total function. To ease this task, we first prove that the difference in $CODI_{\downarrow}$ can be described in terms of parthood alone. We do this by proving the following properties, which with Dif-T4 culminate in the desired property Dif-T5.

(Dif-T1) $\neg ZEX(x-y) \rightarrow P(x-y,x)$ (a nonempty difference $x-y$ is part of x)

(Dif-T2) $PP(y,x) \rightarrow PP(x-y,x)$ (for a proper part y of x, $x-y$ is also a proper part of x)

(Dif-T3) $\neg PO(x-y,y)$ (y and $x-y$ do not partially overlap)

Lemma 7.5. $CODI_{\downarrow} \models$ *{Dif-T1 – Dif-T3}*

Proof. **(Dif-T1)** $\neg ZEX(x-y) \rightarrow P(x-y,x)$.

Assume $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$.

We consider two separate cases.

Case (I): Assume $y <_{\mathbf{dim}} x$.

Then $x-y=x$ and hence $\mathbf{P}(x-y,x)$.

Case (II): Assume $y \geq_{\mathbf{dim}} x$.

Then $\mathbf{Cont}(x-y,x-y)$ and hence by Dif-A3b $\mathbf{Cont}(x-y,x)$ which with $x-y =_{\mathbf{dim}} x$ (Dif-A2) leads to $\mathbf{P}(x-y,x)$.

Because any two entities x,y are either $y <_{\mathbf{dim}} x$ or $y \geq_{\mathbf{dim}} x$, Dif-T1 is valid.

(Dif-T2) $PP(y,x) \rightarrow PP(x-y,x)$.

Consider the following logical derivation:

$$\begin{aligned}
PP(y,x) &\rightarrow \neg ZEX(x) \wedge PP(y,x) && \text{(C-A4)}\\
&\rightarrow \neg ZEX(x) \wedge \neg Cont(x,y) \wedge PP(y,x) && \text{(C-T1,EPP-D,EP-D)}\\
&\rightarrow P(x-y,x) \wedge PP(y,x) && \text{(Dif-T1)}\\
&\rightarrow P(x-y,x) \wedge P(y,x) && \text{(EPP-D)}\\
&\rightarrow P(x-y,x) \wedge P(y,x) \wedge \neg P(y,x-y) && \text{(Dif-A3c)}\\
&\rightarrow P(x-y,x) \wedge x \neq x-y && \text{(inverse of EP-T9)}\\
&\rightarrow PP(x-y,x) && \text{(EPP-D)}
\end{aligned}$$

which proves $PP(y,x) \rightarrow PP(x-y,x)$.

(Dif-T3) $\neg PO(x-y,y)$.

We consider two cases.

Case (I): Assume $x{<_{\mathbf{dim}}}y$.

Then $x-y{=_{\mathbf{dim}}}x{<_{\mathbf{dim}}}y$ by Dif-A2, hence $\neg\mathbf{PO}(x-y,y)$.

Case (II): Assume $x{\geq_{\mathbf{dim}}}y$.

Let us assume that $z\in\mathbf{M}$ is an arbitrary part of $x-y$, i.e., $\mathbf{P}_{\mathcal{M}}z,x-y)$.

Then $z\cdot y{<_{\mathbf{dim}}}z$ by Dif-A3c, hence z cannot be a part of y. Thus $\neg\mathbf{PO}(x-y,y)$.

Because any two entities x,y are either $x{<_{\mathbf{dim}}}y$ or $x{\geq_{\mathbf{dim}}}y$, Dif-T3 is valid.

□

Dif-T1 to Dif-T3 are important since they capture all the parts of the difference that by EP-T9 uniquely characterize each difference. Subsequently, these theorems play a prominent role in proving further properties about the difference and $CODI_{\downarrow}$, in particular the supplementation axioms.

Next, we prove Dif-T4, before we proceed to the key lemma Dif-T5 that we will rely on for the remainder of the chapter.

(Dif-T4) $\neg PO(x-y,x\cdot y)$ ($x\cdot y$ and $x-y$ do not partially overlap)

Lemma 7.6. *$CODI_{\downarrow}\models$ Dif-T4*

Proof. We consider two cases.

Case (I): Assume $x{<_{\mathbf{dim}}}y$.

Then $x\cdot y{<_{\mathbf{dim}}}x{=_{\mathbf{dim}}}x-y$ by Dif-A2, hence $\neg\mathbf{PO}(x-y,x\cdot y)$.

Case (II): Assume $x{\geq_{\mathbf{dim}}}y$.

Let us assume that $z\in\mathbf{M}$ is an arbitrary part of $x-y$, i.e., $\mathbf{P}_{\mathcal{M}}(z,x-y)$.

Then $z\cdot(x\cdot y){\leq_{\mathbf{dim}}}z\cdot y{<_{\mathbf{dim}}}z$, hence z cannot be a part of $x\cdot y$ and thus $\neg\mathbf{PO}(x-y,x\cdot y)$.

Because any two entities x,y are either $x{<_{\mathbf{dim}}}y$ or $x{\geq_{\mathbf{dim}}}y$, Dif-T4 is valid. □

Now we are in a position to prove Dif-T5.

(Dif-T5) $P(z,x-y)\leftrightarrow P(z,x)\wedge\neg PO(z,x\cdot y)$ (parts of the difference $x-y$)

Lemma 7.7. *$CODI_{\downarrow}\models$ Dif-T5*

Proof. We prove the two directions of the biconditional separately:

Direction (a): $P(z,x-y)\rightarrow P(z,x)\wedge\neg PO(z,x\cdot y)$.

Assume z is some part of $x-y$, then $(x-y)\notin\mathbf{ZEX}_{\mathcal{M}}$ and hence $x-y{=_{\mathbf{dim}}}x$ by Dif-A1.

We have two cases:

Case (I): Assume $y{<_{\mathbf{dim}}}x$.

Then $x-y=x$ by Dif-A3b, hence $\mathbf{P}(z,x-y)$ implies $\mathbf{P}(z,x)$.

Case (II): Assume $y{\geq_{\mathbf{dim}}}x$.

Then $\mathbf{P}(z,x-y)$ implies $\mathbf{Cont}(z,x)$ (by Dif-A2) and with $z{=_{\mathbf{dim}}}x-y{=_{\mathbf{dim}}}x$ (Dif-A1) we obtain $\mathbf{P}(z,x)$.

Moreover, $\mathbf{P}(z, x-y)$ and $\neg\mathbf{PO}(x\cdot y, x-y)$ (by Dif-T4) imply $\neg\mathbf{PO}(z, x\cdot y)$.

Direction (b): $P(z,x) \wedge \neg PO(z, x\cdot y) \rightarrow P(z, x-y)$.
Assume z to be an arbitrary part of x with $\neg\mathbf{PO}(z, x\cdot y)$.
Now suppose $\neg\mathbf{P}(z, x-y)$. Then we have two cases.

Case (I): Assume $y <_{\mathbf{dim}} x$.
Then by Dif-A2 $x-y=x$ and thus $\mathbf{P}(z,x)$ if and only if $\mathbf{P}(z,x-y)$ which contradicts our assumption that $\mathbf{P}(z,x)$.

Case (II): Assume $y \geq_{\mathbf{dim}} x$.
Then by Dif-A3c we must have $z\cdot y \not<_{\text{dim}} z$ in order for $\mathbf{P}(z,x)$ and $\neg\mathbf{P}(z,x-y)$ to hold. Since we always have $z\cdot y \leq_{\mathbf{dim}} z$ (by Int-A4), we must have $z\cdot y =_{\mathbf{dim}} z$. Hence some part of z, call it z_P, is also contained in y (compare Int-A4). Since all parts of z are part of x, z_P is also a part of $x\cdot y$ (transitivity of parthood). Hence $\mathbf{PO}(z, x\cdot y)$, which contradicts our assumption that $\neg\mathbf{PO}(z, x\cdot y)$.

The two directions together immediately imply $x-y = x-(x\cdot y)$ by EP-T9. □

7.2.1 Supplementation principles

Our axiomatization of differences forces what is known from equidimensional mereotopology as weak supplementation [compare CV99a], we introduced it as EP-E1 on page 121, in $\mathit{CODI}_{\downarrow}$. Equally, strong supplementation (EP-E2) becomes provable in $\mathit{CODI}_{\downarrow}$. While weak and strong supplementation as expressed in EP-E1 and EP-EP2 specifically only apply to entities of equal dimension, we can generalize strong supplementation to the multidimensional case (EP-E3): any y that is not contained in x has a part z whose intersection with x is of a lower dimension than that of z (and thus y).

(EP-E1) $PP(y,x) \rightarrow \exists z\,[P(z,x) \wedge \neg PO(z,y)]$ (weak supplementation)

(EP-E2) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg P(y,x) \rightarrow \exists z\,[P(z,y) \wedge \neg PO(z,x)]$ (strong supplementation)

(EP-E3) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg Cont(y,x) \rightarrow \exists z\,[P(z,y) \wedge z\cdot x <_{\text{dim}} z]$ (strong supplementation of containment)

Axiom Set 7.6: Extension axioms EP-E1 – EP-E3 of *CODI*, which are provable in $\mathit{CODI}_{\downarrow}$.

Lemma 7.8. $\mathit{CODI}_{\downarrow} \models$ *EP-E1*

Proof. Assume $\mathbf{PP}(y,x)$.
Then by Dif-T2 we have $\mathbf{PP}(x-y,x)$ and thus $\mathbf{P}(x-y,x)$. From Dif-T3 we further conclude that $\neg\mathbf{PO}(x-y,y)$. Therefore $z := x-y$ satisfies the existential in EP-E1. □

Lemma 7.9. $\mathit{CODI}_{\downarrow} \models$ *EP-E2*

Proof. Assume $x, y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\neg\mathbf{P}(y,x)$.
Note that we cannot have $x=y$. The following three cases remain:

Case (I): Assume $x =_{\mathbf{dim}} y$ and $\neg\mathbf{P}(x,y)$.
Then $z := y-x$ satisfies the consequence of the implication in EP-E2: we have $(y-x) \notin \mathbf{ZEX}_{\mathcal{M}}$ by

Dif-A4 (since $\neg\mathbf{Cont}(y,x)$ and $y \notin \mathbf{ZEX}_{\mathcal{M}}$) and thus $\mathbf{P}(y-x,y)$ by Dif-T1. Further, $\neg\mathbf{PO}(y-x,x)$ by Dif-T3.

Case (II): Assume $x{=_{\mathbf{dim}}}y$ and $\mathbf{P}(x,y)$.
Then $\mathbf{PP}(x,y)$ (since $\neg\mathbf{P}(y,x)$) and $\exists z\,[\mathbf{P}(z,y) \wedge \neg\mathbf{PO}(z,x)]$ follows from EP-E1.

Case (III): Assume $x \neq_{\text{dim}} y$.
Then $z := y$ satisfies the consequence of the implication in EP-E2: because of $x \neq_{\text{dim}} y$ we have $\neg\mathbf{PO}(x,y)$ (by PO-T3) but we still have $\mathbf{P}(y,y)$.

The Cases (I) and (II) cover all possible relations between x and y when $x{=_{\mathbf{dim}}}y$, while Case (III) covers any other case. Once formalized as follows, the three cases are automatically provable.

Case (I) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg P(y,x) \wedge x =_{\text{dim}} y \wedge PO(x,y) \wedge \neg PP(y,x)$
$\rightarrow \exists z\,[P(z,y) \wedge \neg PO(z,x)]$

Case (II) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg P(y,x) \wedge x =_{\text{dim}} y \wedge \neg PO(x,y)$
$\rightarrow \exists z\,[P(z,y) \wedge \neg PO(z,x)]$

Case (III) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg P(y,x) \wedge x \neq_{\text{dim}} y$
$\rightarrow \exists z\,[P(z,y) \wedge \neg PO(z,x)]$

□

For atomic theories, Varzi [Var07] proposed an alternative formalization of strong supplementation (EP-E2′, adapted to our theory).

(EP-E2′) $\neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg P(y,x) \rightarrow \exists z\,[Min(z) \wedge P(z,y) \wedge \neg PO(z,x)]$
(strong supplementation for atomic theories: every nonzero entity y that is not a part of x has an atomic, i.e., minimal, part that is not a part of x)

Axiom Set 7.7: Extension axiom EP-E2′ of *CODI*.

We finally prove EP-E3, the multidimensional version of strong supplementation.

Lemma 7.10. *CODI*$_{\downarrow} \vDash$ *EP-E3*

Proof. Assume $\neg\mathbf{Cont}(y,x)$. We consider three cases.

Case (I): Assume $y{>_{\mathbf{dim}}}x$.
Then $z := y$ satisfies the existential formula because we have $\mathbf{P}(y,y)$ and $y \cdot x{\leq_{\mathbf{dim}}}x{<_{\mathbf{dim}}}y$.

Case (II): Assume $y{=_{\mathbf{dim}}}x$.
Then by EP-E2, there exists a $z \in \mathbf{M}$ such that $\mathbf{P}(z,y)$ and $\neg\mathbf{PO}(z,x)$. $\neg\mathbf{PO}(z,x)$ further implies $z \cdot x{<_{\mathbf{dim}}}z$ because $z{=_{\mathbf{dim}}}y{=_{\mathbf{dim}}}x$. Hence the z that satisfies the existential formula in EP-E2 also satisfies the existential formula in EP-E3.

Case (III): Assume $y{<_{\mathbf{dim}}}x$.
Then $z := y - x$ satisfies the existential formula because $(y-x) \notin \mathbf{ZEX}_{\mathcal{M}}$ (from $y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\neg\mathbf{Cont}(y,x)$), so that $\mathbf{P}(y-x,y-x)$. Thus in turn implies $(y-x) \cdot x{<_{\mathbf{dim}}}(y-x)$ by Dif-A3c.

These three cases cover all possible dimension relations between x and y. □

A consequence of strong supplementation is that the relation of partial overlap PO is also extensional (PO-E1), a key property of the theory $CODI_{\downarrow}$ and essential for our subsequent study of the sum operation.

(PO-E1) $\forall z[PO(z,x) \leftrightarrow PO(z,y)] \rightarrow x = y$	(PO extensional)

Axiom Set 7.8: Extension axiom PO-E1 of $CODI$, which is provable in $CODI_{\downarrow}$.

Lemma 7.11. *$CODI_{\downarrow} \models$ PO-E1*

Proof. By EP-E2 and with D-A4 (there is a unique zero region) it suffices to prove the sentence

$$\neg ZEX(x) \wedge \forall z[PO(z,x) \rightarrow PO(z,y)] \rightarrow P(x,y).$$

Now suppose $x \notin \mathbf{ZEX}_{\mathcal{M}}$, for all $z \in \mathbf{M}$ with $\mathbf{PO}(z,x)$ we also have $\mathbf{PO}(z,y)$, and $\neg\mathbf{P}(x,y)$. Then $\mathbf{PO}(x,x)$ and thus $\mathbf{PO}(x,y)$, entailing $y \notin \mathbf{ZEX}_{\mathcal{M}}$. By EP-E2 there exists a $v \in \mathbf{M}$ such that $\mathbf{P}(v,x)$ and $\neg\mathbf{PO}(v,y)$. Because $P(x,y) \rightarrow PO(x,y)$ by PO-D, this contradicts our assumption that for all $z \in \mathbf{M}$ with $\mathbf{PO}(z,x)$ we also have $\mathbf{PO}(z,y)$. □

Now, we can use Dif-T5 and PO-E1 to show that the difference operation is a total function.

Theorem 7.2. *The operation $-$ is a total function in $CODI_{\downarrow}$.*

Proof. We will show that the difference $x-y$ is uniquely defined for arbitrary entities x, y. We distinguish two cases based on the relative dimension between x and y.

Case (I): Assume $x >_{\mathbf{dim}} y$.

By Dif-A2, we have $x - y = x$, hence the difference is defined and unique.

Case (II): Assume $x \leq_{\mathbf{dim}} y$.

If $x \in \mathbf{ZEX}_{\mathcal{M}}$, then by Dif-A4, the difference $x - y$ is uniquely defined as $(x - y) = x \in \mathbf{ZEX}_{\mathcal{M}}$ (recall that there no more than one entity $z \in \mathbf{ZEX}_{\mathcal{M}}$).

Now assume $x \notin \mathbf{ZEX}_{\mathcal{M}}$, then $\mathbf{P}(x - y, x)$ by Dif-T1 and hence $x - y =_{\mathbf{dim}} x$. The difference $x - y$ is, by PO-E1, uniquely defined by its parts. Let z be an arbitrary entity. We distinguish two subcases.

Subcase (II.a): Assume $x \leq_{\mathbf{dim}} y$ and $\neg\mathbf{P}(z,x)$.

Then $\neg\mathbf{P}(z, x - y)$ by Dif-T5.

Subcase (II.b): Assume $x \leq_{\mathbf{dim}} y$ and $\mathbf{P}(z,x)$.

If $\neg\mathbf{PO}(z, x \cdot y)$, then $\mathbf{P}(z, x - y)$ by Dif-T5. The assumption $\mathbf{P}(z,x)$ also implies $\mathbf{PO}(z,x)$, so that we get

$$\mathbf{PO}(z, x \cdot y) \leftrightarrow \mathbf{PO}(z,y).$$

Hence we obtain $\neg\mathbf{P}(z, x - y)$ if $\mathbf{PO}(z,y)$, and $\mathbf{P}(z, x - y)$ otherwise.

Thus, for every z we can determine whether $\mathbf{P}(z, x - y)$ or not, and thus the difference $x - y$ is uniquely determined because of PO-E1.

Hence for arbitrary x and y, the difference $x - y$ is uniquely defined. □

7.2.2 Interaction between intersections and differences

Next, we study the interaction between the difference and the intersection operations in $CODI_\downarrow$. These primarily verify our intuitions about the difference operation, while Dif-T6 will also come in handy for later proofs.

(Dif-T6) $x - y = x - (x \cdot y)$ ($x - y$ and $x - (x \cdot y)$ are identical)

Lemma 7.12. $CODI_\downarrow \models$ *Dif-T6*

Proof. We consider three cases.

Case (I): Assume $(x - y) \in \mathbf{ZEX}_\mathcal{M}$.
Then either $\mathbf{Cont}(x, y)$ or $x \in \mathbf{ZEX}_\mathcal{M}$ by Dif-A4. If $\mathbf{Cont}(x, y)$, then $x \cdot y = x$ and thus $(x-(x \cdot y)) \in \mathbf{ZEX}_\mathcal{M}$ by $(x - x) \in \mathbf{ZEX}_\mathcal{M}$, while if $x \in \mathbf{ZEX}_\mathcal{M}$ we also have $(x - (x \cdot y)) \in \mathbf{ZEX}_\mathcal{M}$.

Case (II): Assume $(x - (x \cdot y)) \in \mathbf{ZEX}_\mathcal{M}$.
Then either $x \in \mathbf{ZEX}_\mathcal{M}$ or $\mathbf{Cont}(x, x \cdot y)$. From $x \in \mathbf{ZEX}_\mathcal{M}$ we obtain $(x - y) \in \mathbf{ZEX}_\mathcal{M}$ by Dif-A4. If $\mathbf{Cont}(x, x \cdot y)$, then $x = x \cdot y$, hence $\mathbf{Cont}(x, y)$ and again $(x - y) \in \mathbf{ZEX}_\mathcal{M}$ Dif-A4.

Case (III): Assume $(x - y) \notin \mathbf{ZEX}_\mathcal{M}$ and $(x - (x \cdot y)) \notin \mathbf{ZEX}_\mathcal{M}$.
To prove $x - y = x - (x \cdot y)$ it suffices (by EP-T9) to show that for all $z \in \mathbf{M}$,

$$\mathbf{P}(z, x - y) \leftrightarrow \mathbf{P}(z, x - (x \cdot y)).$$

By (Dif-T5), this amounts to showing the following equivalence:

$$\mathbf{P}(z, x) \wedge \neg\mathbf{PO}(z, x \cdot y) \leftrightarrow \mathbf{P}(z, x) \wedge \neg\mathbf{PO}(z, x \cdot (x \cdot y))$$

(Int-T11) and (Int-T10) let us simplify the right-hand side to obtain the following equivalent sentence:

$$\mathbf{P}(z, x) \wedge \neg\mathbf{PO}(z, x \cdot y) \leftrightarrow \mathbf{P}(z, x) \wedge \neg\mathbf{PO}(z, x \cdot y)$$

which is a tautology.

It is easy to see that the three cases are exhaustive. □

(Dif-T7) $x - y = x \cdot (x - y)$ ($x - y$ and its intersection with x are identical)

Lemma 7.13. $CODI_\downarrow \models$ *Dif-T7*

Proof. To prove $x - y = x \cdot (x - y)$ it again suffices by EP-T9 to show that for all $z \in \mathbf{M}$,

$$\mathbf{P}(z, x) \wedge \neg\mathbf{PO}(z, x \cdot y) \leftrightarrow \mathbf{P}(z, x \cdot (x - y)).$$

We show each direction individually.

Direction (a): $\mathbf{P}(z, x) \wedge \neg\mathbf{PO}(z, x \cdot y) \rightarrow \mathbf{P}(z, x \cdot (x - y))$.
Let z be an entity so that $\mathbf{P}(z, x)$ and $\neg\mathbf{PO}(z, x \cdot y)$ Then $\neg\mathbf{PO}(z, y)$, otherwise some part of z would be contained in $x \cdot y$ and overlap $x \cdot y$. Then $\mathbf{P}(z, x - y)$ by Dif-T5, i.e., z is a part of both x and $x - y$. Hence $\mathbf{P}(z, x \cdot (x - y)$ by Int-A4.

Direction (a): $\mathbf{P}(z,x) \wedge \neg\mathbf{PO}(z, x\cdot y) \leftarrow \mathbf{P}(z, x\cdot(x-y))$.

We can safely assume that $(x\cdot(x-y)) \notin \mathbf{ZEX}_{\mathcal{M}}$, otherwise it does not contain any parts at all and the implications is vacuously true. Now assume $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$, then $\mathbf{P}(x-y,x)$ (Dif-T1), hence $\mathbf{P}(x-y, x\cdot(x-y))$ and therefore $x{=_{\mathbf{dim}}}x-y{=_{\mathbf{dim}}}x\cdot(x-y)$. Then for an arbitrary z, $\mathbf{P}(z, x\cdot(x-y))$ implies that $\mathbf{P}(z,x)$ and $\mathbf{P}(z,x-y)$, the latter further requiring that $\neg\mathbf{PO}(z,x\cdot y)$ by Dif-T5.

The two directions imply the equivalence of $x-y$ and $x\cdot(x-y)$. ☐

(Dif-T8) $P(y,x) \rightarrow y = x-(x-y)$ (− involutary)

Lemma 7.14. *$CODI_{\downarrow} \vDash$ Dif-T8*

Proof. To prove that $P(y,x) \rightarrow y = x-(x-y)$ we show separately that $P(y,x)$ implies (a) $P(y, x-(x-y))$ and (b) $P(x-(x-y), y)$:

Part (a): $P(y,x) \rightarrow P(y, x-(x-y))$.

By Dif-T5 the implication is equivalent to the sentence

$$P(y,x) \rightarrow P(y,x) \wedge \neg PO(y,(x-y)).$$

Since the relation PO is symmetric, this immediately follows from Dif-T3.

Part (b): $P(y,x) \rightarrow P(x-(x-y), y)$.

Assume $\mathbf{P}(y,x)$, then $x{=_{\mathbf{dim}}}y$.

We will show that for all $z \in \mathbf{M}$, if $\mathbf{P}(z, x-(x-y))$ then $\mathbf{P}(z,y)$].

Let $z \in \mathbf{M}$ be an arbitrary part such that $\mathbf{P}(z, x-(x-y))$. Then $\mathbf{P}(z,x)$ by Dif-A3b and by $(x-(x-y)) \notin \mathbf{ZEX}_{\mathcal{M}}$). We further have $\mathbf{P}(z, x-(x-y))$ if $\mathbf{P}(z,x)$ and $z\cdot(x-y){<_{\mathbf{dim}}}z$. Suppose now $\neg\mathbf{P}(z,y)$, then $z\cdot y{<_{\mathbf{dim}}}y$ and by $\mathbf{P}(z,x)$ we would require $\mathbf{P}(z,x-y)$—a contradiction to $z\cdot(x-y){<_{\mathbf{dim}}}z$. Hence $\mathbf{P}(z,y)$ must hold.

Since $\mathbf{P}(y,x) \rightarrow \mathbf{P}(y, x-(x-y))$ and $\mathbf{P}(y,x) \rightarrow \mathbf{P}(x-(x-y), y)$ we immediately obtain $\mathbf{P}(y,x) \rightarrow y = x-(x-y)$ by EP-T2. ☐

(Dif-T9) $x = y \leftrightarrow ZEX(x-y) \wedge ZEX(y-x)$ (− anticommutative)

Lemma 7.15. *$CODI_{\downarrow} \vDash$ Dif-T9*

Proof. We prove the two directions of the implication individually.

Direction (a): $x = y \rightarrow ZEX(x-y) \wedge ZEX(y-x)$.

Assume $x = y$.

Then $\mathbf{Cont}(x,y)$ and $\mathbf{Cont}(y,x)$, hence $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$ and $(y-x) \in \mathbf{ZEX}_{\mathcal{M}}$ by Dif-A4.

Direction (b): $x = y \leftarrow ZEX(x-y) \wedge ZEX(y-x)$.

Assume $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$ and $(y-x) \in \mathbf{ZEX}_{\mathcal{M}}$.

Suppose $x \in \mathbf{ZEX}_{\mathcal{M}}$, then by C-A4 we have $\neg\mathbf{Cont}(y,x)$ and hence $y \in \mathbf{ZEX}_{\mathcal{M}}$ by Dif-A4. In this case $x = y$ by D-A4.

Now suppose $x \notin \mathbf{ZEX}_{\mathcal{M}}$, then by Dif-A4 we must have $\mathbf{Cont}(x,y)$, which by C-A4 requires $y \notin \mathbf{ZEX}_{\mathcal{M}}$ so that again by Dif-A4 we must have $\mathbf{Cont}(y,x)$. By C-A2 we obtain $x = y$ again.

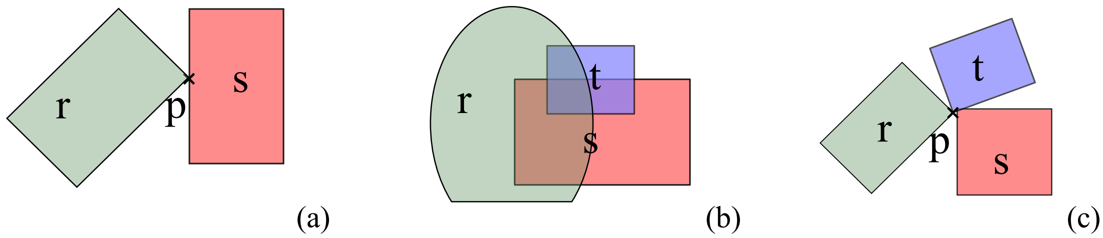


Figure 7.4: Non-distributivity of intersection over differences and of differences over intersections in the models (a)–(c) of $CODI_{\downarrow}$. In (a) intersection does not distributive over the difference because $r \cdot (s - r) = r \cdot s = p \neq \emptyset = p - r = (r \cdot s) - (r \cdot s)$. In example (b), even though r, s, t are of identical dimension the difference $r - (s \cdot t)$ is not left-distributive over the intersection $(s \cdot t)$, since we have $r - (s \cdot t) \supsetneq (r - s) \cdot (r - t)$. Example (c) shows that differences are not always right-distributive, we have $(r \cdot s) - t = p - t = \emptyset \neq p = r \cdot s = (r - t) \cdot (s - t)$.

The two directions together prove the biconditional in Dif-T9. □

(Dif-T10) $SC(x, y) \rightarrow x - y = x$ (difference between entities in superficial contact)

Lemma 7.16. *$CODI_{\downarrow} \models$ Dif-T10*

Proof. Assume $\mathbf{SC}(x, y)$.
Then by Int-T9 we obtain $x \cdot y <_{\mathbf{dim}} z$ and with $\mathbf{P}(x, x)$ we get $\mathbf{P}(x, x - y)$ by Dif-A3a. Hence $(x - y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and together with $\mathbf{P}(x - y, x)$ (by Dif-T1) we immediately conclude $x = x - y$ by C-A2. □

The following corollary is a consequence of Dif-T10. It essentially says that the intersection of two entities x and y that are in superficial contact is contained in both $x - y$ and $y - x$.

(Dif-T10′) $SC(x, y) \rightarrow Cont(x \cdot y, x - y) \wedge Cont(x \cdot y, y - x)$

(the intersection of two entities x, y in superficial contact is in $x - y$ and $y - x$)

Corollary 7.2. *$CODI_{\downarrow} \models$ Dif-T10′*

Proof. Follows directly from Dif-T10 because $Cont(x \cdot y, x)$ and $Cont(x \cdot y, y)$. □

Note that the intersection operation does not always distribute over the difference operation, that is, we may encounter $x \cdot (y - z) \neq (x \cdot y) - (x \cdot z)$, even if x, y, and z are all of the same dimension. The example in Figure 7.4(a) demonstrates this for the more restricted case where $x = z$, which amounts to showing that $x =_{\text{dim}} y \rightarrow x \cdot (y - x) = (x \cdot y) - x$ may not be true. In this restricted case, the right-hand side is always the zero entity because of $Cont(x \cdot y, x)$ but the left-hand side may not be zero, i.e., $CODI_{\downarrow} \not\models ZEX(x \cdot (y - x))$.

Note further that differences do not distribute left over intersections; this is not surprising since it fails even in equidimensional mereotopology, see Figure 7.4(b). That is, we may have $x - (y \cdot z) \neq (x - y) \cdot (x - z)$ even though $x =_{\text{dim}} y =_{\text{dim}} z$. However, contrary to the set theoretic difference, which distributes right over set intersection, in $CODI_{\downarrow}$ differences also do not always distribute right over intersections. We may encounter $(x \cdot y) - z \neq (x - z) \cdot (y - z)$ even if $x =_{\text{dim}} y =_{\text{dim}} z$; consider the example in Figure 7.4(c).

7.2.3 Simple self-connectedness

For a moment, we will deviate from our main line of work in this chapter and define the concept of self-connectedness in $CODI_{\downarrow}$. This subsection will be relevant in later chapters of the thesis, in particular in Chapters 9 and 11. Self-connectedness (we often use the shorter term 'connectedness' as long as it is clear from the context that we deal with a single entity) in equidimensional theories of space relies on the notions of (mereological or topological) sums and often also differences or complements [AV95; Cla85][1]. In the multidimensional mereotopology, self-connectedness becomes definable in the presence of differences, similar to the definition in equidimensional mereotopology. We say an entity x is connected if every proper part y of x is connected to its relative complement, the difference $x - y$. Later, in Section 9.2, we will study other notions of connectedness, in particular internal connectedness, which requires constraining the class of intended models further.

(Con-D) $Con(x) \leftrightarrow \forall y[PP(y,x) \rightarrow C(y, x-y)]$ (self-connectedness)

Axiom Set 7.9: Definition Con-D of $CODI_{\downarrow}$.

It immediately follows that minimal entities as well as the zero entity are connected according to this definition (Con-T1 and Con-T2).

(Con-T1) $Min(x) \rightarrow Con(x)$ (minimal entities connected)

(Con-T2) $ZEX(x) \rightarrow Con(x)$ (zero entity connected)

Lemma 7.17. *$CODI_{\downarrow} \cup$ Con-D $\models$ {Con-T1, Con-T2}*

Moreover, for connected entities, we can prove superficial connection (Con-T3) between a proper part and its relative complement. The more general relationships between the intersection $x \cdot y$ and the difference $x - y$ involving a connected entity x are captured by Con-T4 and Con-T5.

(Con-T3) $Con(x) \wedge PP(y,x) \rightarrow SC(y, x-y)$

(a proper part of a self-connected entity is connected to its relative complement)

(Con-T4) $Con(x) \wedge PO(x,y) \wedge \neg ZEX(x-y) \rightarrow SC(x \cdot y, x-y)$ (if y partially overlaps a self-connected entity x without containing it, then $x \cdot y$ is superficially connected to $x-y$)

(Con-T5) $Con(x) \wedge Inc(x,y) \wedge x - y =_{\text{dim}} x \cdot y \wedge \neg ZEX(x-y) \rightarrow SC(x \cdot y, x-y)$

(if y is incident with a self-connected entity x without containing it, and the difference $x-y$ is of the same dimension as the intersection $x \cdot y$, then $x \cdot y$ is superficially connected to $x-y$)

Lemma 7.18. *$CODI_{\downarrow} \cup$ Con-D $\models$ {Con-T3 – Con-T5}*

Proof. **(Con-T3)** $Con(x) \wedge PP(y,x) \rightarrow SC(y, x-y)$.

Assume $\mathbf{Con}_{\mathcal{M}}(x)$ and $\mathbf{PP}(y,x)$.

By Con-D we derive $\mathbf{C}(y, x-y)$. By Dif-T3 we have $\neg\mathbf{PO}(y, x-y)$ and by Dif-T2 we have $\mathbf{PP}(x-y, x)$, hence $y \mathbf{=_{dim}} x-y$ and $\neg\mathbf{Inc}(y, x-y)$. Then by CD-T4 (contact is either proper overlap, incidence, or superficial contact) we conclude $\mathbf{SC}(y, x-y)$.

[1] Clarke as well as Asher and Vieu use the term 'connected individual', *CON*. Cohn and Varzi [CV03] discuss some of the intricacies of self-connectedness is more detail.

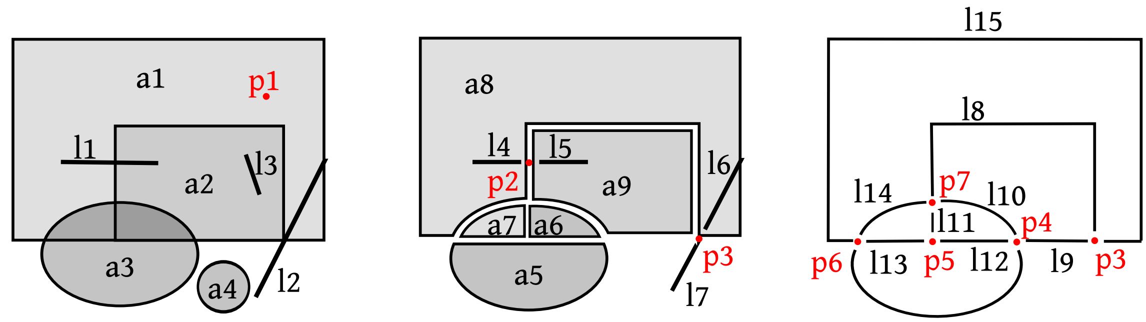


Figure 7.5: A model of $CODI_{\downarrow}$ (left) decomposed by intersections and differences into simple atomic entities (points p1–p7, lines l3–l15, and areas a4–a9). The additional atomic entities that must exist because of decomposability are shown in the figures in the middle and on the right.

(Con-T4) $Con(x) \wedge PO(x, y) \wedge \neg ZEX(x - y) \rightarrow SC(x \cdot y, x - y)$.

Assume $\mathbf{Con}_{\mathcal{M}}(x)$, $\mathbf{PO}(x, y)$, and $(x - y) \notin \mathbf{ZEX}_{\mathcal{M}}$.

Then $\mathbf{Cont}(x \cdot y, x)$ (Int-A2) and $x \cdot y =_{\mathbf{dim}} x$, hence $\mathbf{P}(x \cdot y, x)$. Moreover, we obtain $\mathbf{P}(x - y, x)$ by Dif-T1. Since $\neg\mathbf{PO}(x - y, x \cdot y)$, we actually have proper parthood $\mathbf{PP}(x - y, x)$ and $\mathbf{PP}(x \cdot y, x)$. From the latter, we obtain $\mathbf{SC}(x \cdot y, x - (x \cdot y))$ by Con-T3, which is equivalent to $\mathbf{SC}(x \cdot y, x - y)$ by Dif-T6.

(Con-T5) $Con(x) \wedge Inc(x, y) \wedge x - y =_{\text{dim}} x \cdot y \wedge \neg ZEX(x - y) \rightarrow SC(x \cdot y, x - y)$.

Assume $\mathbf{Con}_{\mathcal{M}}(x)$, $\mathbf{Inc}(x, y)$, $x - y =_{\mathbf{dim}} x \cdot y$, and $(x - y) \notin \mathbf{ZEX}_{\mathcal{M}}$.

Then, by Dif-A1, we have $x =_{\mathbf{dim}} x - y =_{\mathbf{dim}} x \cdot y$. Then $\mathbf{P}(x \cdot y, x)$ and $\mathbf{P}(x - y, x)$ and again (similar to the proof for Con-T2) $\mathbf{PP}(x \cdot y, x)$. Hence we obtain again $\mathbf{SC}(x \cdot y, x - y)$ by Con-T1 and Dif-T6.

□

7.2.4 Decomposability of atomic models of $CODI_{\downarrow}$

With extensionality of PO and a proper understanding of the interaction between intersections and differences, we are now able to prove that any entity in an atomic model of $CODI_{\downarrow}$ can be decomposed into a set of minimal nonoverlapping parts, compare Figure 7.5. This property, called decomposability [compare Var07], follows in mereology and equidimensional mereotopology from strong supplementation (EP-E2) and PO-E1. Varzi [Var07] offers a detailed discussion of decomposition principles in mereology and equidimensional mereotopology. Here, we want to confirm that this kind of decomposability of atomic models also works in our multidimensional mereotopology.

An atomic model is one that satisfies ME-E1 from page 105: every nonzero entity contains *some* minimal, i.e., indivisible, part. To show that atomic models are decomposable, it suffices to show that atomic models are atomistic, i.e., are the sum of a set of minimal parts. Even though we have not yet introduced sums, we can express that a model is atomistic if every other entity that partially overlaps a given entity, overlaps in some minimal part. Then, the set of minimal parts an entity overlaps (or for that matter, contains) uniquely defines the entity. In Section 7.3, we can express decomposability using the sum operation in Sum-T9 on page 141.

The proofs comes in several parts. First, we prove that for any part y of x, every minimal entity contained in x is either contained in y or in the difference $x - y$ (Dif-T11).

(Dif-T11) $Min(z) \wedge Cont(z,x) \wedge P(y,x) \rightarrow [Cont(z, x-y) \vee Cont(z,y)]$

(minimal entities contained in x are contained in a part y of x or in its relative complement $x - y$)

Lemma 7.19. $CODI_{\downarrow} \vDash$ *Dif-T11*

Proof. Proof by contradiction.
Suppose $\mathbf{P}(y,x)$, $\mathbf{Cont}(z,x)$, $\neg\mathbf{Cont}(z,x-y)$, $\neg\mathbf{Cont}(z,y)$, and $z \in \mathbf{Min}_{\mathcal{M}}$.
Because we have $\mathbf{Cont}(z,x)$ and $\neg\mathbf{Cont}(z,x-y)$ we must have $z \cdot y \not<_{\text{dim}} z$ by Dif-A3. Since we always have $z \cdot y \leq_{\mathbf{dim}} z$, $z \cdot y_{\text{dim}} z$ is the only option. But then some v must exist such that $\mathbf{P}(v,z)$ and $\mathbf{Cont}(v, z \cdot y)$. Since $z \in \mathbf{Min}_{\mathcal{M}}$, z has no other parts than itself; hence $\mathbf{Cont}(z, z \cdot y)$. By the definition of intersections we conclude that $\mathbf{Cont}(z,y)$ must hold, which contradicts our assumption. □

As a special case of Dif-T11 we have Dif-T11′.

(Dif-T11′) $P(y,x) \wedge Min(z) \wedge P(z,x) \rightarrow P(z, x \cdot y) \vee P(z, x-y)$

(any minimal part z of x is either contained in part y of x or in the difference $x - y$)

Lemma 7.20. $CODI_{\downarrow} \cup$ *Dif-T11* $\vDash$ *Dif-T11′*

Now, we establish that every entity in an atomic model is uniquely defined by its minimal parts.

Theorem 7.3. *Let $\mathcal{M}$ be an atomic model of $CODI_{\downarrow}$, i.e., a model of $CODI_{\downarrow} \cup$ ME-E1. Then every entity in the domain of $\mathcal{M}$ is uniquely determined by its set of minimal parts, which are exhaustive and pairwise non-overlapping.*

Proof. Let $\mathcal{M}$ be an atomic model of $CODI_{\downarrow}$ with domain $\mathbf{M}$. Because it is atomic we have (by ME-E1):

$$\mathcal{M} \vDash \forall x \big[\neg ZEX(x) \rightarrow \exists y[P(y,x) \wedge Min(y)]\big].$$

Recall that by PO-E1, every entity $x \in \mathbf{M}$ in the model is uniquely defined by the extension of PO involving x:

$$\{\langle x,w\rangle \mid w \in \mathbf{M}\} \subseteq \mathbf{PO}_{\mathcal{M}}.$$

We want to prove that any entity $x \in \mathbf{M}$ is uniquely defined by the subset of the extension of PO relating x to minimal entities:

$$\{\langle x,w\rangle \mid w \in \mathbf{Min}_{\mathcal{M}}\} \subseteq \mathbf{PO}_{\mathcal{M}}.$$

We have two cases.

Case I: x is itself a minimal entity.

The only minimal entity x can partially overlap is itself, i.e., we have $\neg\mathbf{PO}(x,y)$ for any $y \neq x$. Suppose x partially overlaps a minimal entity $y \neq x$. Then by ME-D1 and PO-D for all $z \in \mathbf{M}$, $\neg\mathbf{PP}(z,x)]$ and there exists a $z \in \mathbf{M}$ such that $\mathbf{P}(z,x)$ and $\mathbf{P}(z,y)$. Hence the only z that can be contained in both x and y is x itself. But then $\mathbf{P}(x,y)$ and thus either y is nonminimal or equivalent to x.

Case II: x is not a minimal entity.

By ME-D1, x contains at least some minimal part, call it y. By strong supplementation (EP-E2), there exists a part z of x that does not partially overlap y. Because this entity z contains some minimal part, x contains at least two distinct minimal parts. Therefore, x's extension of PO with respect to minimal entities must be different than that of any minimal entity.

Now suppose there exists another entity $x' \neq x$ that partially overlaps the same set of minimal entities as x. From $x' \neq x$, by EP-E2 there must exist a w such that $\mathbf{P}(w, x') \wedge \neg\mathbf{PO}(w, x)$. Such a w would—by ME-E1—contain a minimal entity v. Then $\langle x, v\rangle \notin \mathbf{PO}_{\mathcal{M}}$ but $\langle x', v\rangle \in \mathbf{PO}_{\mathcal{M}}$—a contradiction to our assumption that x' and x partially overlap the same set of minimal entities.

Clearly, the two cases are exhaustive. Then any entity is uniquely defined by its part that are minimal entities.

Because (1) two distinct minimal entities cannot overlap and (2) any two entities that partially overlap must share a minimal part, the set of minimal parts of an entity are exhaustive and pairwise nonoverlapping. □

In particular, we can recursively decompose any nonminimal entity x into a minimal part y (which must exist) and its 'remainder'—the difference $x - y$. If $x - y$ is nonminimal, we can repeat this step. Dif-T12 formalizes this notion that is dual to Dif-T11.

(Dif-T12) $Cont(z, x) \wedge Cont(y - z, x) \rightarrow Cont(y, x)]$ (if the difference $y - z$ is contained in x, and z is a minimal entity z that is also contained in x, then y is contained in x)

Lemma 7.21. *$CODI_{\downarrow} \cup$ ME-E1 $\vDash$ Dif-T12*

Proof. Proof by contradiction.
Assume x, y, z to be arbitrary entities that satisfy $\mathbf{Cont}(z, x)$ and $\mathbf{Cont}(y - z, x)$.
Now suppose $\neg\mathbf{Cont}(y, x)$, then some $v \in \mathbf{M}$ exists such that $v \in \mathbf{Min}_{\mathcal{M}}$, $\mathbf{P}(v, y)$, and $\neg\mathbf{Cont}(v, x)$ by EP-E3 and ME-E1.
We distinguish two cases.

Case (I): Assume $\mathbf{PO}(v, z)$.

Then, because v is minimal, the only part v and z can share is v itself, thus $\mathbf{P}(v, z)$. From $\mathbf{Cont}(z, x)$ we obtain $\mathbf{Cont}(v, x)$ by C-A3.

Case (II): Assume $\neg\mathbf{PO}(v, z)$.

Then $\mathbf{P}(v, y - z)$ by Dif-A3a and $\mathbf{Cont}(v, x)$ from $\mathbf{Cont}(y - z, x)$ and C-A3.

Clearly, the two cases are exhaustive, hence in any case we have $\mathbf{Cont}(v, x)$ in contradiction to our supposition $\neg\mathbf{Cont}(v, x)$. Hence we must have $\mathbf{Cont}(y, x)$. □

Because our intended structures are constructed from finite sets of manifolds, the resulting models are necessarily of finite domain and thus atomic. Hence, the models of $CODI_{\downarrow}$ already satisfy one property of the intended structures: they are decomposable into sets of atomic entities.

7.2.5 Satisfiability of $CODI_\downarrow$

In Chapter 5 we introduced the intended structures, formally defined as the class $\mathbb{M}$, that we want to capture with $CODI_\downarrow$. Let $\mathfrak{M}^m$ be such an intended structure; we claim that its finite collection of atomic m-manifolds of $\mathfrak{M}^m$ is then captured by the set $\mathbf{Min}_\mathcal{M}$ in the corresponding model of $CODI_\downarrow$. Any composite m-manifold in $\mathfrak{M}^m$ that is not an atomic m-manifold is composed of a set of at least two atomic m-manifolds joined only in their boundaries, i.e., superficially, compare Definition 5.6. Those composite manifolds must consist of two or more nonoverlapping parts; Theorem 7.3 then guarantees that those are composed of minimal parts, which correspond to its atomic m-manifolds. Here, we will show that any structure $\mathfrak{M}^m$ in the class $\mathbb{M}$ yields a model of $CODI_\downarrow$ in this way. That is, we show the satisfiability of $CODI_\downarrow$ with respect to the intended structures.

Theorem 7.4 (Satisfiability of $CODI_\downarrow$)**.** *Let $\mathfrak{M}$ be a complex manifold (a collection of composite manifolds) in the class $\mathbb{M}$ of intended structures with domain $Dom(\mathfrak{M})$ (as defined in Definition 5.11) and $\emptyset \in \mathrm{Dom}(\mathfrak{M})$. Then there exists a corresponding model $\mathcal{M}$ of $CODI_\downarrow$ with finite domain $\mathbf{M}$ such that*

1. *$\mu : \mathrm{Dom}(\mathfrak{M}) \to \mathbf{M}$ is a bijection;*

and for all $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$,

2. *$\mu(d) \in \mathbf{ZEX}_\mathcal{M} \iff \Sigma d = \emptyset$;*

3. *$\langle \mu(d_1), \mu(d_2)\rangle \in (<_{\mathbf{dim}})_\mathcal{M} \iff \big(\dim(d_1) < \dim(d_2)\big)$ or $\big(d_1 = \emptyset$ and $d_2 \neq \emptyset\big)$;*

4. *$\langle \mu(d_1), \mu(d_2)\rangle \in \mathbf{Cont} \iff \Sigma d_1 \subseteq \Sigma d_2$ and $d_1 \neq \emptyset$.*

Proof. Let $\mathfrak{M}$ be an arbitrary structure in $\mathbb{M}$. We then construct an interpretation $\mathcal{M}$ of $CODI_\downarrow$ according to the conditions (1)–(4). In particular, we choose the domain $\mathbf{M}$ of $\mathcal{M}$ such that a one-to-one mapping μ between entities in $Dom(\mathfrak{M})$ (including the empty set) and entities in $\mathbf{M}$ exists (condition (1)). The conditions (2)–(4) define the extensions $\mathbf{ZEX}_\mathcal{M}$, $(<_{\mathbf{dim}})_\mathcal{M}$, and $\mathbf{Cont}_\mathcal{M}$, respectively.

It remains to show that this interpretation is a model of $CODI_\downarrow$, i.e., that is satisfies the axioms D-A1 – D-A6, C-A1 – C-A4, CD-A1, Int-A1 – Int-A4, Dif-A1, Dif-A2, Dif-A3a, Dif-A3b, Dif-A3c, Dif-A4, Z-A1 with the definitions D-D1 – D-D7, EP-D, C-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2.

D-A1 – D-A6 are satisfied because the dimensions in Euclidean spaces form a discrete linear order with a lowest and maximal dimension for any finite set of manifolds.

From the mapping of $\subseteq$ between the areas of manifolds to the relation *Cont* between the corresponding entities in condition (4), it is also straightforward to verify C-A1 – C-A4 and CD-A1.

What remains to be shown is that the axioms Int-A1 – Int-A4 and Dif-A1 – Dif-A4 are satisfied by $\mathcal{M}$. That is, we must show that for arbitrary $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$, there exist $d_3, d_4 \in \mathrm{Dom}(\mathfrak{M})$ such that $\mu(d_3) = \mu(d_1) \cdot \mu(d_2)$ satisfies Int-A1 – Int-A4 and such that $\mu(d_4) = \mu(d_1) - \mu(d_2)$ satisfies Dif-A1 – Dif-A4. To achieve this, we give explicit definitions of the operations $\cdot$ and $-$ in $\mathfrak{M}$.

Part (I): $\mu(d_1) \cdot \mu(d_2) = \mu(d_3)$ satisfies Int-A1 – Int-A4 for some $d_3 \in \mathrm{Dom}(\mathfrak{M})$.
Assume d_1, d_2 are arbitrary entities in $Dom(\mathfrak{M})$.

Case (I.a): Assume $d_1 = \emptyset$ or $d_2 = \emptyset$.
Then we define $d_3 = \emptyset$. Clearly, $\Sigma d_3 \in \mathrm{Dom}(\mathfrak{M})$ and $\mu(d_3) \in \mathbf{M}$. By condition (2), we have

$(\mu(d_3)) \in \mathbf{ZEX}_{\mathcal{M}}$. By condition (4) we have

$$\text{for all } d \in \text{Dom}(\mathfrak{M}) \; [\neg\mathbf{Cont}(\mu(d), \mu(d_3))] \text{ and}$$

$$\text{for all } d \in \text{Dom}(\mathfrak{M}) \; [\neg\mathbf{Cont}(\mu(d), \mu(d_1)) \text{ or } \neg\mathbf{Cont}(\mu(d), \mu(d_2))]$$

so that Int-A1 – Int-A4 are trivially satisfied.

Case (I.b): Assume $d_1 \neq \emptyset$ and $d_2 \neq \emptyset$.

Then $d_1 \cap d_2 \neq \emptyset$ and by Definition 5.11(2), there exists a composite manifold $d_3 \in \mathfrak{M}$ such that $\Sigma d_3 \subseteq \Sigma d_1 \cap \Sigma d_2$ and such that for all $d_4 \in \mathfrak{M}$ either $d_4 \subseteq d_3$ or $\dim(d_4) \lneq \dim(d_3)$.

We will use this d_3 to define $\mu(d_3) = \mu(d_1) \cdot \mu(d_2)$ and show that this satisfies Int-A1 – Int-A4. Notice that $\Sigma d_3 \subseteq \Sigma d_1$ and $\Sigma d_3 \subseteq \Sigma d_2$ and thus $\mathbf{Cont}(\mu(d_3), \mu(d_1))$ and $\mathbf{Cont}(\mu(d_3), \mu(d_2))$.

(Int-A1): $\neg\mathbf{C}(\mu(d_1), \mu(d_2)) \to (\mu(d_3)) \in \mathbf{ZEX}_{\mathcal{M}}$.

$\mathbf{C}(\mu(d_1), \mu(d_2))$ follows from $\mathbf{Cont}(\mu(d_3), \mu(d_1))$ and $\mathbf{Cont}(\mu(d_3), \mu(d_2))$ by C-D, so that the antecedent of Int-A1 is falsified and Int-A1 is satisfied.

(Int-A2): $(\mu(d_3)) \notin \mathbf{ZEX}_{\mathcal{M}} \to \mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{Cont}(\mu(d_3), \mu(d_2))$.

We already have both consequents of Int-A2: $\mathbf{Cont}(\mu(d_3), \mu(d_1))$ and $\mathbf{Cont}(\mu(d_3), \mu(d_2))$.

(Int-A3): $\mathbf{Cont}(z, \mu(d_1)) \wedge \mathbf{Cont}(z, \mu(d_2)) \to z \leq_{\mathbf{dim}} \mu(d_3)$.

Let z be an arbitrary entity such that $\mathbf{Cont}(z, \mu(d_1)) \wedge \mathbf{Cont}(z, \mu(d_2))$. Then $\Sigma z \subseteq \Sigma d_1$ and $\Sigma z \subseteq \Sigma d_2$ and thus $\Sigma z \subseteq \Sigma d_1 \cap \Sigma d_2$. Hence by Definition 5.11(2) we must have $\Sigma z \subseteq \Sigma d_3$ or $\dim(z) < \dim(d_3)$. In the first case, we derive $\mathbf{Cont}(\mu(d), \mu(d_3))$ and thus $\mu(d) \leq_{\mathbf{dim}} \mu(d_3)$ by CD-A1. In the later case we obtain $\mu(d) <_{\mathbf{dim}} \mu(d_3)$ by condition (3).

(Int-A4) $\mathbf{Cont}(z, \mu(d_1)) \wedge \mathbf{Cont}(z, \mu(d_2)) \wedge z =_{\mathbf{dim}} \mu(d_3) \leftrightarrow \mathbf{P}(z, \mu(d_3))$.

Direction (i): $\mathbf{Cont}(z, \mu(d_1)) \wedge \mathbf{Cont}(z, \mu(d_2)) \wedge z =_{\mathbf{dim}} \mu(d_3) \to \mathbf{P}(z, \mu(d_3))$.

Let z be an arbitrary entity such that

$$\mathbf{Cont}(z, \mu(d_1)) \wedge \mathbf{Cont}(z, \mu(d_2)) \wedge z =_{\mathbf{dim}} \mu(d_3).$$

Then there exists a $d \in \text{Dom}(\mathfrak{M})$ such that $\mu(d) = z$. Then $\Sigma d \subseteq \Sigma d_1$ and $\Sigma d \subseteq \Sigma d_2$ by condition (4) and $\dim(d) = \dim(d_3)$ by condition (3). By Definition 5.11(2) we must have $\Sigma d \subseteq \Sigma d_3$ and thus $\mathbf{Cont}(\mu(d), \mu(d_3))$ by condition (4). Together with $z = \mu(d) =_{\mathbf{dim}} \mu(d_3)$ we obtain $P(\mu(d), \mu(_3))$ by EP-D.

Direction (ii): $\mathbf{P}(z, \mu(d_3)) \to \mathbf{Cont}(z, \mu(d_1)) \wedge \mathbf{Cont}(z, \mu(d_2)) \wedge z =_{\mathbf{dim}} \mu(d_3)$.

Let z be an arbitrary entity such that $\mathbf{P}(z, \mu(d_3))$.

Then there exists a $d \in \text{Dom}(\mathfrak{M})$ such that $\mu(d) = z$. From EP-D we obtain $\mathbf{Cont}(\mu(d), \mu(d_3))$ and $z = \mu(d) =_{\mathbf{dim}} \mu(d_3)$. From the former, we obtain $\Sigma d \subseteq \Sigma d_3 \subseteq \Sigma d_1 \cap \Sigma d_2$ and thus $\mathbf{Cont}(\mu(d), \mu(d_1))$ and $\mathbf{Cont}(\mu(d), \mu(d_2))$, all by condition (4).

We have thereby shown that in the case $d_1, d_2 \neq \emptyset$ the d_3 that is guaranteed to exist by Definition 5.11(2) satisfies all axioms governing the intersection $\mu(d_3) = \mu(d_1) \cdot \mu(d_2)$.

Consequently, for any structure $\mathfrak{M}$ for arbitrary $d_1, d_2 \in \text{Dom}(\mathfrak{M})$ there exists an entity $d_3 \in \text{Dom}(\mathfrak{M})$ so that $\mu(d_3) = \mu(d_1) \cdot \mu(d_2)$ satisfies Int-A1 – Int-A4.

Part (II): $\mu(d_1) - \mu(d_2) = \mu(d_3)$ satisfies Dif-A1 – Dif-A4 for some $d_3 \in \mathrm{Dom}(\mathfrak{M})$.

Case (II.a): $d_1 = \emptyset$.

Choose $d_3 = \emptyset$. Then $(\mu(d_1)) \in \mathbf{ZEX}_{\mathcal{M}}$ and $(\mu(d_3)) \in \mathbf{ZEX}_{\mathcal{M}}$ by condition (2) so that Dif-A4 is satisfied. By condition (4), for all $d \in \mathrm{Dom}(\mathfrak{M})$ we have $[\neg\mathbf{Cont}(\mu(d), \mu(d_3))]$, hence Dif-A1, Dif-A2, Dif-A1(a)–(c) are trivially satisfied.

Case (II.b): $\dim(\Sigma d_1 \setminus \Sigma d_2) < \dim(d_1)$.

Again we choose $d_3 = \emptyset$. By the same argument as in case (II.a) Dif-A1, Dif-A2, Dif-A1(a)–(c) are satisfied. Because by condition (4) we further have $\mathbf{Cont}(\mu(d_1), \mu(d_2))$, Dif-A4 is also satisfied.

Case (II.c): $d_1 \neq \emptyset$, $\dim(\Sigma d_1 \setminus \Sigma d_2) = \dim(d_1)$, and $\dim(\Sigma d_1 \cap \Sigma d_2) < \dim(d_1)$.

We choose $\mu(d_1) - \mu(d_2) = \mu(d_1)$. Clearly, $d_1 \in \mathrm{Dom}(\mathfrak{M})$ by our assumption. Dif-A1, Dif-A2, Dif-A3(a), Dif-A3(b) and Dif-A4 are trivially satisfied.

It remains to prove Dif-A3(c). Assume $\mathbf{P}(z, \mu(d_1))$ with $z = \mu(d)$ for some $d \in \mathrm{Dom}(\mathfrak{M})$. Then $\mathbf{Cont}(\mu(d), \mu(d_1))$ and $\mu(d) {=_{\mathbf{dim}}} \mu(d_1)$ by EP-D. Hence $\Sigma d \subseteq \Sigma d_1$ and $\dim(\mu(d)) = \dim(\mu(d_1))$. We then have $\Sigma d \cap \Sigma d_2 \subseteq \Sigma d_1 \cap \Sigma d_2$. By our assumption we obtain $\dim(\Sigma d \cap \Sigma d_2) < \dim(\Sigma d_1) = \dim(\Sigma d)$. Hence we conclude $\mu(d) \cdot \mu(d_2) {<_{\mathbf{dim}}} \mu(d)$, the consequent of Dif-A3(c).

Case (II.d): $d_1 \neq \emptyset$, $\dim(\Sigma d_1 \setminus \Sigma d_2) = \dim(d_1)$, and $\dim(\Sigma d_1 \cap \Sigma d_2) = \dim(d_1)$.

Then by Definition 5.11(4), there exists a composite manifold $d_3 \in \mathfrak{M}$ such that $d_3 = \overline{\Sigma d_1 \setminus \Sigma d_2}$. We will show that then $\mu(d_3) = \mu(d_1) - \mu(d_2)$ satisfies all axioms Dif-A1 – Dif-A4.

(Dif-A1): $(\mu(d_3)) \notin \mathbf{ZEX}_{\mathcal{M}} \to \mu(d_3) {=_{\mathbf{dim}}} \mu(d_1)$.

We have $\dim(d_3) = \dim(\Sigma d_1 \setminus \Sigma d_2) = \dim(d_1)$ by assumption and thus $\mu(d_3) {=_{\mathbf{dim}}} \mu(d_1)$ by condition (3).

(Dif-A2): $\mu(d_2) {<_{\mathbf{dim}}} \mu(d_1) \to \mu(d_3) = \mu(d_1)$.

Because by assumption $\dim(\Sigma d_1 \cap \Sigma d_2) = \dim(d_1)$ and thus $dim(d_2) \geq \dim(d_1)$. By condition (3) we obtain $\mu(d_2) {\geq_{\mathbf{dim}}} \mu(d_1)$, thus violating the antecedent of Dif-A2 and thus trivially satisfying Dif-A2.

(Dif-A3a): $\mu(d_1) {\leq_{\mathbf{dim}}} \mu(d_2) \to [\mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge z \cdot \mu(d_2) {<_{\mathbf{dim}}} z \to \mathbf{Cont}(z, \mu(d_3))]$.

Proof by contradiction: Assume $d_4 \in \mathrm{Dom}(\mathfrak{M})$ is an entity such that

$$\mathbf{Cont}(\mu(d_4), \mu(d_1)) \wedge \mu(d_4) \cdot \mu(d_2) {<_{\mathbf{dim}}} \mu(d_4)$$

and suppose

$$\neg\mathbf{Cont}(\mu(d_4), \mu(\overline{\Sigma d_1 \setminus \Sigma d_2})).$$

Then $\Sigma d_4 \subseteq \Sigma d_1$ and $\dim(d_4 \cap d_2) < \dim(d_4)$. Then $\Sigma d_4 \subseteq \overline{\Sigma d_1 \setminus \Sigma d_2}$ because the topological closure $\overline{X}$ restores all lower-dimensional sets in $\Sigma d_4 \cap \Sigma d_1$ removed by Σd_2. Hence $\mathbf{Cont}(\mu(d_4), \mu(d_1 \setminus d_2))$ contrary to our supposition.

(Dif-A3b): $\mu(d_1) {\leq_{\mathbf{dim}}} \mu(d_2) \to [\mathbf{Cont}(z, \mu(d_3)) \to \mathbf{Cont}(z, \mu(d_1))]$.

Assume $d_4 \in \mathrm{Dom}(\mathfrak{M})$ such that

$$\mu(d_4) \cdot \mu(d_2) {<_{\mathbf{dim}}} \mu(d_4) \text{ and } \mathbf{Cont}(\mu(d_4), \mu(\overline{\Sigma d_1 \setminus \Sigma d_2})).$$

Then $\Sigma d_4 \subseteq \overline{\Sigma d_1 \setminus \Sigma d_2} \subseteq \Sigma d_1$ and hence $\mathbf{Cont}(\mu(d_4), \mu(d_1))$.

(Dif-A3c): $\mu(d_1) \leq_{\mathbf{dim}} \mu(d_2) \rightarrow [\mathbf{P}(z, \mu(d_3)) \rightarrow z \cdot \mu(d_2) <_{\mathbf{dim}} z]$.

Assume $d_4 \in \mathrm{Dom}(\mathfrak{M})$ such that

$$\mu(d_4) \cdot \mu(d_2) <_{\mathbf{dim}} \mu(d_4) \text{ and } \mathbf{P}(\mu(d_4), \mu(\overline{\Sigma d_1 \setminus \Sigma d_2})).$$

Then $\Sigma d_4 \subseteq \overline{\Sigma d_1 \setminus \Sigma d_2}$ and thus $\dim(\Sigma d_4 \cap \Sigma d_2) < \dim(d_4)$ (see proof of Dif-A3(a)). Then some $d_5 \in \mathrm{Dom}(\mathfrak{M})$ exists with $\Sigma d_5 \subseteq \Sigma d_4 \cap \Sigma d_2$ and $\mu(d_5) = \mu(d_4) \cdot \mu(d_2)$ for which $\dim(d_5) = dim(\Sigma d_4 \cap \Sigma d_2) < \dim(d_4)$ holds. Thus $\mu(d_4) \cdot \mu(d_2) <_{\mathbf{dim}} \mu(d_4)$.

(Dif-A4): $(\mu(d_3)) \notin \mathbf{ZEX}_{\mathcal{M}} \leftrightarrow (\mu(d_1)) \in \mathbf{ZEX}_{\mathcal{M}} \vee \mathbf{Cont}(\mu(d_1), \mu(d_2))$.

$(\mu(d_1)) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\neg\mathbf{Cont}(\mu(d_1), \mu(d_2))$ follow by conditions (2) and (4) from our assumption. By Dif-A1 we also have $\mu(d_3) =_{\mathbf{dim}} \mu(d_1)$ and thus $(\mu(d_3)) \notin \mathbf{ZEX}_{\mathcal{M}}$.

Hence for arbitrary $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$ with $d_1 \neq \emptyset$, $\dim(\Sigma d_1 \setminus \Sigma d_2) = \dim(d_1)$, and $\dim(\Sigma d_1 \cap \Sigma d_2) = \dim(d_1)$, $d_3 = \overline{\Sigma d_1 \setminus \Sigma d_2}$ satisfies all axioms for $\mu(d_3) = \mu(d_1) - \mu(d_2)$.

The cases (II.a) to (II.d) are clearly exhaustive, thus there always exists an entity $d_3 \in \mathrm{Dom}(\mathfrak{M})$ with $\mu(d_3) = \mu(d_1) - \mu(d_2)$ that satisfies the axioms Dif-A1−Dif-A4.

This completes the satisfiability proof of $CODI_{\downarrow}$. □

Importantly, sharing a point between two manifolds d_1 and d_2 in $\mathfrak{M}$ then really means their representations $\mu(d_1)$ and $\mu(d_2)$ in $\mathcal{M}$ are in contact.

Notice that the Theorem 7.4(3) may "compact" the dimensions: if a complex 4-manifold $\mathfrak{M}_1$ only contains 1-manifolds, 2-manifolds, and 4-manifolds its representation as a model of $CODI_{\downarrow}$ may be identical to that of some complex 3-manifold. Moreover, other properties of the intended structures, such as that any two atomic m-manifolds within a composite m-manifold may only intersect in their boundaries, cannot yet be ensured in $CODI_{\downarrow}$ simply because we cannot define this relation in $CODI_{\downarrow}$, compare our discussion in Section 7.5. In Chapter 9 we will address this shortcoming. Not being able to define boundary contact does not necessarily mean that $CODI_{\downarrow}$ is missing axioms that prevent axiomatizability.

Axiomatizability could only be proved for the theory $CODI_{\downarrow} \cup$ ME-E1. It requires that every model of $CODI_{\downarrow}$ corresponds to some structure $\mathfrak{M}^m$. In the axiomatizability proof we can rely on the mappings as defined in the conditions (1)–(4) of the Satisfiability Theorem. While we can easily verify the conditions (1), (2), and (4) of Definition 5.11 from any model of $CODI_{\downarrow}$, it is much more difficult to show that for any model $\mathcal{M}$, for any $d \in \mathbf{M}$ the corresponding entity $\varrho(d) \in \mathrm{Dom}(\mathfrak{M})$ is indeed a manifold and that the conditions (1) and (2) of Definition 5.6 are satisfied. Also condition (3) of Definition 5.11 cannot be proved from the theory $CODI_{\downarrow}$—a shortcoming that we will address in Chapter 9 by introducing the relation *BCont* that adequately captures the nonzero intersection that is contained in $\Delta\mathrm{MF}_2$ when $\Sigma\mathrm{MF}_1 \cap \Delta\mathrm{MF}_2 \neq \emptyset$.

The following model of $CODI_{\downarrow}$ and its attempted mapping to a structure in $\mathbb{M}$ will demonstrate this, before we present a different mapping to overcome this problem.

Let $\mathcal{M}$ be a model of $CODI_{\downarrow}$ with domain

$$\mathbf{M} = \{a_1, a_2, a_3, a_4, l_{12}, l_{13}, l_{14}, p, \text{ze}\} \text{ and}$$
$$\mathbf{ZEX}_{\mathcal{M}} = \{\text{ze}\}$$
$$\mathbf{Min}_{\mathcal{M}} = \mathbf{Max}_{\mathcal{M}} = mathbfM \setminus \mathbf{ZEX}_{\mathcal{M}}.$$

Let further

$$\mathbf{MaxDim}_{\mathcal{M}} = \{a_1, a_2, a_3, a_4\}$$
$$\mathbf{MinDim}_{\mathcal{M}} = \{p\}$$
$$\langle p, l_{12}\rangle, \langle p, l_{13}\rangle, \langle p, l_{14}\rangle \in \prec_{\mathbf{dim}},$$

the latter completely defining $\prec_{\mathbf{dim}}$ in the presence of the extensions $\mathbf{MinDim}_{\mathcal{M}}$ and $\mathbf{MaxDim}_{\mathcal{M}}$. We define

$$\mathbf{Cont}_{\mathcal{M}} = \{\langle l_{12}, a_1\rangle, \langle l_{13}, a_1\rangle, \langle l_{14}, a_1\rangle, \langle l_{12}, a_2\rangle, \langle l_{13}, a_3\rangle, \langle l_{14}, a_4\rangle, \} \cup \{\langle p, d\rangle \mid d \in \mathbf{M} \setminus \mathbf{ZEX}_{\mathcal{M}}\}.$$

It is easily verified that $\mathcal{M}$ is indeed a model of $CODI_{\downarrow}$.

Intuitively, because there are three different dimensions in this model, we would attempt to construct a structure $\mathfrak{M}^2$ that corresponds to this model. In such a structure the atomic 2-manifold $\varrho(a_1)$ intersects three other atomic 2-manifolds (areas) $\varrho(a_2)$, $\varrho(a_3)$, and $\varrho(a_4)$ each in atomic 1-manifolds (line segments), namely $\varrho(l_{12})$, $\varrho(l_{13})$, and $\varrho(l_{14})$ which all intersect in a common 0-manifold (a point) $\varrho(p)$. Trying to imagine such a spatial configuration suffices to understand that it is only possible if one of $\varrho(l_{12})$, $\varrho(l_{13})$, and $\varrho(l_{14})$ is not entirely in the boundary of $\varrho(a_1)$ (because no more than two nonoverlapping line segments that are part of the boundary of a 2-manifold can meet in a single point), which violates condition (1) or (2) of Definition 5.6. Hence the so-constructed structure $\mathfrak{M}^3$ is not in $\mathbb{M}$.

However, we can find a structure $\mathfrak{M}^3$ in $\mathbb{M}$ that corresponds to $\mathcal{M}$. We construct it as previously, with the exception that $\varrho(a_1)$ to $\varrho(a_4)$ are now 3-manifolds (volumes), which intersect in line segments, not in surfaces. Then, all of $\varrho(l_{12})$, $\varrho(l_{13})$, and $\varrho(l_{14})$ can be in the boundary of $\varrho(a_1)$ at the same time, thus satisfying Definition 5.6(1) and (2).

For this particular problem, we found a corresponding structure $\mathfrak{M}$ in the class of intended structures $\mathbb{M}$. However, it is far from clear how this can be generalized in order to prove axiomatizability for the finite models of $CODI_{\downarrow}$. Additionally, we still need to prove that we can always find a structure $\mathfrak{M}$ in which all the singleton sets {MF} in $\mathfrak{M}$ can be realized as manifolds without violating any of the other axioms of $CODI_{\downarrow}$. We leave this as an open question (**Question 2**).

7.3 Sums

The previous two sections have focused on downwards mereological closures. Similarly, we can close models of the multidimensional mereotopology upwards by introducing sums. Postulating the existence of sums of arbitrary entities may again lead to entities of nonuniform dimension. For example, the sum consisting of a region, a line, and a set of points in which the line and the points are not contained in the region is not of uniform dimension and as such cannot be an entity in a model of our multidimensional

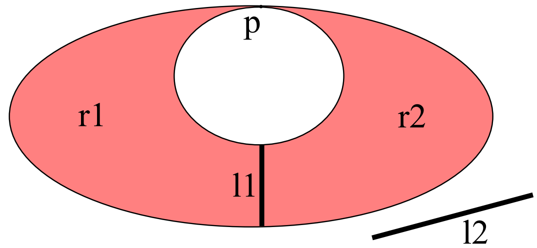


Figure 7.6: A nontrivial model of $CODI_{\downarrow} \cup \{\text{Sum-A1} - \text{Sum-A4}\}$ with 8 entities including a zero entity. We have three areal features $r1$, $r2$ and $r3 = r1 + r2$, three linear features $l1$, $l2$, and $l3 = l1 + l2$, and a single point p. In this model, we have all three kinds of contact: $\mathbf{Inc}(r3, l3)$ (but $\neg\mathbf{Cont}(l3, r3)$), $\mathbf{SC}(r1, r2)$, and, e.g., $\mathbf{PO}(r1, r3)$. See codi/consistency/tptp/codi_down_sum_nontrivial.tptp.out for details of the model, in particular the extensions of the intersection, difference, and sum operations.

theory. But again, we can treat sums in the way we treated intersections and differences: sums are defined as expected for entities of identical dimensions, while sums of entities of different dimensions are defined as the sum of the entities of highest dimension among them (Sum-A2). Then, if $x \neq_{\text{dim}} y$, not everything contained in x or in y is automatically contained in $x + y$. In other words, the sum loses everything that is not contained in an entity of its dimension. However, if $x \leq_{\text{dim}} y$ then everything contained in y must also be contained in $x + y$ (Sum-A3). Our axiomatization of closure under sums requires models to be closed under intersections to work as expected.

(Sum-A1) $x + y = y + x$ (+ commutative)

(Sum-A2) $x <_{\text{dim}} y \rightarrow x + y = y$ ($x + y = y$ if x is of higher dimension than y)

(Sum-A3) $x \leq_{\text{dim}} y \wedge Cont(z, y) \rightarrow Cont(z, x + y)$

(+ monotone under containment for entities of the highest dimension)

(Sum-A4) $Cont(z, x + y) \wedge \neg Cont(z, x) \rightarrow Cont(z - x, y)$ (everything contained in the sum $x + y$ is either contained in y or has a nonzero difference $z - x$ contained in y)

Axiom Set 7.10: Axioms Sum-A1 – Sum-A4 of $CODI_{\updownarrow}$.

Figure 7.6 gives a nontrivial model of $CODI_{\downarrow}$ extended by Sum-A1 – Sum-A4. In the extended theory we can prove various properties (Sum-T1 – Sum-T6) which allow us to verify that the sum operation—as previously shown for the intersection and difference operations—is also a total function (Theorem 7.5). The properties not directly necessary for the proof of Theorem 7.5 will help us prove associativity of sums (Sum-T7).

(Sum-T1) $Cont(z, x + y) \rightarrow \exists v[P(v, z) \wedge [Cont(v, x) \vee Cont(v, y)]]$

(every z contained in $x + y$ has some part contained in x or in y)

Lemma 7.22. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}$\models$ Sum-T1*

Proof. Assume $\mathbf{Cont}(z, x + y)$.
Suppose $\mathbf{Cont}(z, x)$ then $v := z$ is a part of z and contained in y. Equally if $\mathbf{Cont}(z, y)$.

Now suppose for all v with $\mathbf{P}(v, z)$ we had $\neg\mathbf{Cont}(v, x) \wedge \neg\mathbf{Cont}(v, y)$.

Then $x{=_{\mathbf{dim}}}y$ (otherwise it would violate Sum-A2). Moreover, by Sum-A4, we have $\mathbf{Cont}(v-x,y)$. This implies $(v-x) \notin \mathbf{ZEX}_{\mathcal{M}}$ and thus $v-x{=_{\mathbf{dim}}}v{=_{\mathbf{dim}}}z$ (by Dif-A1). Hence by Dif-A3c, some part of v, call it v_P, must exist whose intersection with x is of a lower dimension than v and which is thereby also a part of $v-x$ (by Dif-A3a). Since $\mathbf{Cont}(v-x,y)$ we would have $\mathbf{Cont}(v_P,y)$—a contradiction to our supposition that all v with $\mathbf{P}(v,z)$ are neither contained in x nor in y.

Hence the supposition was false and the consequence of Sum-T1 must be true. □

(Sum-T2) $(x+x)=x$ (+ idempotent)

Lemma 7.23. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}$\models$ Sum-T2*

Proof. From Sum-A3 we obtain $\mathbf{Cont}(x,x+x)$.
Now suppose $\exists y\ [Cont(y,x+x) \wedge \neg\mathbf{Cont}(y,x)]$.

Then by Sum-A4 we must have $\mathbf{Cont}(y-x,x)$, i.e., every part of y not contained in x must still be in x—a contradiction in itself. Hence our supposition was false, and by antisymmetry of *Cont* (C-A2) we obtain $(x+x)=x$. □

(Sum-T3) $ZEX(x+y) \leftrightarrow ZEX(x) \wedge ZEX(y)$ (zero sum of zero entities)

Lemma 7.24. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}$\models$ Sum-T3*

Proof. Assume $x,y \in \mathbf{M}$ with $(x+y) \in \mathbf{ZEX}_{\mathcal{M}}$ but $x \notin \mathbf{ZEX}_{\mathcal{M}}$.
Suppose $y \notin \mathbf{ZEX}_{\mathcal{M}}$, then $\mathbf{Cont}(x,x)$ and $\mathbf{Cont}(y,y)$. Moreover, we have $x(\leq_{\mathbf{dim}})_{\mathcal{M}}y$ or $y(\leq_{\mathbf{dim}})_{\mathcal{M}}x$. Then by Sum-A3, we have either $\mathbf{Cont}(x,x+y)$ or $\mathbf{Cont}(y,x+y)$, which contradicts C-A4.
Now suppose $y \in \mathbf{ZEX}_{\mathcal{M}}$. Then $y(\leq_{\mathbf{dim}})_{\mathcal{M}}x$ and with $\mathbf{Cont}(x,x)$ we again derive $\mathbf{Cont}(x,x+y)$ by Sum-A3, which contradicts C-A4. □

(Sum-T4) $ZEX(y) \rightarrow x+y=x$ (sum with zero entity)

Lemma 7.25. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}$\models$ Sum-T4*

(Sum-T5) $x =_{\dim} y \rightarrow x =_{\dim} x+y$ (dimension of the sum of equidimensional entities)

Lemma 7.26. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}$\models$ Sum-T5*

Proof. Assume $x{=_{\mathbf{dim}}}y$.
We consider two cases.

Case (I): Assume $x \in \mathbf{ZEX}_{\mathcal{M}}$.
Then $y \in \mathbf{ZEX}_{\mathcal{M}}$ and, by Sum-T3, $(x+y) \in \mathbf{ZEX}_{\mathcal{M}}$ and thus $x+y{=_{\mathbf{dim}}}x$.

Case (II): Assume $x \notin \mathbf{ZEX}_{\mathcal{M}}$.
Then we have, by Sum-A3, $\mathbf{Cont}(x,x+y)$ and thus $x{\leq_{\mathbf{dim}}}x+y$. By Sum-T1, if we had $x+y{>_{\mathbf{dim}}}x,y$, then some part v of $x+y$ must be contained in x or y which contradicts CD-A1: $Cont(v,x) \rightarrow v \leq_{\dim} x$. Thus $x{=_{\mathbf{dim}}}x+y$.

This two cases exhaustively proof Sum-T5. □

Next we will essentially show that in the equidimensional case, our definition of sums is the definition of sums as usually found in equidimensional mereotopologies:

(Sum-T6) $x =_{\text{dim}} y \rightarrow \forall z[PO(z,x) \lor PO(z,y) \leftrightarrow PO(z,x+y)]$

($x+y$ partially overlaps anything that overlaps either x or y)

Lemma 7.27. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T6*

Proof. We prove the two directions of the inner biconditional separately.

Direction (a): $x =_{\text{dim}} y \land (PO(z,x) \lor PO(z,y)) \rightarrow PO(z,x+y)$.

Assume $x =_{\mathbf{dim}} y$ and—without loss of generality—$\mathbf{PO}(z,x)$.

Then by Sum-A3, we immediately obtain $\mathbf{PO}(z,x+y)$.

Direction (b): $x =_{\text{dim}} y \land PO(z,x+y) \rightarrow PO(y,x) \land PO(z,x) \lor PO(z,y)$.

Assume $x =_{\mathbf{dim}} y$ and $\mathbf{PO}(z,x+y)$ and $\neg\mathbf{PO}(z,x)$.

By $\mathbf{PO}(z,x+y)$ there must be some part v that z and $x+y$ have in common. By Sum-T1, some part of v, call it v_P, must be contained in either x or y. Because $\neg\mathbf{PO}(z,x)$, this part cannot be in x and hence must be in y. But then $\mathbf{PO}(v,y)$ and consequently $\mathbf{PO}(z,y)$.

The two directions immediately imply the biconditional in Sum-T6. □

Now we can easily show that the sum operation as axiomatized in $CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} is a total function.

Theorem 7.5. *The operation + is a total function in $CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}.*

Proof. Let x and y be arbitrary entities. If $x \neq_{\text{dim}} y$, i.e., they are of differing dimension, their sum $x+y$ is uniquely defined by Sum-A2 (note that by Sum-A1 the sum operation is commutative). Now assume $x =_{\mathbf{dim}} y$, then we can apply Sum-T6 which together with extensionality of PO (PO-E1, compare Theorem 7.11) uniquely defines the sum $x+y$. □

From Sum-T6 we can also prove associativity of + (Sum-T7).

(Sum-T7) $(x+y)+z = x+(y+z)$ (+ associative)

Lemma 7.28. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T7*

Proof. For the proof we distinguish several cases depending on the relative dimensions of x, y, and z. The primary distinction lies with the relative dimension between x and y, that is, whether $x <_{\mathbf{dim}} y$, $x >_{\mathbf{dim}} y$, or $x =_{\mathbf{dim}} y$. We use a secondary distinction based on the relative dimension between x and z as necessary. All cases are straightforward applications of Sum-A2 except for the case $x =_{\mathbf{dim}} y =_{\mathbf{dim}} z$ with all of x, y, z being of nonzero dimension, we employ Sum-T6 in this case.

Case (I): Assume $x <_{\mathbf{dim}} y$. Then

$$
\begin{aligned}
(x+y)+z &= y+z && \text{(Sum-A2: } x <_{\mathbf{dim}} z) \\
&= x+(y+z) && \text{(Sum-A2: } x <_{\mathbf{dim}} y \leq_{\mathbf{dim}} y+z)
\end{aligned}
$$

Case (II): Assume $x >_{\mathbf{dim}} y$.

Subcase (II.a): Assume $x >_{\mathbf{dim}} y$ and $x >_{\mathbf{dim}} z$. Then

$$\begin{aligned}(x+y)+z &= x+z && \text{(Sum-A2: } x >_{\mathbf{dim}} y)\\ &= x && \text{(Sum-A2: } x >_{\mathbf{dim}} z)\\ &= x+(y+z) && \text{(Sum-A2: } x >_{\mathbf{dim}} y+z)\end{aligned}$$

Subcase (II.b): Assume $x >_{\mathbf{dim}} y$ and $x \leq_{\mathbf{dim}} z$. Then $z >_{\mathbf{dim}} y$ and

$$\begin{aligned}(x+y)+z &= x+z && \text{(Sum-A2: } x >_{\mathbf{dim}} y)\\ &= x+(y+z) && \text{(Sum-A2: } z >_{\mathbf{dim}} y)\end{aligned}$$

Case (III): Assume $x =_{\mathbf{dim}} y$.

Subcase (III.a): Assume $x =_{\mathbf{dim}} y$ and $x >_{\mathbf{dim}} z$. Then $y >_{\mathbf{dim}} z$ and

$$\begin{aligned}(x+y)+z &= x+y && \text{(Sum-A2: } x+y >_{\mathbf{dim}} z)\\ &= x+(y+z) && \text{(Sum-A2: } y >_{\mathbf{dim}} z)\end{aligned}$$

Subcase (III.b): Assume $x =_{\mathbf{dim}} y$ and $x <_{\mathbf{dim}} z$. Then $y <_{\mathbf{dim}} z$ and

$$\begin{aligned}(x+y)+z &= z && \text{(Sum-A2: } x+y <_{\mathbf{dim}} z)\\ &= x+z && \text{(Sum-A2: } x <_{\mathbf{dim}} z)\\ &= x+(y+z) && \text{(Sum-A2: } y <_{\mathbf{dim}} z)\end{aligned}$$

Subcase (III.c): Assume $x =_{\mathbf{dim}} y =_{\mathbf{dim}} z$ and $x \in \mathbf{ZEX}_{\mathcal{M}}$.
Then $y \in \mathbf{ZEX}_{\mathcal{M}}$, $z \in \mathbf{ZEX}_{\mathcal{M}}$ and thus $((x+y)+z) \in \mathbf{ZEX}_{\mathcal{M}}$ and $(x+(y+z)) \in \mathbf{ZEX}_{\mathcal{M}}$ by Sum-T3, so $(x+y)+z = x+(y+z)$.

Subcase (III.d): Assume $x =_{\mathbf{dim}} y =_{\mathbf{dim}} z$ and $x \notin \mathbf{ZEX}_{\mathcal{M}}$.
Then also $y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $z \notin \mathbf{ZEX}_{\mathcal{M}}$. We can use extensionality of *PO* (PO-E1) to prove

$$\forall s[\mathbf{PO}(s,(x+y)+z) \leftrightarrow \mathbf{PO}(s,x+(y+z))]$$

Consider the following equivalences:

$$\begin{aligned}& \mathbf{PO}(s,(x+y)+z) &&\\ \leftrightarrow\ & \mathbf{PO}(s,x+y) \vee \mathbf{PO}(s,z) && \text{(Sum-T6)}\\ \leftrightarrow\ & \mathbf{PO}(s,x) \vee \mathbf{PO}(s,y) \vee \mathbf{PO}(s,z) && \text{(Sum-T6)}\\ \leftrightarrow\ & \mathbf{PO}(s,x) \vee \mathbf{PO}(s,y+z) && \text{(Sum-T6)}\\ \leftrightarrow\ & \mathbf{PO}(s,x+(y+z)) && \text{(Sum-T6)}\end{aligned}$$

These cases cover all possible relative dimension constraints among x, y, and z, hence for all x, y, z we have $(x+y)+z = x+(y+z)$. □

7.3.1 Containment in sums

We now proceed to prove that every part in the sum $x+y$ can be split into parts u and v so that $P(u,x)$ and $P(v,y)$ but not $PO(u,v)$ (Sum-T9) and, second, that the sum of two parts of z is also a part of z (Sum-T10). In other words, the sum $x+y$ does not contain a part that is neither in x, y, nor can be divided into parts in x and y. To prove Sum-T9, we first prove Sum-T8—an important relationship between the sum and difference for equidimensional entities—which complements the property Dif-T8.

(Sum-T8) $P(y,x) \to x = y + (x-y)$ (any part y of x and the difference $x-y$ cover x)

Lemma 7.29. *$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4} $\vDash$ Sum-T8*

Proof. We can utilize extensionality of PO (PO-E1) in $CODI_\downarrow$ and show the following two sentences, which together imply Sum-T8:

$$P(y,x) \wedge PO(z,x) \to PO(z, y+(x-y)) \tag{a}$$

$$P(y,x) \wedge PO(z, y+(x-y)) \to PO(z,x) \tag{b}$$

Part (a): $P(y,x) \wedge PO(z,x) \to PO(z,y) \vee PO(z,x-y)$.

By Sum-T6 we can rewrite (a) as

$$P(y,x) \wedge PO(z,x) \to PO(z,y) \vee PO(z,x-y)$$

which is equivalent to

$$P(y,x) \wedge PO(z,x) \wedge \neg PO(z,x-y) \to PO(z,y) \tag{a*}$$

To prove (a*), assume $\mathbf{P}(y,x)$, $\mathbf{PO}(z,x)$, and $\neg\mathbf{PO}(z,x-y)$.

Then there exists a v that is a part of x and z and thus part of $x \cdot z$ (note that we always have $x \cdot z \leq_{\mathbf{dim}} x, z$). Now we use the contrapositive of one of the directions of the implication in Dif-T5, namely

$$P(v,x) \wedge \neg P(v,x-y) \to PO(v,x \cdot y)$$

From our assumption $\neg\mathbf{PO}(z,x-y)$ we conclude $\neg\mathbf{P}(v,x-y)$. Together with $\mathbf{P}(v,x)$, it implies $\mathbf{PO}(v,x\cdot y)$. Then we must have $\mathbf{PO}(v,y)$ (recall that by $\mathbf{P}(y,x)$ we have $y =_{\mathbf{dim}} x$), which with $\mathbf{P}(v,z)$ results in $\mathbf{PO}(z,y)$, the consequent of (a*).

Part (b): $P(y,x) \wedge PO(z,y+(x-y)) \to PO(z,x)$.

We prove instead (b*), from which (b) follows by Sum-T6

$$P(y,x) \wedge [PO(z,y) \vee PO(z,x-y)] \to PO(z,x) \tag{b*}$$

We will consider the two cases resulting from the disjunction in (b*) individually

$$P(y,x) \wedge PO(z,y) \to PO(z,x) \tag{b*.1}$$

$$P(y,x) \wedge PO(z, x-y) \rightarrow PO(z,x) \tag{b*.2}$$

(b*.1) Assume $\mathbf{P}(y,x)$ and $\mathbf{PO}(z,y)$.
Then by transitivity of parthood: if a v exists that is part of y and z, then v is also part of x and thus $\mathbf{PO}(z,x)$.

(b*.2) Assume $\mathbf{P}(y,x)$ and $\mathbf{PO}(z,x-y)$.
$\mathbf{P}(y,x)$ implies $\mathbf{P}(x-y,x)$ by Dif-T1; hence $\mathbf{PO}(z,x-y)$ implies, again by transitivity of parthood, $\mathbf{PO}(z,x)$.

Hence (b*) holds.

The two part, (a) and (b), together imply that x and $(x \cdot y) + (x-y)$ are equivalent. □

An important case of Sum-T8 concerns the intersection $x \cdot y$ for which we can prove $x = (x \cdot y) + (x-y)$ even if $x \cdot y$ is not a part of x (note that it is always contained in x):

(Sum-T8') $x = (x \cdot y) + (x-y)$ ($x \cdot y$ and $x-y$ cover x)

Corollary 7.3. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T8'*

Proof. Because $x \cdot y \leq_{\text{dim}} x$ always holds, it suffices to consider the following two cases.

Case (I): Assume $x \cdot y <_{\mathbf{dim}} x$.
Then

$$\begin{aligned} x \cdot y + (x-y) &= x - y && \text{(Sum-A2: } x \cdot y <_{\mathbf{x}}) \\ &= x - (x \cdot y) && \text{(Dif-T6)} \\ &= x && \text{(Dif-A2: } x \cdot y <_{\mathbf{x}}) \end{aligned}$$

Case (II): Assume $x \cdot y =_{\mathbf{dim}} x$.
Then $\mathbf{P}(x \cdot y, x)$, which implies

$$\begin{aligned} x &= (x \cdot y) + (x - (x \cdot y)) && \text{(Sum-T8)} \\ &= (x \cdot y) + (x - y) && \text{(Dif-T6)} \end{aligned}$$

In either case we have $w = (w \cdot x) + (w - x)$. □

Sum-T8' essentially proves the first property of the u, v that Sum-T9 claims to exist when a part z is not contained in x or y but is contained in their sum $x + y$.

(Sum-T9) $x =_{\text{dim}} y \wedge Cont(w, x+y) \wedge \neg Cont(w,x) \wedge \neg Cont(w,y)$
$\rightarrow \exists u, v[w = u + v \wedge \neg PO(u,v) \wedge P(u,w) \wedge Cont(u,x) \wedge P(v,w) \wedge Cont(v,y)]$
(everything contained in $x+y$ but contained neither in x nor in y is the sum of two nonoverlapping parts contained in x and y)

Lemma 7.30. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T9*

Proof. Assume $x =_{\mathbf{dim}} y$, $\mathbf{Cont}(w, x+y)$, $\neg\mathbf{Cont}(w, x)$, and $\neg\mathbf{Cont}(w, y)$.
The assumption further implies $x =_{\mathbf{dim}} y \geq_{\mathbf{dim}} z$ and we have $x \notin \mathbf{ZEX}_{\mathcal{M}}$ and $y \notin \mathbf{ZEX}_{\mathcal{M}}$ by Cont-A4.

We want to show that the consequent of Sum-T9 is satisfied for $u := w \cdot x$ and $v := w - x$. In order to do so we must show (a) to (f):

(a) $w = (w \cdot x) + (w - x)$

(b) $\neg\mathbf{PO}(w \cdot x, w - x)$

(c) $\mathbf{P}(w \cdot x, w)$

(d) $\mathbf{Cont}(w \cdot x, x)$

(e) $\mathbf{P}(w - x, w)$

(f) $\mathbf{Cont}(w - x, y)$

Part (a): $w = (w \cdot x) + (w - x)$.
By Sum-T8'.

Part (b): $\neg\mathbf{PO}(w \cdot x, w - x)$.
By Dif-T4.

Part (c): $\mathbf{P}(w \cdot x, w)$.
By Sum-A4, we can use $\mathbf{Cont}(w, x+y)$ together with $\neg\mathbf{Cont}(w, y)$ to conclude $\mathbf{Cont}(w - y, x)$. Hence $w - y$ is nonempty (by C-A4) and further $w - y =_{\mathbf{dim}} w$ by Dif-A1. Since $\mathbf{P}(w - y, w)$ (by Dif-T1), $w - y$ must be a part of the intersection $(w - y) \cdot x$ and thus also of the intersection $w \cdot x$ (which cannot be of larger dimension than w, i.e., of larger dimension than $w - y$) by Int-A4.

Part (d): $\mathbf{Cont}(w \cdot x, x)$.
By Int-A2 because $(w \cdot x) \notin \mathbf{ZEX}_{\mathcal{M}}$ by Part (c).

Part (e): $\mathbf{P}(w - x, w)$.
By Sum-A4, we can use $\mathbf{Cont}(w, x+y)$ together with $\neg\mathbf{Cont}(w, x)$ to conclude $\mathbf{Cont}(w - x, y)$. Hence $(w - x) \notin \mathbf{ZEX}_{\mathcal{M}}$ and then $\mathbf{P}(w - x, w)$ by Dif-T1.

Part (f): $\mathbf{Cont}(w - x, y)$.
Follows by Sum-A4 from $\mathbf{Cont}(w, x+y)$ together with $\neg\mathbf{Cont}(w, x)$.

We have shown that some u and v exists, namely $u := w \cdot x$ and $v := w - x$, that satisfy all conditions of the consequent of Sum-T9. □

We actually proved something stronger than Sum-T9, namely that the entities $w \cdot x$ and $w - x$ split any w not contained in x or y into entities that are contained in x and y, respectively. This gives us the following stronger property:

Corollary 7.4. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}* $\models$

$$\begin{aligned} Cont(w, x+y) \quad \rightarrow \quad & Cont(w, x) \vee Cont(w, y) \quad \vee \\ & [w = (w \cdot x) + (w - x) \wedge P(w \cdot x, w) \wedge P(w - x, w) \wedge \\ & \neg PO(w \cdot x, w - x) \wedge Cont(w \cdot x, x) \wedge Cont(w - x, y)]\,. \end{aligned}$$

This guarantees that everything contained in the sum $x + y$ is either entirely in x or in y, or can be split into $w \cdot x$ and $w - x$ which are nonoverlapping parts of w that cover w and are contained in x and y, respectively.

Sum-T9 will also be helpful for the next theorem (Sum-T10), which shows that the sum of two parts of z is also a part of z and vice versa.

(Sum-T10) $x =_{\dim} y \wedge \neg ZEX(x) \rightarrow \forall z[Cont(x, z) \wedge Cont(y, z) \leftrightarrow Cont(x + y, z)]$

(the sum $x + y$ is part of z iff x and y are part of z)

Lemma 7.31. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T10*

Proof. We prove the two directions of the inner biconditional separately.

Direction (a): $x =_{\dim} y \wedge \neg ZEX(x) \wedge \neg ZEX(y) \rightarrow \forall z(Cont(x, z) \wedge Cont(y, z) \leftarrow Cont(x + y, z))$.
Assume $x \mathbf{=_{dim}} y$, $x \notin \mathbf{ZEX}_{\mathcal{M}}$, and $\mathbf{Cont}(x + y, z)$.
Then $x \mathbf{=_{dim}} y \mathbf{=_{dim}} x + y$ (by Sum-T5) implies $\mathbf{P}(x, x + y)$, which together with $\mathbf{Cont}(x + y, z)$ implies $\mathbf{Cont}(x, z)$ by transitivity of parthood (because $x \notin \mathbf{ZEX}_{\mathcal{M}}$). The same argument works for y to conclude $\mathbf{Cont}(y, z)$ (because of $x \mathbf{=_{dim}} y$ and $x \notin \mathbf{ZEX}_{\mathcal{M}}$ we also have $y \notin \mathbf{ZEX}_{\mathcal{M}}$).

Direction (b): $x =_{\dim} y \wedge \neg ZEX(x) \wedge \neg ZEX(y) \rightarrow \forall z(Cont(x, z) \wedge Cont(y, z) \rightarrow Cont(x + y, z))$.
Assume $x \mathbf{=_{dim}} y$, $x \notin \mathbf{ZEX}_{\mathcal{M}}$, and $\mathbf{Cont}(x, z)$, $\mathbf{Cont}(y, z)$.
Suppose $\neg\mathbf{Cont}(x + y, z)$ then $\exists w[\mathbf{P}(w, x + y) \wedge \neg\mathbf{Cont}(w, z)]$. By Sum-T9, we can split w into the parts $w \cdot x$ and $w - x$ which are parts of x, y, respectively. In particular, the greatest part of $x + y$, namely $x + y$ itself, can be split in that way: $(x + y) \cdot x$ and $(x + y) - x$; the former clearly being x and the latter being y (it will have exactly all parts of y), each of which is contained in z. Hence, no part of $x + y$ can be not contained in z and our supposition is false, and the consequent of Sum-T10 must be true.

These two directions immediately entail the biconditional in Sum-T10. □

7.3.2 Interaction between sums and intersections

Next, we will look at properties of the sum operation $+$ in interaction with the other two mereological closure operations, intersection $\cdot$ and difference $-$. First we study the interaction with intersections. Since the intersection and sum operations are commutative, we do not have to distinguish left- from right-distributivity in our study.

Contrary to the set-theoretic operations, our intersection operation is not distributive over the sum operation. In other words, we do not always have $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$, see Figure 7.7(a) for an example. However, we do have distributivity when y and z are of the same dimension (Sum-T11), but not if $y \neq_{\dim} z$ but x has the same dimension as either y or z, compare Figure 7.7(b). Note that if $x \mathbf{<_{dim}} y$ even $x \cdot (x + y) = x$ (the consequent of Sum-T12) may fail, compare Figure 7.7(c).

(Sum-T11) $y =_{\dim} z \rightarrow x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

($\cdot$ distributive over $y + z$ when y and z are of equal dimension)

Lemma 7.32. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T11*

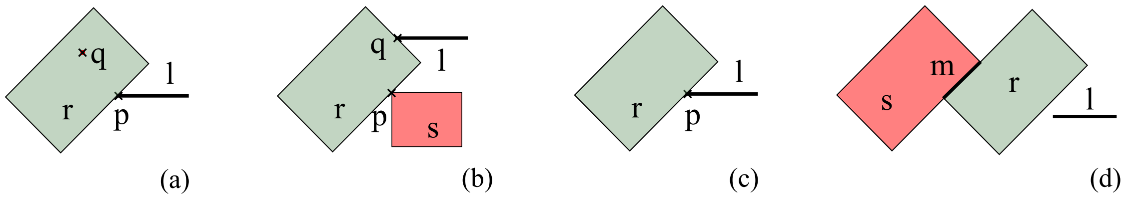


Figure 7.7: Non-distributivity of intersection over sums. Example (a) shows that intersections do not always distribute over sums; we have $r \cdot (l+q) = r \cdot l = p \neq p+q = (r \cdot l)+(r \cdot q)$. (b) gives an example with two entities of equal dimension still not being distributive because the entities inside the sum on the left side are not of equal dimension: $r \cdot (s+l) = r \cdot s = p \neq p+q = (r \cdot s)+(r \cdot l)$. In (c) the intersection does not absorb the sum, i.e., we have $l \cdot (l+r) = l \cdot r = p \neq l$. Example (d) shows that sums do not always distribute over intersections: $l+(r \cdot s) = l+m \neq m = r \cdot s = (l+r) \cdot (l+s)$, even though r and s are of equal dimension.

Proof. We consider three cases, (I)–(III), based on the relative dimension of $x \cdot y$ and $x \cdot z$. For each case we show the following two directions

$$\forall w \, [P(w, x \cdot (y+z)) \rightarrow P(w, (x \cdot y)+(x \cdot z))] \tag{a}$$

$$\forall w \, [P(w, (x \cdot y)+(x \cdot z)) \rightarrow P(w, x \cdot (y+z))] \tag{b}$$

which, by EP-T2 amount to $x \cdot (y+z) = (x \cdot y)+(x \cdot z)$.

For all cases, assume $y =_{\mathbf{dim}} z$ (the antecedent of Sum-T11).

Case (I): Assume $x \cdot y =_{\mathbf{dim}} x \cdot z$.

Direction (I.a):

By Sum-T5 we also have $x \cdot y =_{\mathbf{dim}} x \cdot z =_{\mathbf{dim}} x \cdot (y+z)$.

Let w be an arbitrary part of $x \cdot (y+z)$. Then $\mathbf{Cont}(w,x)$ and $\mathbf{Cont}(w,y+z)$ by Int-A4. If we have $\mathbf{Cont}(w,x)$ or $\mathbf{Cont}(w,y)$ we can immediately conclude $\mathbf{P}(w,(x \cdot y)+(x \cdot z))$. Now suppose neither $\mathbf{Cont}(w,x)$ nor $\mathbf{Cont}(w,y)$, then by Corollary 7.4 we must have

$$\begin{aligned}
&\mathbf{Cont}(w,x) \wedge \mathbf{Cont}(w \cdot y, y) \wedge \mathbf{Cont}(w-y, z) \wedge w = (w \cdot y)+(w-y) \\
\Rightarrow &\mathbf{Cont}(w \cdot y, x \cdot y) \wedge \mathbf{Cont}(w-y, x \cdot z) \wedge w = (w \cdot y)+(w-y) \\
\Rightarrow &\mathbf{Cont}(w \cdot y, (x \cdot y)+(x \cdot z)) \wedge \mathbf{Cont}(w-y, (x \cdot y)+(x \cdot z)) \\
&\wedge w = (w \cdot y)+(w-y) \\
\Rightarrow &\mathbf{Cont}((w \cdot y)+(w-y), (x \cdot y)+(x \cdot z)) \wedge w = (w \cdot y)+(w-y) && \text{(Sum-T10)} \\
\Rightarrow &\mathbf{Cont}(w, (x \cdot y)+(x \cdot z))
\end{aligned}$$

Direction (I.b):

Let w be an arbitrary part of $(x \cdot y)+(x \cdot z)$.

If $\mathbf{Cont}(w, x \cdot y)$ or $\mathbf{Cont}(w, x \cdot z)$ then $\mathbf{Cont}(w, x \cdot (y+z))$ because $y =_{\mathbf{dim}} z =_{\mathbf{dim}} y+z$. Now suppose neither $\mathbf{Cont}(w, x \cdot y)$ nor $\mathbf{Cont}(w, x \cdot z)$, then by Corollary 7.4,

$$\begin{aligned}
&\mathbf{Cont}(w \cdot (x \cdot y), (x \cdot y)) \wedge \mathbf{Cont}(w - (x \cdot y), (x \cdot z)) \wedge \\
&w = (w \cdot (x \cdot y)) + (w - (x \cdot y)) \\
\Rightarrow &\mathbf{Cont}(w \cdot (x \cdot y), x \cdot (y + z)) \wedge \mathbf{Cont}(w - (x \cdot y), x \cdot (y + z)) \wedge \qquad (*) \\
&w = (w \cdot (x \cdot y)) + (w - (x \cdot y)) \\
\Rightarrow &\mathbf{Cont}(w, x \cdot (y + z)) \qquad \text{(Sum-T10)}
\end{aligned}$$

For the step $(*)$ we use the fact that $\mathbf{P}(y, y + z)$ (recall that $y =_{\mathbf{dim}} z$), hence if $\mathbf{Cont}(w \cdot (x \cdot y), (x \cdot y))$ we also have $\mathbf{Cont}(w \cdot (x \cdot y), (x \cdot (y + z)))$. Equally, from $\mathbf{P}(z, y + z)$ and $\mathbf{Cont}(w - (x \cdot y), (x \cdot z))$ we obtain $\mathbf{Cont}(w - (x \cdot y), (x \cdot (y + z)))$.

Case (II): Assume $x \cdot y <_{\mathbf{dim}} x \cdot z$.

Direction (II.a):

From $x \cdot y <_{\mathbf{dim}} x \cdot z$ we know by Sum-A2 that $(x \cdot y) + (x \cdot z) = x \cdot z$. Hence it suffices to show that

$$\forall w\ [\mathbf{P}(w, x \cdot (y + z)) \rightarrow \mathbf{P}(w, x \cdot z)]$$

Let w be an arbitrary part of $x \cdot (y + z)$. Then $\mathbf{Cont}(w, x)$ and $\mathbf{Cont}(w, y + z)$. Suppose $y >_{\mathbf{dim}} z$, then $y + z = y$ (by Sum-A2) and $x \cdot y \geq_{\mathbf{dim}} x \cdot z$ in contradiction to our assumption. Hence $y \leq_{\mathbf{dim}} z$ must hold.

Subcase (II.a.i): Assume $y <_{\mathbf{dim}} z$.

Then $y + z = z$ and $\mathbf{Cont}(w, y + z)$ imply $\mathbf{Cont}(w, z)$, hence $\mathbf{Cont}(w, x \cdot z)$ and with $y + z =_{\mathbf{dim}} z$ also $\mathbf{P}(w, x \cdot z)$.

Subcase (II.a.ii): Assume $y =_{\mathbf{dim}} z$.

Suppose $\neg\mathbf{Cont}(w, z)$. Then because $x \cdot y <_{\mathbf{dim}} x \cdot z$, w cannot be a part of $x \cdot (y + z)$ even if $\mathbf{Cont}(w, y)$ because the dimension of $x \cdot y$ is lower than the dimension of $x \cdot (y + z)$ which has to be at least the dimension of $x \cdot z$. However, $\neg\mathbf{P}(w, x \cdot (y + z))$ contradicts our initial assumption. Hence our supposition $\neg\mathbf{Cont}(w, z)$ was wrong and we must have $\mathbf{Cont}(w, z)$. Consequently we also have $\mathbf{P}(w, x \cdot z)$.

(Direction II.b):

Let w be an arbitrary entity so that $\mathbf{P}(w, (x \cdot y) + (x \cdot z))$, then from $x \cdot y <_{\mathbf{dim}} x \cdot z$ we have $\mathbf{P}(w, (x \cdot z))$ which directly implies $\mathbf{P}(w, (x \cdot (y + z)))$ when $y =_{\mathbf{dim}} z$.

Case (III): Assume $x \cdot y <_{\mathbf{dim}} x \cdot z$.

The proof is analogous to that of case (II).

These cases cover all possible relative dimension constraints between $x \cdot y$ and $x \cdot z$. Hence in any case, (a) and (b) hold and therefore

$$\mathbf{P}(x \cdot (y + z), (x \cdot y) + (x \cdot z)) \wedge P((x \cdot y) + (x \cdot z), x \cdot (y + z)),$$

and thus $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ if $y =_{\mathbf{dim}} z$. $\square$

(Sum-T12) $x \geq_{\text{dim}} y \rightarrow x \cdot (x + y) = x$ ($\cdot$ absorbs $+$ for $x \geq_{\text{dim}} y$)

Lemma 7.33. *$CODI_{\downarrow} \cup \{$Sum-A1 – Sum-A4$\} \vDash$ Sum-T12*

Proof. We have two cases.

Case (I): Assume $x \in \mathbf{ZEX}_{\mathcal{M}}$.
Then $\neg\mathbf{C}(x, x + y)$ by C-T4 and $(x \cdot (x + y)) \in \mathbf{ZEX}_{\mathcal{M}}$ by Int-A1. Hence $x = x \cdot (x + y)$.

Case (II): Assume $x \notin \mathbf{ZEX}_{\mathcal{M}}$.
Let us further assume $x \mathbf{\geq_{dim}} y$, the antecedent of Sum-T12.
We want to show that $\mathbf{Cont}(x \cdot (x+y), x)$ and $\mathbf{Cont}(x, x \cdot (x+y))$ which together imply $x = x \cdot (x+y)$ by antisymmetry of *Cont*.

$\mathbf{Cont}(x \cdot (x + y), x)$ trivially holds by Int-A2 since $(x + y) \notin \mathbf{ZEX}_{\mathcal{M}}$.

$\mathbf{Cont}(x, x \cdot (x + y))$ iff $\mathbf{Cont}(x, x + y)$, which holds by Sum-A3 because $x \mathbf{\geq_{dim}} y$.

The two cases are trivially exhaustive, hence in any case $x = x \cdot (x + y)$. ☐

Sum-T13 shows when the intersection operation does not absorb the sum operation.

(Sum-T13) $x <_{\text{dim}} y \rightarrow x \cdot (x + y) = x \cdot y$ (nonabsorption of $+$ by $\cdot$ when $x <_{\text{dim}} y$)

Lemma 7.34. *$CODI_{\downarrow} \cup \{$Sum-A1 – Sum-A4$\} \vDash$ Sum-T13*

Proof. We consider two cases.

Case (I): Assume $x \in \mathbf{ZEX}_{\mathcal{M}}$.
Then $\neg\mathbf{C}(x, x + y)$ by C-T4 and $(x \cdot (x + y)) \in \mathbf{ZEX}_{\mathcal{M}}$ by Int-A1. Equally, $(x \cdot y) \in \mathbf{ZEX}_{\mathcal{M}}$ since $\neg\mathbf{C}(x, y)$. Thus $x \cdot (x + y) = x \cdot y$.

Case (II): Assume $x \notin \mathbf{ZEX}_{\mathcal{M}}$.
Then $x + y = y$ (by Sum-A2 and $x \mathbf{<_{dim}} y$) and thus $x \cdot (x + y) = x \cdot y$.

The two cases are trivially exhaustive, hence $x \cdot (x + y) = x \cdot y$ always holds if $x \mathbf{<_{dim}} y$. ☐

Moreover, sums only distribute over intersections, i.e., $x + (y \cdot z) = (x + y) \cdot (x + z)$ if y and z are of the same dimension and x is either of a lower dimension than their intersection or of the same or greater dimension than both of y and z (Sum-T14). Because we always have $y \cdot z \leq_{\text{dim}} y, z$, this essentially rules out that x is in between the dimension of y, z and the dimension of $y \cdot z$. Consider Figure 7.7(d) which show that in such cases distributivity may fail.

(Sum-T14) $y =_{\text{dim}} z \wedge [x <_{\text{dim}} y \cdot z \vee x \geq_{\text{dim}} y] \rightarrow x + (y \cdot z) = (x + y) \cdot (x + z)$

(+ distributive over $y \cdot z$ when y and z are of equal dimension)

Lemma 7.35. *$CODI_{\downarrow} \cup \{$Sum-A1 – Sum-A4$\} \vDash$ Sum-T14*

Proof. Assume $y \mathbf{=_{dim}} z$ and $(x \mathbf{<_{dim}} y \cdot z$ or $x \mathbf{\geq_{dim}} y)$.
We distinguish the following three cases.

Case (I): Assume $x >_{\mathbf{dim}} y, z \geq_{\mathbf{dim}} y \cdot z$. Then

$$\begin{aligned} x + (y \cdot z) &= (x + y) \cdot (x + z) \\ x &= x \cdot x && \text{(Sum-A2: } x >_{\mathbf{dim}} y, z (y \cdot z)) \end{aligned}$$

Case (II): Assume $x =_{\mathbf{dim}} y, z \geq_{\mathbf{dim}} y \cdot z$.

Consider the following computation which applies Corollary 7.4, Sum-T8', Sum-T10, and the assumption $x =_{\mathbf{dim}} y =_{\mathbf{dim}} z$.

$$\begin{aligned} & \mathbf{P}(w, (x + y) \cdot (x + z))) \\ \Leftrightarrow\ & \mathbf{P}(w, x + y) \wedge \mathbf{P}(w, x + z)) \\ \Leftrightarrow\ & \mathbf{P}(w, x) \vee [\mathbf{P}(w, y) \wedge \mathbf{P}(w, z)] \vee \\ & [w = (w \cdot x) + (w - x) \wedge \mathbf{P}(w \cdot x, x) \wedge \mathbf{P}(w - x, y) \wedge \mathbf{P}(w - x, y)] \\ \Leftrightarrow\ & \mathbf{P}(w, x) \vee \mathbf{P}(w, y \cdot z) \vee [w = (w \cdot x) + (w - x) \wedge \mathbf{P}(w \cdot x, x) \wedge \mathbf{P}(w - x, y \cdot z)] \\ \Leftrightarrow\ & \mathbf{P}(w, x + (y \cdot z)) \end{aligned}$$

Case (III): Assume $x <_{\mathbf{dim}} y \cdot z \leq_{\mathbf{dim}} y, z$. Then

$$\begin{aligned} x + (y \cdot z) &= (x + y) \cdot (x + z) \\ y \cdot z &= y \cdot z && \text{(Sum-A2: } x <_{\mathbf{dim}} y, z (y \cdot z)) \end{aligned}$$

These three cases cover all cases where $y =_{\mathbf{dim}} z$ and $[x < y \cdot z \vee x \geq_{\mathbf{dim}} y]$. Case (I) covers the first condition of the disjunction while cases (II) and (III) cover the second condition, splitting it into $x =_{\mathbf{dim}} y$ and $x >_{\mathbf{dim}} y$. Hence, we proved

$$y =_{\mathbf{dim}} z \wedge [x <_{\mathbf{dim}} y \cdot z \vee x >_{\mathbf{dim}} y] \rightarrow x + (y \cdot z) = (x + y) \cdot (x + z).$$

□

Sum-T15 proves that the sum operation always absorbs the intersection operation.

(Sum-T15) $x + (x \cdot y) = x$ (+ absorbs $\cdot$)

Lemma 7.36. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\vDash$ Sum-T15*

Proof. We consider two cases.

Case (I): Assume $x \in \mathbf{ZEX}_{\mathcal{M}}$.

Then $\neg\mathbf{C}(x, y)$ and thus $(x, y) \in \mathbf{ZEX}_{\mathcal{M}}$ and thus $x + (x \cdot y) = x$.

Case (II): Assume $x \notin \mathbf{ZEX}_{\mathcal{M}}$.

Then by $\mathbf{Cont}(x \cdot y, x)$ we immediately obtain $x + (x \cdot y) = x$ no matter whether $x \cdot y <_{\mathbf{dim}} x$ or $x \cdot y =_{\mathbf{dim}} x$.

The two cases are trivially exhaustive; in any case $x + (x \cdot y) = x$. □

Summarily, Sum-T11 and Sum-T14 establish that for equidimensional entities x, y, and z, the intersection and sum operation are distributive over the other. While Sum-T12 and Sum-T13 show how the intersection operation absorbs a sum, Sum-T15 shows how the sum operation absorbs a product.

7.3.3 Interaction between sums and differences

Next, we study the interaction between sums and differences. While differences do not distribute left over sums even in set theory or in the equidimensional case, differences do distribute right over sums for equidimensional entities. This also works in cases when z is of a different dimension than x and y as long as x and y are of the same dimension (Sum-T16).

(Sum-T16) $x =_{\text{dim}} y \rightarrow (x + y) - z = (x - z) + (y - z)$ (− right-distributive over +)

Lemma 7.37. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T16*

Proof. We split the proof into four cases.

Case (I): Assume $x \in \mathbf{ZEX}_{\mathcal{M}}$.

Then $y \in \mathbf{ZEX}_{\mathcal{M}}$; together they imply $((x + y) - z) \in \mathbf{ZEX}_{\mathcal{M}}$ and $((x - z) + (y - z)) \in \mathbf{ZEX}_{\mathcal{M}}$.

Case (II): Assume $x \notin \mathbf{ZEX}_{\mathcal{M}}$ and $z <_{\mathbf{dim}} x$. Consider the following computation:

$$\begin{aligned}
& (x + y) - z = (x - z) + (y - z) && \\
\Leftrightarrow\ & (x + y) - z = x + y && (\text{Sum-A2: } z <_{\mathbf{dim}} x, y) \\
\Leftrightarrow\ & x + y = x + y && (\text{Sum-A2: } z <_{\mathbf{dim}} x + y =_{\mathbf{dim}} x, y)
\end{aligned}$$

Case (III): Assume $x \notin \mathbf{ZEX}_{\mathcal{M}}$ and $z =_{\mathbf{dim}} x$.

To show the equivalence of $(x + y) - z$ and $(x - z) + (y - z)$ it suffices by EP-T9 to prove parthood in two directions, (a) and (b).

Direction (III.a): $\forall w\, [\mathbf{P}(w, (x + y) - z) \rightarrow \mathbf{P}(w, (x - z) + (y - z))]$.

Assume w is an arbitrary entity such that $\mathbf{P}(w, (x + y) - z)$.

Then

$$\begin{aligned}
& \mathbf{P}(w, (x + y) - z) \\
\Rightarrow\ & \neg \mathbf{PO}(w, z) \wedge \mathbf{P}(w, x + y) \\
\Rightarrow\ & \neg \mathbf{PO}(w, z) \wedge [\mathbf{P}(w, x) \vee \mathbf{P}(w, y) \vee [w = (w \cdot x) + (w - x) \wedge \\
& \mathbf{P}(w \cdot x, w) \wedge \mathbf{P}(w - x, w) \wedge \mathbf{P}(w \cdot x, x) \wedge \mathbf{P}(w - x, y)]]
\end{aligned}$$

The last step follows from Corollary 7.4.

Subcase (III.a.i): Assume $\mathbf{P}(w, x)$.

Then $\mathbf{P}(w, x - z)$ and hence we immediately conclude $\mathbf{P}(w, (x - z) + (y - z))$.

Subcase (III.a.ii): Assume $\mathbf{P}(w, y)$.

Then $\mathbf{P}(w, y - z)$ and hence we immediately conclude $\mathbf{P}(w, (x - z) + (y - z))$.

Subcase (III.a.iii): Assume $\neg \mathbf{P}(w, x)$ and $\neg \mathbf{P}(w, y)$.

Then the last case of the disjunction must hold; we have $\neg \mathbf{PO}(w \cdot x, z)$ and $\neg \mathbf{PO}(w - x, z)$

because of $\neg\mathbf{PO}(w,z)$. Together with $\mathbf{P}(w\cdot x,x)$ and $\mathbf{P}(w-x,y)$ we obtain $\mathbf{P}(w\cdot x,x-z)$ and $\mathbf{P}(w-x,y-z)$. In particular, $x-z$ and $y-z$ must be nonzero. Hence $(x-z)+(y-z)$ is nonzero and $x{=_{\mathbf{dim}}}y{=_{\mathbf{dim}}}x-z{=_{\mathbf{dim}}}y-z{=_{\mathbf{dim}}}(x-z)+(y-z)$. By transitivity of parthood, we get

$$\mathbf{P}(w\cdot x,(x-z)+(y-z)) \text{ and } \mathbf{P}(w-x,(x-z)+(y-z))$$

and consequently $\mathbf{P}(w,(x-z)+(y-z))$ by Sum-T10.

In either case $\mathbf{P}(w,(x+y)-z)$ implies $\mathbf{P}(w,(x-z)+(y-z))$].

Direction (III.b): $\forall w\,[\mathbf{P}(w,(x-z)+(y-z))\rightarrow\mathbf{P}(w,(x+y)-z)]$.

Assume w is an arbitrary entity such that $\mathbf{P}(w,(x-z)+(y-z))$.

Then

$$\begin{aligned}
&\mathbf{P}(w,(x-z)+(y-z))\\
\Leftrightarrow\ &\mathbf{P}(w,x-z)\vee\mathbf{P}(w,y-z)\vee\exists u,v[w=u+v\wedge\mathbf{P}(u,x-z)\wedge\mathbf{P}(v,y-z)]\\
\Leftrightarrow\ &[\mathbf{P}(w,x)\wedge\neg\mathbf{PO}(w,z)]\vee[\mathbf{P}(w,y)\wedge\neg\mathbf{PO}(w,z)]\vee\\
&\exists u,v[w=u+v\wedge\mathbf{P}(u,x)\wedge\neg\mathbf{PO}(u,z)\wedge\mathbf{P}(v,y)\wedge\neg\mathbf{PO}(v,z)]\\
\Leftrightarrow\ &[\mathbf{P}(w,x)\wedge\neg\mathbf{PO}(w,z)]\vee[\mathbf{P}(w,y)\wedge\neg\mathbf{PO}(w,z)]\vee\\
&\exists u,v[w=u+v\wedge\mathbf{P}(u,x+y)\wedge\mathbf{P}(v,x+y)\neg\mathbf{PO}(u+v,z)]\\
\Leftrightarrow\ &[\mathbf{P}(w,x)\wedge\neg\mathbf{PO}(w,z)]\vee[\mathbf{P}(w,y)\wedge\neg\mathbf{PO}(w,z)]\vee[\mathbf{P}(w,x+y)\wedge\neg\mathbf{PO}(w,z)]\\
\Leftrightarrow\ &\neg\mathbf{PO}(w,z)\wedge\mathbf{P}(w,x+y)\\
\Leftrightarrow\ &\mathbf{P}(w,(x+y)-z)
\end{aligned}$$

Case (IV): Assume $z{>_{\mathbf{dim}}}x$.

The proof is analogous to case (III) except that we now use $\neg\mathbf{Inc}(w,z)$ instead of $\neg\mathbf{PO}(w,z)$ throughout, leaving the conclusions unchanged.

These cases cover all possible relative dimension constraints between $x{=_{\mathbf{dim}}}y$ and z. □

We can also prove two cancellations properties between sums and differences (Sum-T17, Sum-T18).

(Sum-T17) $x<_{\text{dim}}y\rightarrow ZEX((x+y)-y)$ (cancellation of + and −)

(Sum-T18) $x<_{\text{dim}}y\rightarrow ZEX(y-(x+y))$ (cancellation of + and −)

Lemma 7.38. *$CODI_{\downarrow}\cup$ {Sum-A1 – Sum-A4} $\models$ {Sum-T17, Sum-T18}*

Sum-T19 guarantees that for some cases ($z\leq_{\text{dim}}x$), sums distribute left or right (recall commutativity of sums) over differences while obeying the special property of differences.

(Sum-T19) $z\leq_{\text{dim}}x\rightarrow x+(y-z)=(x+y)-(z-x)$ (if $z\leq_{\text{dim}}$ then $x+$ distributes over $y-z$)

Lemma 7.39. *$CODI_{\downarrow}\cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T19*

Proof. We consider a total of six cases. The first case covers $z\in\mathbf{ZEX}_{\mathcal{M}}$, while the remaining cases with $z\notin\mathbf{ZEX}_{\mathcal{M}}$ make a distinction based on the relative dimension between x and y, i.e., $y{<_{\mathbf{dim}}}x$, $y{=_{\mathbf{dim}}}x$, and $y{>_{\mathbf{dim}}}x$. For all cases we use the common assumption $z{\leq_{\mathbf{dim}}}x$. We will use a secondary distinction between $z{<_{\mathbf{dim}}}x$ and $z{=_{\mathbf{dim}}}x$ as necessary.

Note that $z \notin \mathbf{ZEX}_{\mathcal{M}}$ also implies $x \notin \mathbf{ZEX}_{\mathcal{M}}$.

Case (I): Assume $z \in \mathbf{ZEX}_{\mathcal{M}}$.

Let us further assume $z{\leq_{\mathbf{dim}}}x$, the antecedent of Sum-T19.

Then $x + (y - z) = x + y = (x + y) - (z - x)$ by Sum-T4 and Dif-A4 because $(z - x) \in \mathbf{ZEX}_{\mathcal{M}}$.

Case (II): Assume $z \notin \mathbf{ZEX}_{\mathcal{M}}$, $y{<_{\mathbf{dim}}}x$, and $z{<_{\mathbf{dim}}}x$.

$$
\begin{aligned}
x + (y - z) &= (x + y) - (z - x) \\
x &= x - (z - x) && (\text{Sum-A2: } y - z{=_{\mathbf{dim}}}y{<_{\mathbf{dim}}}x) \\
x &= x && (\text{Sum-A2: } z - x{=_{\mathbf{dim}}}z{<_{\mathbf{dim}}}x)
\end{aligned}
$$

Case (III): Assume $z \notin \mathbf{ZEX}_{\mathcal{M}}$, $y{<_{\mathbf{dim}}}x$, and $z{=_{\mathbf{dim}}}x$.

$$
\begin{aligned}
x + (y - z) &= (x + y) - (z - x) \\
x &= x - (z - x) && (\text{Sum-A2: } y - z{=_{\mathbf{dim}}}y{<_{\mathbf{dim}}}x)
\end{aligned}
$$

It remains to show that $x = x - (z - x)$. $\mathbf{P}(x - (z - x), x)$ is trivial, to prove $\mathbf{P}(x, x - (z - x))$ let w be an arbitrary part of x, i.e., $\mathbf{P}(w, x)$. Then $\neg\mathbf{PO}(w, z - x)$ and thus $\mathbf{P}(w, x - (z - x))$ (both by Dif-A1).

Case (IV): Assume $z \notin \mathbf{ZEX}_{\mathcal{M}}$, $y{=_{\mathbf{dim}}}x$, and $z{<_{\mathbf{dim}}}x$.

$$
\begin{aligned}
x + (y - z) &= (x + y) - (z - x) \\
x + y &= x + y && (\text{Sum-A2: } z - x{=_{\mathbf{dim}}}z{<_{\mathbf{dim}}}y{=_{\mathbf{dim}}}x + y)
\end{aligned}
$$

Case (V): Assume $z \notin \mathbf{ZEX}_{\mathcal{M}}$, $y{=_{\mathbf{dim}}}x$, and $z{=_{\mathbf{dim}}}x$.

To show the equivalence of $(x + y) - z$ and $(x - z) + (y - z)$ it suffices by EP-T9 to prove parthood in either direction, (a) and (b).

Direction (V.a): $\forall w\, (\mathbf{P}(w, x + (y - z)) \rightarrow \mathbf{P}(w, (x + y) - (z - x)))$.

Assume that w is an arbitrary entity so that $\mathbf{P}(w, x + (y - z))$.

Then:

$$
\begin{aligned}
& \mathbf{P}(w, x + (y - z)) \\
\Rightarrow\ & \mathbf{P}(w, x) \vee \mathbf{P}(w, y - z) \vee \\
& \exists u, v[w = u + v \wedge \mathbf{P}(u, w) \wedge \mathbf{P}(v, w) \wedge \mathbf{P}(u, x) \wedge \mathbf{P}(v, y - z)]]
\end{aligned}
$$

We split the proof into three subcases (i)–(iii).

Subcase (V.a.i): Assume $\mathbf{P}(w, x)$.

Then we obtain $\mathbf{P}(w, x{+}y)$ and $\neg\mathbf{PO}(w, z{-}x)$ by Dif-A3c because $x{=_{\mathbf{dim}}}y$. We conclude $\mathbf{P}(w, (x + y) - (z - x))$ by Dif-A3a.

Subcase (V.a.ii): Assume $\mathbf{P}(w, y - z)$.

We must have $\mathbf{P}(w, y) \wedge \neg\mathbf{PO}(w, z)$ by Dif-A3b and Dif-A3c. We then obtain $\mathbf{P}(w, x + y)$ and $\neg\mathbf{PO}(w, z - x)$ by Dif-A3c because $x {=_{\mathbf{dim}}} y$ and conclude $\mathbf{P}(w, (x + y) - (z - x))$ by Dif-A3a.

Subcase (V.a.iii): Assume there exists u, v such that $w = u + v$, $\mathbf{P}(u, w)$, $\mathbf{P}(v, w)$, $\mathbf{P}(u, x)$, and $\mathbf{P}(v, y - z)]]$.

We have $\mathbf{P}(u, x + y)$ and $\mathbf{P}(v, y - z)$. The latter implies $\mathbf{P}(v, y)$ and as such $\mathbf{P}(v, x + y)$. Then we must have $\mathbf{P}(w, x+y)$. $\mathbf{P}(v, y-z)$ also entails $\neg\mathbf{PO}(v, z)$. Hence $\neg\mathbf{PO}(v, z-x)$ while from $\mathbf{P}(u, x)$ we entail $\neg\mathbf{PO}(u, z - x)$. Together those imply $\neg\mathbf{PO}(w, z - x)$ which lets us conclude $\mathbf{P}(w, (x + y) - (z - x))$.

Direction (V.b): $\forall w\ (\mathbf{P}(w, (x + y) - (z - x)) \rightarrow \mathbf{P}(w, x + (y - z)))$.

Assume w to be an entity with $\mathbf{P}(w, (x + y) - (z - x))$.

Then $\mathbf{P}(w, x + y)$ and $\neg\mathbf{PO}(w, (x + y) \cdot (z - x)$, so that $\neg\mathbf{PO}(w, (z - x))$ must hold.

By Sum-T9, it suffices to prove that $\mathbf{P}(w \cdot z, x + (y - z))$ and $\mathbf{P}(w - z, x + (y - z))$, because all other parts of $x + (y - z)$ are sums of parts of $w \cdot z$ and $w - z$.

Part (V.b.i): $\mathbf{P}(w \cdot z, x + (y - z))$.

We know $\mathbf{Cont}(w \cdot z, w)$ and we have two subcases: either $w \cdot z {<_{\mathbf{dim}}} w$ or $\mathbf{P}(w \cdot z, w)$.

If $w \cdot z {<_{\mathbf{dim}}} w$, then $\mathbf{P}(w, x + (y - z))$ if and only if $\mathbf{P}(w, x + y)$. Since the later holds by our assumption, we obtain $\mathbf{P}(w, x + (y - z))$.

Now assume $\mathbf{P}(w \cdot z, w)$.

We then obtain $\mathbf{P}(w \cdot z, x + y)$ and $\neg\mathbf{PO}(w \cdot z, z - x)$ (by transitivity of P and by Dif-T5. The later requires $\mathbf{P}(w \cdot z, x)$, again by Dif-T5, because $\mathbf{P}(w \cdot z, z)$. Hence, $\mathbf{P}(w \cdot z, x + (y - z))$.

Part (V.b.ii): $\mathbf{P}(w - z, x + (y - z))$.

Again, we have $\mathbf{P}(w - z, x + y)$, but we also have $\neg\mathbf{PO}(w - z, z)$ by Dif-T3 and hence $\mathbf{P}(w - z, x + (y - z))$ if and only if $\mathbf{P}(w - z, x + y)$. Since we assumed $\mathbf{P}(w - z, x + y)$, we obtain $\mathbf{P}(w - z, x + (y - z))$.

Case (VI): Assume $z \notin \mathbf{ZEX}_{\mathcal{M}}$, $y {>_{\mathbf{dim}}} x$, and $z {\leq_{\mathbf{dim}}} x$.

$$
\begin{aligned}
x + (y - z) &= (x + y) - (z - x) && \\
y - z &= y - (z - x) && \text{(Sum-A2: } y - z {=_{\mathbf{dim}}} y {>_{\mathbf{dim}}} x) \\
y &= y && \text{(Sum-A2: } y {>_{\mathbf{dim}}} z \wedge y {>_{\mathbf{dim}}} z - x)
\end{aligned}
$$

Note that case (VI) covers both $y {>_{\mathbf{dim}}} x {=_{\mathbf{dim}}} z$ and $y {>_{\mathbf{dim}}} x {>_{\mathbf{dim}}} z$. Hence, these six cases cover all possible relative dimension constraints between x, y and z in which $x {\leq_{\mathbf{dim}}} z$ (as by the antecedent of Sum-T19). □

Sum-T20 considers the case of Sum-T19 where $z = x$.

(Sum-T20) $x + (y - x) = (x + y)$ (special case when $+$ distributes over $-$)

Lemma 7.40. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\vDash$ Sum-T20*

Proof. follows from Sum-T19 by the following computation using $z = x$ ($z \leq_{\mathbf{dim}} x$ is trivially satisfied in this case):

$$x + (y - z) = (x + y) - (z - x) \quad \text{(Sum-T19)}$$
$$x + (y - x) = (x + y) - (x - x) \quad (x = z)$$
$$x + (y - x) = (x + y) \quad \text{(Dif-A4)}$$

□

7.3.4 Interaction between sums, differences, and intersections

As a final step of our study of how the sum axioms interact with all the previous axioms, we will look at the mutual interaction between intersections, differences, and sums for equidimensional entities as expressed in Sum-T21 and Sum-T22.

(Sum-T21) $x =_{\text{dim}} y =_{\text{dim}} z \to x - (y \cdot z) = (x - y) + (x - z)$

Lemma 7.41. *$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4} $\models$ Sum-T21*

Proof. Assume x, y, z are arbitrary entities such that $x =_{\mathbf{dim}} y =_{\mathbf{dim}} z$.
We will prove $\forall v[PO(v, x - (y \cdot z)) \leftrightarrow PO(v, (x - y) + (x - z))]$ in six cases.

Case (I): Assume $\neg\mathbf{PO}(v, x)$.
Then both $\neg\mathbf{PO}(v, x - (y \cdot z))$ and $\neg\mathbf{PO}(v, (x - y) + (x - z))$.

Case (II): Assume $\mathbf{PO}(v, x)$, $\neg\mathbf{PO}(v, y)$, and $\neg\mathbf{PO}(v, z)$.
Then $\mathbf{PO}(v, x - (y \cdot z)) \leftrightarrow \mathbf{PO}(v, x) \leftrightarrow \mathbf{PO}(v, (x - y) + (x - z))$.

Case (III): Assume $\mathbf{PO}(v, x)$, $\neg\mathbf{PO}(v, y)$, and $\mathbf{PO}(v, z)$.
Then $\neg\mathbf{PO}(v, y \cdot z)$ and hence $\mathbf{PO}(v, x - (y \cdot z))$. But we also have $\mathbf{PO}(v, x - y)$ and hence $\mathbf{PO}(v, (x - y) + (x - z))$.

Case (IV): Assume $\mathbf{PO}(v, x)$, $\mathbf{PO}(v, y)$, and $\neg\mathbf{PO}(v, z)$.
Argument analogue to case (III).

Case (V): Assume $\mathbf{PO}(v, x)$, $\mathbf{PO}(v, y)$, $\mathbf{PO}(v, z)$, and $y \cdot z <_{\mathbf{dim}} x$.
We have $x = x - (y \cdot z)$ by Dif-A2 because $y \cdot z <_{\mathbf{dim}} x$.

We can split v into $v \cdot y$ and $v - y$. We then have $\neg\mathbf{PO}(v \cdot y, z)$, otherwise $y \cdot z <_{\mathbf{dim}} x$ would be falsified. Hence $\mathbf{P}(v \cdot y, x - z)$ and thus $\mathbf{P}(v \cdot y, (x - y) + (x - z))$. We further have $\neg\mathbf{PO}(v - y, y)$ by Dif-T3. Hence $\mathbf{P}(v - y, x - y)$ by Dif-A3a and thus $\mathbf{P}(v - y, (x - y) + (x - z))$. Because $v = (v \cdot y) + (v - y)$, we can apply Sum-T8' and obtain $\mathbf{P}(v, (x - y) + (x - z))$. In other words, every part contained in x is also contained in $(x - y) + (x - z)$. Since trivially $\mathbf{P}(x - y) + (x - z), x)$, we obtain $x = (x - y) + (x - z)$ as well.

Case (VI): Assume $\mathbf{PO}(v, x)$, $\mathbf{PO}(v, y)$, $\mathbf{PO}(v, z)$, and $y \cdot z =_{\mathbf{dim}} x$.
We can split v into $v \cdot y$ and $v - y$. The former can be further split into $(v \cdot y) \cdot z$ and $(v \cdot y) - z$, while the latter can be further split into $(v - y) \cdot z$ and $(v - y) - z$.

We will prove separately that these four entities partially overlap $x - (y \cdot z)$ if and only if they partially overlap $(x - y) + (x - z)$.

Part (VI.a): $(v \cdot y) \cdot z$.

We have $\mathbf{P}((v{\cdot}y){\cdot}z, y)$ and $\mathbf{P}((v{\cdot}y){\cdot}z, z)$, thus $\mathbf{P}((v{\cdot}y){\cdot}z, y{\cdot}z)$ and hence $\neg\mathbf{PO}((v{\cdot}y){\cdot}z, x{-}(y{\cdot}z))$. And at the same time, $\neg\mathbf{PO}((v \cdot y) \cdot z, (x - y) + (x - z))$.

Part (VI.b): $(v \cdot y) - z$.

Consider the following two derivations:

$$\begin{aligned}\neg\mathbf{PO}((v \cdot y) - z, z) \Rightarrow & \neg\mathbf{PO}((v \cdot y) - z, y \cdot z)\\ & \Rightarrow \mathbf{PO}((v \cdot y) - z, x - (y \cdot z)) \leftrightarrow \mathbf{PO}((v \cdot y) - z, x)\end{aligned}$$

and

$$\begin{aligned}\neg\mathbf{PO}((v \cdot y) - z, z) \Rightarrow & \mathbf{PO}((v \cdot y) - z, x - z) \leftrightarrow \mathbf{PO}(v, x)\\ & \Rightarrow \mathbf{PO}((v \cdot y) - z, (x - y) + (x - z)) \leftrightarrow \mathbf{PO}((v \cdot y) - z, x)\end{aligned}$$

Hence,

$$\mathbf{PO}((v \cdot y) - z, x - (y \cdot z)) \leftrightarrow \mathbf{PO}((v \cdot y) - z, x) \leftrightarrow \mathbf{PO}((v \cdot y) - z, (x - y) + (x - z)).$$

Part (VI.c): $(v - y) \cdot z$.

Consider the following two derivations:

$$\begin{aligned}\neg\mathbf{PO}((v - y) \cdot z, y) \Rightarrow & \neg\mathbf{PO}((v - y) \cdot z, y \cdot z)\\ & \Rightarrow \mathbf{PO}((v - y) \cdot z, x - (y \cdot z)) \leftrightarrow \mathbf{PO}(v, x)\end{aligned}$$

and at the same time

$$\begin{aligned}\neg\mathbf{PO}((v - y) \cdot z, y) \Rightarrow & \mathbf{PO}((v - y) \cdot z, x - y) \leftrightarrow \mathbf{PO}(v, x))\\ & \Rightarrow \mathbf{PO}((v - y) \cdot z, (x - y) + (x - z)) \leftrightarrow \mathbf{PO}(v, x))\end{aligned}$$

Hence, $\mathbf{PO}((v - y) \cdot z, x - (y \cdot z)) \leftrightarrow \mathbf{PO}((v - y) \cdot z, x) \leftrightarrow \mathbf{PO}((v - y) \cdot z, (x - y) + (x - z))$.

Part (VI.d): $(v - y) - z$.

By the same derivations used in Part (IV.b) and (IV.c), we obtain

$$\mathbf{PO}((v - y) - z, x - (y \cdot z)) \leftrightarrow \mathbf{PO}((v - y) - z, x) \leftrightarrow \mathbf{PO}((v - y) - z, (x - y) + (x - z)))$$

Since each of $(v \cdot y) \cdot z$, $(v \cdot y) - z$, $(v - y) \cdot z$, and $(v - y) - z$ is a part of $x - (y \cdot z)$ iff it is a part of $(x - y) + (x - z)$, and they sum up to v (by repeated application of Sum-T8'), this implies that for all v with $\mathbf{PO}(v, y) \wedge \mathbf{PO}(v, z) \wedge y \cdot z {=_{\mathbf{dim}}} x$ we have $\mathbf{PO}(v, x - (y \cdot z)) \leftrightarrow \mathbf{PO}(v, (x - y) + (x - z))$.

These six cases cover all possible relationships between v and y, z (note that we cannot have $y \cdot z {>_{\mathbf{dim}}} x$ because $y \cdot z {\leq_{\mathbf{dim}}} y {=_{\mathbf{dim}}} x$), in either case we have $\mathbf{PO}(v, x - (y \cdot z)) \leftrightarrow \mathbf{PO}(v, (x - y) + (x - z))$. □

(Sum-T22) $x =_{\dim} y =_{\dim} z \wedge \neg Cont(x, y + z) \to x - (y + z) = (x - y) \cdot (x - z)$

Lemma 7.42. *CODI*$_{\downarrow} \cup$ *{Sum-A1 – Sum-A4}* $\vDash$ *Sum-T22*

Proof. Note that $x =_{\text{dim}} x-(y+z) =_{\text{dim}} (x-y)\cdot(x-z)$ as long as $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $(x-z) \notin \mathbf{ZEX}_{\mathcal{M}}$. If $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$, we have $x \in \mathbf{ZEX}_{\mathcal{M}}$ or $\mathbf{Cont}(x,y)$ by Dif-A4. If $(x-z) \in \mathbf{ZEX}_{\mathcal{M}}$, we have $x \in \mathbf{ZEX}_{\mathcal{M}}$ or $\mathbf{Cont}(x,z)$ by Dif-A4. We can neither have $\mathbf{Cont}(x,y)$ nor have $\mathbf{Cont}(x,z)$ because of $\neg\mathbf{Cont}(x,y+z)$; while $x \in \mathbf{ZEX}_{\mathcal{M}}$ requires $(x-(y+z)) \in \mathbf{ZEX}_{\mathcal{M}}$ as well.

For the remainder we can safely assume $(x-y),(x-z) \notin \mathbf{ZEX}_{\mathcal{M}}$ as well as $x \mathbf{=_{dim}} x-(y+z) \mathbf{=_{dim}} (x-y)\cdot(x-z)$.
We will show that then

$$x \mathbf{=_{dim}} y \mathbf{=_{dim}} z \wedge \neg\mathbf{Cont}(x,y+z) \rightarrow \forall v(\mathbf{P}(v,x-(y+z)) \leftrightarrow \mathbf{P}(v,(x-z)\cdot(x-z)))$$

which follows from the following equivalences for an arbitrary v:

$$\begin{aligned}\mathbf{P}(v,x-(y+z) &\leftrightarrow \mathbf{P}(v,x) \wedge \neg\mathbf{PO}(v,y+z)\\ &\leftrightarrow \mathbf{P}(v,x) \wedge \neg\mathbf{PO}(v,y) \wedge \neg\mathbf{PO}(v,z)\\ &\leftrightarrow \mathbf{P}(v,x-y) \wedge \mathbf{P}(v,x-z)\\ &\leftrightarrow \mathbf{P}(v,(x-y)\cdot(x-z))\end{aligned}$$

The direction $\leftarrow$ of the last step only works because of $\neg\mathbf{Cont}(x,y+z)$ which requires a part of x to exist that is neither contained in y nor in z and thus both in $x-y$ and in $x-z$, and consequently in $(x-y)\cdot(x-z)$. ☐

This concludes our study of the interaction of the three mereological closure operations intersection, difference, and sum.

7.3.5 A model-theoretic characterization of $CODI_{\downarrow} \cup$ {Sum-A1 − Sum-A4}

We can now strengthen Theorem 6.1, which established that a model of *CODI* can be partitioned into two substructures: one of containment within a dimension (parthood) and another one of lower-dimensional containment, to models equipped with downwards and upwards mereological closures for all entities. We can show that the extension of the containment relation of a model of $CODI_{\downarrow} \cup$ {Sum-A1 − Sum-A4} is always partitioned into jointly exhaustive, pairwise disjoint substructures of parthood, each of them being a Boolean lattice. Contact within each structure arises only as partial overlap *PO*, while contact across structures arises as incidence *Inc*. Superficial contact *SC* may arise within or across structures, but is only identifiable across structures, since the shared entity cannot be in the same structure.

More formally, each entity x that is maximal in its dimension, i.e., each $x \in \mathbf{Max}_{\mathcal{M}}$, in a model of $CODI_{\downarrow} \cup$ {Sum-A1 − Sum-A4} has a parthood structure that is a mereological field, i.e., a Boolean algebra with the least ('zero') entity removed.

Theorem 7.6. *Let $\mathcal{M}$ be a model of $CODI_{\downarrow} \cup$ {Sum-A1 − Sum-A4} with domain* $\mathbf{M}$ *of finite size* ≥ 2.
For each max $\in \mathbf{Max}_{\mathcal{M}}$ *we define the following equivalence class of equidimensional entities*

$$\mathbf{B}_{\max} = \{d \mid d \in \mathbf{M} \textit{ and } \langle d, max\rangle \in (\mathbf{=_{dim}})_{\mathcal{M}}\} \cup \mathbf{ZEX}_{\mathcal{M}}.$$

Then

1. *Each set* $\mathbf{B}_{\max}$ *defines a structure* $\mathcal{M}_{\max} = \langle \mathbf{B}_{\max}, \times, +, -, \mathrm{ze}, \max \rangle$ *that is a substructure of* $\mathcal{M}$ *closed under* $-$*,* $+$*, and* $\times$ *where:*

$$x \times y = \begin{cases} x \cdot y & \text{if } \langle x, y \rangle \in \mathbf{PO}_{\mathcal{M}}, \\ \mathrm{ze} & \text{otherwise.} \end{cases}$$

2. *Each structure* $\mathcal{M}_{\max}$ *is a Boolean algebra with unique complementation defined as* $x' = \max - n$*.*

3. *For any* $d \in \mathbf{M}$*, there exists some* max *such that* $d \in \mathbf{B}_{\max}$*,*

4. *Any* $d \in \mathbf{M} \setminus \{\mathrm{ze}\}$ *is in at most one* $\mathbf{B}_{\max}$*.*

Proof. 1. We need to show that $(x \times y), (x - y), (x + y) \in \mathbf{B}_{\max}$ for any $x, y \in \mathbf{B}_{\max}$.

Assume x, y to be arbitrary entities in $\mathbf{B}_{\max}$.
Then $x =_{\mathbf{dim}} y$ by definition.

If $\mathbf{PO}(x, y)$, we have $x \times y = x \cdot y$ which has the same dimension as x and y by Int-T7, hence $(x \times y) \in \mathbf{B}_{\max}$. Otherwise, $x \times y = \mathrm{ze}$ with $\mathrm{ze} \in \mathbf{B}_{\max}$ by definition.

By Dif-A1, we have $x - y =_{\mathbf{dim}} x$ and thus $(x - y) \in \mathbf{B}_{\max}$ unless $(x - y) \in \mathbf{ZEX}_{\mathcal{M}}$ in which case $x - y = \mathrm{ze} \in \mathbf{B}_{\max}$.

By Sum-T5, we have $x + y =_{\mathbf{dim}} x$ if $x =_{\mathbf{dim}} y$ and thus $(x + y) \in \mathbf{B}_{\max}$.

2. Any distributive bounded lattice equipped with an operation of unique complementation is a Boolean lattice.

First we show that each of the structures $\mathcal{M}_{\max} = \langle \mathbf{B}_{\max}, \times, +, -, \mathrm{ze}, \max \rangle$ is a bounded distributive lattice if we define for all $x, y \in \mathbf{B}_{\max}$,

$$x \leq y \Leftrightarrow \mathbf{P}(x, y) \vee x \in \mathbf{ZEX}_{\mathcal{M}}.$$

Let a, b, c be arbitrary entities in $\mathbf{B}_{\max}$. Then $a =_{\mathbf{dim}} b =_{\mathbf{dim}} c$ and thereby:

- $a \leq b + b$ ($+$ defines a supremum or join operation),
- $a \times b \leq a$ ($\times$ defines a infimum or meet operation),
- $\mathrm{ze} \leq a$ (ze is a lower bound),
- $a \leq \max$ (max is an upper bound),
- $a \times (b + c) = (a \times b) + (a \times c)$ ($\times$ distributive; by $\cdot$ being distributive by Sum-T11 together with $\mathcal{M}_{\max}$ being closed under $\times$ and $+$),
- $a + (b \times c) = (a + b) \times (a + c)$ ($+$ distributive; by Sum-T14 together with $\mathcal{M}_{\max}$ being closed under $\times$ and $+$).

It remains to show that the structure $\mathcal{M}_{\max} = \langle \mathbf{B}_{\max}, \times, +, -, \mathrm{ze}, \max \rangle$ defines a unique complementation operation. Choose $x' = \max - x$ for all $x \in \mathbf{B}_{\max}$, then:

- $x + x' = \max$ (by Sum-T20: $x + (\max - x) = x + \max = \max$),
- $x \cdot x' = \mathrm{ze}$ (by Dif-T3 $\neg\mathbf{PO}(x, \max - x)$).

Since $\max - x$ is uniquely defined by Dif-T3 through a complete description of its part, $x' = \max - x$ is a unique complementation operation.

Hence $\mathcal{M}_{\max} = \langle \mathbf{B}_{\max}, \times, +, -, \mathrm{ze}, \max \rangle$ is a Boolean lattice.

3. We will show that for any $d \in \mathbf{M}$ there exists a $\max \in \mathbf{M}$ such that $\max \in \mathbf{Max}_{\mathcal{M}}$ and $\max \boldsymbol{=}_{\mathbf{dim}} d$.

   Assume $d \notin \mathbf{ZEX}_{\mathcal{M}}$.
   Then if $d \in Max_{\mathcal{M}}$ we immediately have $d \in \mathbf{B}_{\mathrm{d}}$. Suppose $d \notin \mathbf{Max}_{\mathcal{M}}$, then by the definition of *Max* (ME-D1), there exists some entity d_1 such that $\mathbf{PP}(d, d_1)$. Either $d_1 \in \mathbf{Max}_{\mathcal{M}}$, or there exists a d_2 such that $\mathbf{PP}(d_1, d_2)$. Since the domain is finite, some $d_n \in \mathbf{Max}_{\mathcal{M}}$ must exist. Since $\mathbf{PP}(d_{\mathrm{i}-1}, d_i)$ implies $d_{\mathrm{i}-1} \boldsymbol{=}_{\mathbf{dim}} d_i$, we obtain also $d \boldsymbol{=}_{\mathbf{dim}} d_n$ by transitivity of $=_{\mathrm{dim}}$. Hence $d \in \mathbf{B}_{\mathrm{d_n}}$. Consequently, for any $d \notin \mathbf{ZEX}_{\mathcal{M}}$ some $\max \in \mathbf{M}$ exists such that $d \in \mathbf{B}_{\max}$.

   Now assume $d \in \mathbf{ZEX}_{\mathcal{M}}$.
   Since the domain contains some $e \notin \mathbf{ZEX}_{\mathcal{M}}$ we have $d \in \mathbf{B}_{\mathrm{e}}$ by definition.

4. Suppose there exists a $d \in \mathbf{M} \setminus \{ze\}$ such that $d \in \mathbf{B}_{\max}$ and $d \in \mathbf{B}_{\max'}$. Then we have $max \boldsymbol{=}_{\mathbf{dim}} d \boldsymbol{=}_{\mathbf{dim}} max'$ and as such $\mathbf{B}_{\max} = \mathbf{B}_{\max'}$ by the definition of the sets $\mathbf{B}_{\max}$ and $\mathbf{B}_{\max'}$.

Thus the sets $\mathbf{B}_{\max}$ partition the domain $\mathbf{M}$ of a model of $CODI_{\downarrow} \cup \{$Sum-A1 – Sum-A4$\}$, with each structure $\mathcal{M}_{\max} = \langle \mathbf{B}_{\max}, \times, +, -, \mathrm{ze}, \max \rangle$ defining a Boolean algebra that is a substructure of $\mathcal{M}$. □

By the Satisfiability Theorem (Theorem 7.4), every structure $\mathfrak{M}$ in the class of intended structures $\mathbb{M}$ corresponds to a model of $CODI_{\downarrow}$. If such structure $\mathfrak{M}$ corresponds to a model of $CODI_{\downarrow} \cup \{$Sum-A1 – Sum-A4$\}$, the collections $\mathrm{MF}^m \in \mathfrak{M}$ with identical m form a Boolean algebra. A complex manifold $\mathfrak{M}^m$ is then a collection of Boolean structures of composite manifolds for each dimension $n \leq m$. Note though that not every structure $\mathfrak{M}$ in the class of intended structures $\mathbb{M}$ is a model of $CODI_{\downarrow} \cup \{$Sum-A1 – Sum-A4$\}$, since the structures in $\mathbb{M}$ are not required to be closed under sums of equidimensional manifolds.

Note further that the characterization in Theorem 7.6 is restricted to models with finite domains because we cannot guarantee the existence of sums of infinitely many entities (at least not without employing infinite axiom schemas or second-order logic). However, it is straightforward to see that the theorem extends to infinite models of $CODI_{\downarrow} \cup \{$Sum-A1 – Sum-A4$\}$ in which there exists a maximal entity of each dimension. However, more commonly, only a maximal entity of greatest dimension, a so-called universal entity, is postulated to exist. Next, we will extend our theory in such a way.

7.4 Universals

As a consequence of Theorem 7.6 all finite models of $CODI_{\downarrow} \cup \{$Sum-A1 – Sum-A4$\}$ have a unique maximal element in every dimension. However, this does not mean that the maximal entity of maximal dimension will contain all other entities in a finite model. To ensure that such a universal entity exist, a constant u can be introduced to denote a universal entity which must contain every other entity (except the zero entity).

We can now combine the downwards closed multidimensional mereotopology $CODI_{\downarrow}$ with upwards closures of sums and a universal to obtain the upwards and downwards closed theory

$$CODI_{\updownarrow} = CODI_{\downarrow} \cup \{\text{Sum-A1} - \text{Sum-A4, U-A1}\}.$$

(U-A1) $\neg ZEX(x) \rightarrow Cont(x, \mathrm{u})$ (universal u contains everything)

Axiom Set 7.11: Axiom U-A1 of $CODI_{\updownarrow}$.

In $CODI_{\updownarrow}$ the universal is implicitly of highest dimension (U-T1) and maximal in its dimension (U-T2).

(U-T1) $MaxDim(\mathrm{u})$ (universal u is of highest dimension)

(U-T2) $Max(\mathrm{u})$ (universal u is maximal in its dimension)

Lemma 7.43. $CODI_{\updownarrow} \models$ *{U-T1, U-T2}*

By satisfying U-T1, all models of $CODI_{\updownarrow}$ also satisfy D-A7 and are thus models of $DI_{\text{linear}-\text{bounded}}$.

Lemma 7.44. $CODI_{\updownarrow} \models DI_{\text{linear}-\text{bounded}}$

Proof. Recall that $DI_{\text{linear}-\text{bounded}} = DI_{\text{linear}} \cup$ D-A7. With $CODI_{\updownarrow}$ being an extension of DI_{linear} it suffices to prove D-A7, which follows immediately from U-T1. □

However, this does not work in the other direction, that is U-A1 is a stronger axiom than D-A7, that is,

$$CODI_{\downarrow} \cup \{\text{Sum-A1} - \text{Sum-A4}\} \cup \text{D-A7} \not\models \text{U-A1}$$

Obviously, we can identify the maximal dimension in a model of $CODI_{\downarrow} \cup \{\text{Sum-A1} - \text{Sum-A4}\} \cup$ D-A7. However, U-A1 may fail for two separate reasons. First, entities of a lower dimension might not necessarily be contained in an entity of highest dimension. Secondly, we can construct a model with an infinite number of entities of highest dimension with no explicit maximal entity in that dimension. Even though the sum for every pair of entities of highest dimension may exist, not necessarily the sum of all entities of highest dimension exists. This is essentially the distinction drawn between first-order definable Closure Mereotopology (**CMT**) and General Closure Mereotopology (**GMT**).

Just as we showed that the mereological closure operations are uniquely defined (and total), for completeness we must prove that the universal is a uniquely defined constant.

Theorem 7.7. *The constant* u *is uniquely defined in* $CODI_{\updownarrow}$.

Proof. For any model with a nonzero entity x, U-A1 requires some entity u to exist. Now, suppose it is not uniquely defined, that is two distinct entities $u_1, u_2 \in \mathbf{M}$ satisfy U-A1. Since $u_1, u_2 \notin \mathbf{ZEX}_{\mathcal{M}}$, we automatically have $\mathbf{Cont}(u_1, u_2)$ and $\mathbf{Cont}(u_2, u_1)$ by U-A1, which in turn leads to $u_1 = u_2$ by *Cont* being antisymmetric (C-A2). This contradicts our assumption of u_1 and u_2 being distinct. □

Next, we confirm that the universal does behave as expected with respect to the mereological closure operations.

(U-T3) $x \cdot \mathrm{u} = x$ (the intersection of x with the universal is x)

(U-T4) $ZEX(x - \mathrm{u})$ (empty difference between an entity and the universal)

(U-T5) $x + \mathrm{u} = \mathrm{u}$ (the sum involving the universal is again the universal)

Lemma 7.45. $CODI_{\updownarrow} \models$ *{U-T3 – U-T5}*

Proof. For $x \in \mathbf{ZEX}_{\mathcal{M}}$ these are trivial. For $x \notin \mathbf{ZEX}_{\mathcal{M}}$ all three properties follow directly from $\mathbf{Cont}(x, \mathrm{u})$. □

Moreover, we can show that the De Morgan laws hold for the equidimensional entities of highest dimension in $CODI_{\updownarrow}$. Note that the complement x of an entity of highest dimension can be represented as $\mathrm{u} - x$, since we have:

- $x + (\mathrm{u} - x) = x + \mathrm{u} = \mathrm{u}$ by Sum-T20 ($x + x' = \top$),
- $\neg\mathbf{PO}(x, \mathrm{u} - x)$ by Dif-T3 ($x \cdot x' = \bot$),
- $\mathrm{u} - (\mathrm{u} - x) = x$ by Dif-T8 ($x'' = x$).

Hence every entity of highest dimension, i.e., every $x \in \mathbf{MaxDim}_{\mathcal{M}}$ has in $\mathrm{u} - x$ a complement defined, which is unique since all parts of $\mathrm{u} - x$ are uniquely defined by Dif-A3 and hence $\mathrm{u} - x$ is unique by EP-T9. Then the DeMorgan laws can be stated as U-T6 and U-T7 (U-T7 only guarantees it for two entities which together do not cover the entire space).

(U-T6) $MaxDim(x) \wedge MaxDim(y) \rightarrow \mathrm{u} - (x \cdot y) = (\mathrm{u} - x) + (\mathrm{u} - y)$

(U-T7) $MaxDim(x) \wedge MaxDim(y) \wedge x + y \neq \mathrm{u} \rightarrow \mathrm{u} - (x + y) = (\mathrm{u} - x) \cdot (\mathrm{u} - y)$

Lemma 7.46. $CODI_{\updownarrow} \models$ *{U-T6, U-T7}*

Proof. Follow directly from Sum-T21 and Sum-T22 if we choose $x := \mathrm{u}$, $y := x$, and $z := y$. Note that because of $x, y, U \in \mathbf{MaxDim}_{\mathcal{M}}$ we have $x \mathbf{=_{dim}} y \mathbf{=_{dim}} \mathrm{u}$. □

This essentially means that the theorems Sum-T21 and Sum-T22 are generalized versions of the De Morgan laws, i.e., they are the De Morgan laws relative to a given entity, which may differ from the unique greatest entity. Both theorems are only guaranteed to work for any three equidimensional entities, with Sum-T22 additionally presupposing $\neg Cont(x, y + z)$.

If we want to ensure that for every dimension a unique maximal entity exists, we can extend $CODI_{\updownarrow}$ by U-E1. Note that it does not suffice to postulate

$$\forall x[\neg ZEX(x) \rightarrow \exists y(x =_{\mathrm{dim}} y \wedge Max(y))]$$

because this allows for multiple maximal entities that not are in parthood relation to one another.

(U-E1) $\forall x[\neg ZEX(x) \rightarrow \exists y \forall z(z =_{\mathrm{dim}} x \rightarrow P(z, y))]$	(unique maximal entity in each dimension)
(U-E2) $Con(U)$	(universal entity self-connected)

Axiom Set 7.12: Extension axioms U-E1 and U-E2 of $CODI_{\updownarrow}$.

U-E1 trivially holds in all finite models of $CODI_{\downarrow} \cup \{$ Sum-A1 – Sum-A4$\}$ and thus in all finite models of $CODI_{\updownarrow}$—we used this property for the proof of Theorem 7.6 on page 154. Then it is also easy to see how Theorem 7.6 extends to infinite models of $CODI_{\downarrow} \cup \{$ Sum-A1 – Sum-A4, U-E1$\}$.

But in $CODI_{\downarrow}$ U-E1 is still not sufficient to prove U-A1 because even though U-E1 requires a unique greatest entity of maximal dimension to exist, it is not guaranteed that this entity contains all entities of lower dimensions.

A final possible extension requires the universal entity to be self-connected (U-E2). Ideally, we want to say that it is internally self-connected, but we cannot yet define this stronger condition in the *CODI* hierarchy.

7.5 Summary

The theories in this and the previous chapter and the metatheoretical relationships among them are illustrated in Figure 7.8. The two main theories we discussed in this chapter are $CODI_{\downarrow}$, which is closed under intersections and differences, and $CODI_{\updownarrow}$, which is closed under intersections, differences, sums, and universals. The merits of closure under sums are debatable in mereotopology, an abundance of undesirable or irrelevant sums has been given, for example in [CV99a]. But we have a much stronger reason to not enforce closure under sums: the class $\mathbb{M}$ of intended structures is not compatible with closure under sums. This is for two reasons: not all structures in $\mathbb{M}$ are closed under sums—something we could easily fix by extending the definition of a complex manifold—but more importantly, closure under sums may require entities to exist that are not composite manifolds because constituent atomic manifolds may intersect in their interiors. To be clear, we do not intend to disallow spatial configuration in which two atomic or composite manifolds intersect in their interiors, but we do not want to call the sum of two such manifolds a composite manifold itself, in line with Definition 5.6. That said, such a sum cannot correspond to an entity in the corresponding model of $CODI_{\downarrow}$. For these reasons we will work primarily with $CODI_{\downarrow}$, sometimes extended by U-A1.

In Section 7.2.5 we were able to show the satisfiability of $CODI_{\downarrow}$. Because of the inherent problems with closures under sums, we do not attempt to provide a satisfiability theorem for $CODI_{\updownarrow}$. To do so, we would need to restrict sums to entities which do not intersect in the interior of their constituent atomic manifolds, which in turn requires us to distinguish the interior from the boundary of minimal entities (the entities that correspond to atomic manifolds) in $CODI_{\downarrow}$. Presently, two nonisomorphic structures—one in the class $\mathbb{M}$ and one that violates Definition 5.6(1) or (2)—may be elementarily equivalent models of $CODI_{\downarrow}$. In other words, two such models may be indistinguishable by $CODI_{\downarrow}$ (and any extension thereof with the same lexicon such as $CODI_{\updownarrow}$), that is, no first-order sentence in the language of $CODI_{\downarrow}$ can distinguish those models. See Figure 7.9 for an example. To address this problem, we need to introduce a new primitive relation that, in combination with $CODI_{\downarrow}$, can define how any two of Figure 7.9(b) to Figure 7.9(e) are different. Obviously, the difference lies in whether two entities share an interior or boundary point. An extension of $CODI_{\downarrow}$ by an adequate additional primitive relation that allows such distinction is presented in Chapter 9. Once we can distinguish between the spatial configurations of Figure 7.9(b) to Figure 7.9(e), we can revisit the axioms Sum-A1 to Sum-A4 and constrain them so that sums which cannot be represented by composite m-manifolds are disallowed. In essence, we will be able to logically distinguish sums that represent composite manifolds as defined in Definition 5.6 in Chapter 5 from sums that do not represent composite manifolds.

Meanwhile, it is still helpful to maintain the sum notation without forcing sums to exist for every pair of entities. For this purpose, we can rewrite the axioms Sum-A1 to Sum-A4 using a ternary relation $Sum(x, y, z)$ that means $x + y = z$ if the sum $x + y$ exists. This results in Sum′-A1 to Sum′-A4. We

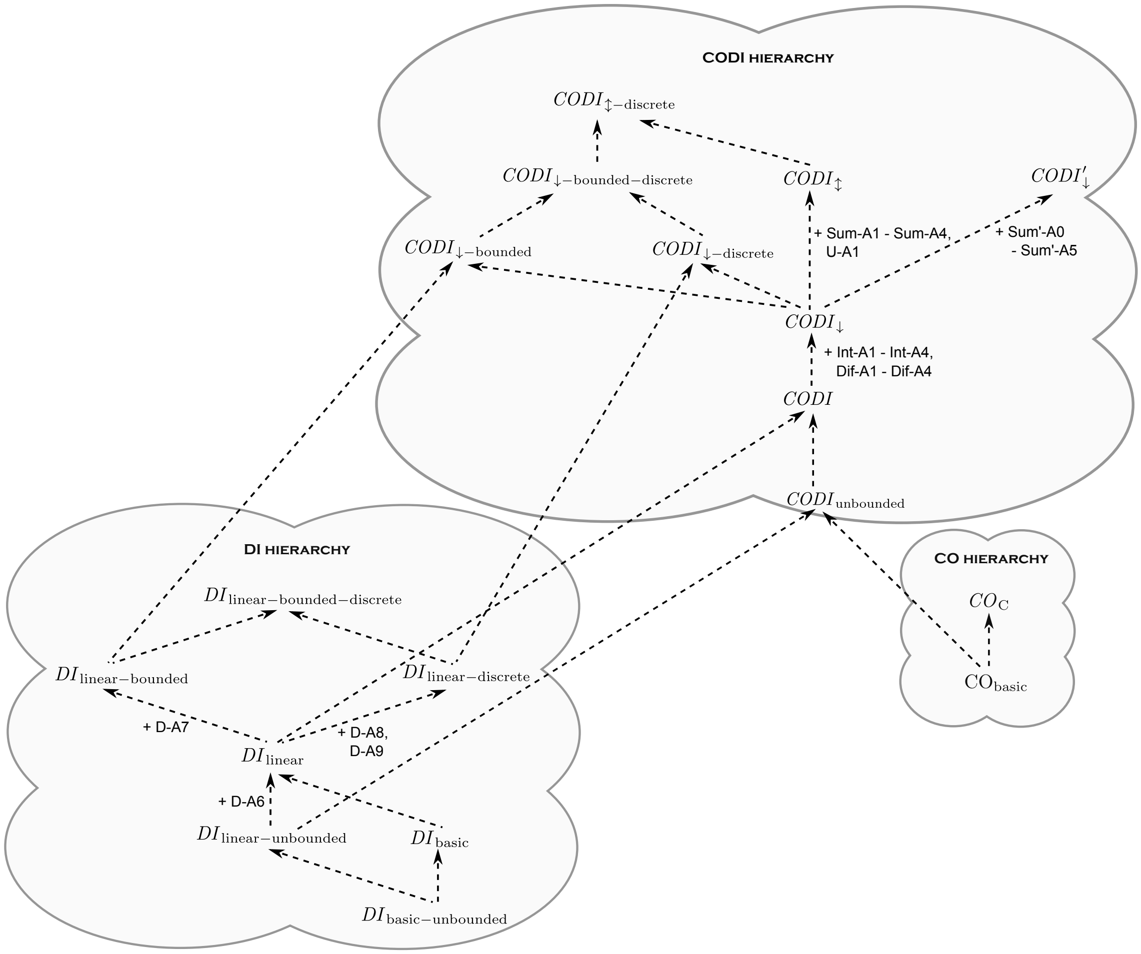


Figure 7.8: The theories of the hierarchy *CODI* and their relationship to theories of the dimension hierarchy *DI* and the containment hierarchy *CO*. The different closure operations further extend the *CODI* hierarchy developed in Chapter 6. The theories $CODI_{\downarrow}$ and $CODI_{\updownarrow}$ can be again combined with stronger theories of dimension.

add Sum′-A0 to ensure that the sum of any two entities is unique (if it exists)—a property that was implicit in the sum function. We further add Sum′-A5 as necessary condition of when a given entity z can be called the sum of two equidimensional entities x and y: everything that partially overlaps z must partially overlap either x or y (and vice versa).

We use + as a nontotal function defined as $x + y = z \leftrightarrow Sum(x, y, z)$ in the theory

$$CODI'_{\downarrow} = CODI_{\downarrow} \cup \{\text{Sum}'\text{-A0}-\text{Sum}'\text{-A5}\}.$$

(Sum′-A0) $Sum(x, y, z) \wedge Sum(x, y, v) \rightarrow v = z$

(Sum′-A1) $Sum(x, y, z) \rightarrow Sum(y, x, z)$

(Sum′-A2) $x <_{\text{dim}} y \rightarrow Sum(x, y, y)$

(Sum′-A3) $Sum(x, y, z) \wedge x \leq_{\text{dim}} y \wedge Cont(v, y) \rightarrow Cont(v, z)$

(Sum′-A4) $Sum(x, y, z) \wedge Cont(v, z) \wedge \neg Cont(v, x) \rightarrow Cont(v - x, y)$

(Sum′-A5) $x =_{\text{dim}} y \wedge Sum(x, y, z) \rightarrow \forall v[PO(v, z) \leftrightarrow PO(v, x) \vee PO(v, y)]$

Axiom Set 7.13: Axioms Sum′-A0 – Sum′-A5 of $CODI'_{\downarrow}$.

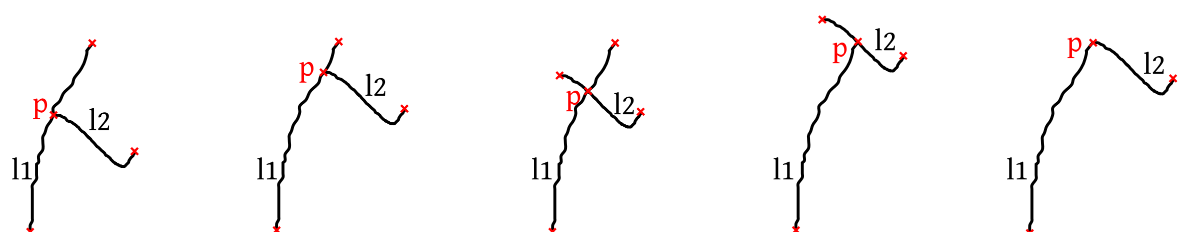


Figure 7.9: Five spatial configurations that are equivalent models $\mathcal{M}$ of $CODI_{\updownarrow}$ with domain $M = \{\text{ze}, p, l1, l2, l1{+}l2\}$ and $\mathbf{ZEX}_{\mathcal{M}} = \{\text{ze}\}$. We want to discriminate four of them—the are different intended structures. The extension of relative dimension is defined as $(<_{\mathbf{dim}})_{\mathcal{M}} = \{\langle \text{ze}, p\rangle, \langle \text{ze}, l1\rangle, \langle \text{ze}, l2\rangle, \langle \text{ze}, l1{+}l2\rangle, \langle p, l1\rangle, \langle p, l2\rangle, \langle p, l1{+}l1\rangle\}$ and the extension of containment as $\mathbf{Cont}_{\mathcal{M}} = \{\langle p, p\rangle, \langle p, l1\rangle, \langle p, l2\rangle, \langle p, l1{+}l2\rangle, \langle l1, l1\rangle, \langle l1, l1 + l2\rangle, \langle l2, l2\rangle, \langle l2, l1 + l2\rangle, \langle l1 + l2, l1 + l2\rangle\}$. $l2 + l2$ is the universal entity in each configuration. While (a)–(d) are unintended spatial configurations because $l1$ and $l2$ are fused at interior points, i.e., $l1 + l2$ does not represent a complex manifold, (e) is in the intended class of structures. In other words, the entity $l1 + l2$ is only in (e) a composite manifold as defined in Chapter 5, while in (a)–(d) it is a complex manifold, but not a composite manifold. While we eventually want to distinguish between any pair out of configurations (b)–(e), we do not aim to distinguish (a) from (b) since their difference is not qualitative in nature.

Chapter 8

Relationship to other mereotopologies[1]

In this chapter we relate the theories of the *CODI* hierarchy to other spatial theories with similar expressiveness. In particular, we show how to extend *CODI* theories to reconstruct two mereotopologies—the well-known equidimensional Region Connection Calculus (RCC) [Coh+97b] and the multidimensional INCH Calculus [Got96].

This work serves several purposes. Firstly, we confirm that the *CODI* theories really mereotopologies. On the one side, they are axiomatically less restricted than the RCC as shown in Section 8.1S, and, by the axioms we need to extend *CODI* to restrict models to the RCC and the results from Chapter 4, consequently less restricted than any equidimensional mereotopologies. Of course, the nonlogical language of the *CODI* theories must be more expressive than that of equidimensional mereotopologies, because we want to deal with multidimensionality. On the other side, the language of *CODI* is equally expressive as the language of the *INCH* calculus, an independently developed multidimensional mereotopology. In fact, our main result in Section 8.2 shows that the INCH Calculus and $CODI_{\updownarrow}$ have only minor differences in their axiomatic restrictiveness. We prove this by showing that extensions of the INCH Calculus and of $CODI_{\updownarrow}$ are definably equivalent; the axioms we need to add in those extensions capture the logical difference between the two theories.

Secondly, we cross-verify our axiomatizations with other independently developed spatial theories. For the relationship to the Region Connection Calculus, this verifies our axiomatization as the RCC has been extensively studied and is well-understood. For the relationship to the INCH Calculus, whose models are much less understood, it is more of a two-way verification: we can identify logical sentences entailed by one of the two theories but not the other and then contemplate whether those are reasonable assumptions or whether those demonstrate a problem with the axiomatization.

Thirdly, the results in this chapter are steps towards full integration of the different equidimensional and multidimensional mereotopologies. It makes their common and differing assumptions explicit and also establishes that *CODI* and *INCH* have equally expressive languages, the *CODI* language is strictly more expressive than that of the RCC: two distinct models of *CODI* may have elementarily equivalent RCC representations, i.e., the RCC cannot distinguish those models.

Finally, our work makes previously implicit dimension constraints explicit, such as those of the RCC

[1] The work in this chapter extends work previously published as [HG11a].

relations *external contact* EC, or of the INCH relations 'includes an equidimensional chunk of' $INCH$ and 'lower-dimensional element' EL.

The chapter is structured as follows. Section 8.1 studies the relationship to the RCC: Section 8.1.1 and 8.1.2 review the first-order axiomatization and its algebraic representation as Boolean Contact Algebras, Section 8.1.3 presents the main result—the construction of Boolean Contact Algebras as substructures of models of $CODI_{\updownarrow}$ that satisfy C-extensionality, and Section 8.1.4 discusses the reverse interpretation. Section 8.1.5 summarizes our findings about the relationship between $CODI_{\updownarrow}$ and the RCC. Section 8.2 studies the relationship to the INCH Calculus: Section 8.2.1 reviews the original axiomatization and Section 8.2.2 proposes a correction to match the intuitive understanding. The main results are contained in Sections 8.2.3 to 8.2.5, first testing of which axioms of $CODI$ are satisfied by the INCH Calculus and vice versa, and then introducing extensions to each theory to finalize the definable equivalence between extensions of $CODI_{\updownarrow}$ and the INCH Calculus. Section 8.2.6 summarizes our findings about the INCH Calculus as well as the relationship between the INCH Calculus and the $CODI$ theories.

8.1 Equidimensional mereotopology: The Region Connection Calculus

The Region Connection Calculus (RCC) is one of the most prominent equidimensional mereotopologies. However, there are two different theories that are both called RCC, a first-order theory arising from the complete set of axioms proposed in [RCC92] and a spatial calculi that only considers the jointly exhaustive, pairwise disjoint binary qualitative relations between regions. Of course, the spatial calculus can be expressed in first-order logic, but relies on fewer axioms and definitions than the original full first-order theory. We will focus here on the first-order theory and show how it can be constructed as an extension of $CODI_{\updownarrow}$.

8.1.1 The first-order theory of RCC

RCC as a first-order theory defines various relations using connection C as only primitive relation. In the first-order theory, additional functions, that is, a constant u denoting the unique universal element, a unary function $\mathrm{compl}(x)$, and the binary functions $\mathrm{sum}(x,y)$, $\mathrm{prod}(x,y)$, and $\mathrm{diff}(x,y)$, are defined [RCC92] which are not part of the spatial calculi. Moreover, a unary relation $Con(x)$ can be introduced to denote when an entity is self-connected; we do not include this definition here. The first-order theory comes in a continuous and a discrete version. The original continuous theory includes the axioms RCC8, which requires each region to have some nontangential proper part, and RCC4′, which requires a region x to be disconnected from all regions of whose complement x is a nontangential part, i.e., $\neg C(x,y) \leftrightarrow NTPP(x, \mathrm{compl}(y))$. The discrete version of RCC leaves RCC8 and RCC4′ out.

While the original theory allows different definitions of equivalence [Ste00], with the help of the additional axiom RCC-Ext [DWM99; DWM01] the theory is restricted to the intended mereological identity. The theory used in this section leaves out the axioms RCC8 and RCC′ since we are primarily concerned with atomic models, but we include RCC-Ext. In that way, the following definition has been adopted from [Ste00] to fit our purposes here. Notice that it does not include the function $\mathrm{diff}(x,y)$, which is definable as $\mathrm{prod}(x, \mathrm{compl}(y))$.

Definition 8.1. *A model of the RCC consists of a base set* $\mathbf{U} = \mathbf{R} \cup \{\mathrm{n}\}$ *with* $\{\mathrm{n}\} \notin \mathbf{R}$*, a distinguished* *element* $\mathrm{u} \in \mathbf{R}$*, a unary operation* $\mathrm{compl} : \mathbf{R} \setminus \{\mathrm{u}\} \to \mathbf{R} \setminus \{\mathrm{u}\}$*, binary operations* $\mathrm{sum} : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *and* $\mathrm{prod} : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \cup \{\mathrm{n}\}$*, and a binary relation* C *on* $\mathbf{R}$ *satisfying the axioms RCC1 – RCC7, RCC-Ext with the definitions RCC-P, RCC-PP, RCC-O, RCC-EC, and RCC-NTPP.*

(RCC1) $\forall x \in \mathbf{R}\ [C(x, x)]$

(RCC2) $\forall x, y \in \mathbf{R}\ [C(x, y) \to C(y, x)]$

(RCC3) $\forall x \in \mathbf{R}\ [C(x, \mathrm{u})]$

(RCC4) $\forall x \in \mathbf{R}, y \in \mathbf{R} \setminus \{\mathrm{u}\}\ [O(x, \mathrm{compl}(y)) \leftrightarrow \neg P(x, y)]$

(RCC5) $\forall x, y, z \in \mathbf{R}\ [C(x, \mathrm{sum}(y, z)) \leftrightarrow C(x, y) \lor C(x, z)]$

(RCC6) $\forall x, y, z \in \mathbf{R}\ [C(x, \mathrm{prod}(y, z)) \leftrightarrow \exists w \in \mathbf{R}\ [P(w, y) \land P(w, z)) \land C(x, w)]]$

(RCC7) $\forall x, y \in \mathbf{R}\ [\mathrm{prod}(x, y) \in \mathbf{R} \leftrightarrow O(x, y)]$

(RCC-Ext) $\forall x, y \in \mathbf{R}\ [P(x, y) \land P(y, x) \to x = y]$

(RCC-P) $\forall x, y \in \mathbf{R}\ [P(x, y) \leftrightarrow \forall z \in \mathbf{R}(C(z, x) \to C(z, y))]$

(RCC-PP) $\forall x, y \in \mathbf{R}\ [PP(x, y) \leftrightarrow (P(x, y) \land \neg P(y, x))]$

(RCC-O) $\forall x, y \in \mathbf{R}\ [O(x, y) \leftrightarrow \exists z \in \mathbf{R}\ (P(z, x) \land P(z, y))]$

(RCC-EC) $\forall x, y \in \mathbf{R}\ [EC(x, y) \leftrightarrow (C(x, y) \land \neg O(x, y))]$

(RCC-NTPP) $\forall x, y \in \mathbf{R}\ [NTPP(x, y) \leftrightarrow (PP(x, y) \land \neg \exists z \in \mathbf{R}\ (EC(z, x) \land EC(z, y))]$

Axiom Set 8.1: Axioms RCC1 – RCC7 and RCC-Ext and definitions RCC-P, RCC-PP, RCC-O, RCC-EC, and RCC-NTPP of the theory *RCC*.

We define the first-order theory as

$$RCC = \{\text{RCC1 – RCC7, RCC-Ext, RCC-P,RCC-PP, RCC-O, RCC-EC, RCC-NTPP}\}.$$

Other defined relations are not of primary interest here.

RCC as spatial calculus

The first-order theory *RCC* gives rise to a lattice of binary relation based on eight different base relations *DC*, *EC*, *PO*, *TPP*, *NTPP*, TPP^{-1}, $NTPP^{-1}$, and the equivalence relation *EQ*. The lattice of relations in depicted in Figure 8.1. Those base relations together with a weak composition table form what is known as a qualitative spatial calculus, usually referred to as RCC-8 [CCR93], while a subset of those relations consisting of *DC*, *EC*, *PO*, *PP*, and PP^{-1} defines the RCC-5 calculus, see e.g.R̃enz-Topology-02. Other sets of relations are also possible, see [DWM01]. Reasoning in this setting is restricted to consistency checking of a subset of the powerset of the binary relations, very much like a Constraint Satisfaction Problem (CSP). In this setting, all non-binary relations as well as all functions from the first-order theory *RCC* are completely ignored; we cannot reason about them. For our purposes here, we are not further concerned with spatial calculi.

8.1.2 The algebraic structures of the first-order RCC

Both [Ste00] and [DWM99] showed that the models of the strict RCC are Boolean algebras equipped with a special contact relation. If a model satisfies axiom RCC8, then its Boolean algebra is atomless and

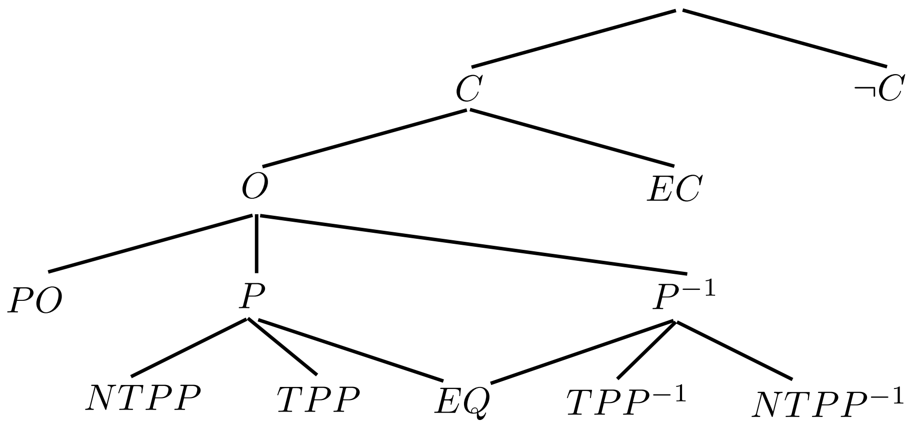


Figure 8.1: The lattice of jointly exhaustive, pairwise disjoint binary base relations from the RCC.

its contact relation becomes connected, i.e., the contact relation satisfies the connection axiom [DW04].

(RCC4′) $\forall x \in \mathbf{R}, y \in \mathbf{R} \setminus \{\mathrm{u}\}\ [C(x, \mathrm{compl}(y)) \leftrightarrow \neg NTPP(x, y)]$
(RCC8) $\forall x \in \mathbf{R}\, \exists y \in \mathbf{R}\ [NTPP(y, x)]$

Axiom Set 8.2: Axioms RCC4′ and RCC8 extending the theory *RCC*.

These algebraic structures provide an alternative way of studying the models of the first-order theory *RCC*. Each model of *RCC* is a Boolean contact algebra (BCA), that is, a Boolean algebra $\mathcal{L} = \langle L, 0, 1, +, \cdot \rangle$ equipped with a contact relation $\mathbf{C}$ that satisfies the axioms C0 − C4 and C-Ext, compare Definition 4.23. 0 and 1 denote the empty and the universal region; $^{\perp}$ denotes the unary function of complementation compl, and $+$ and $\cdot$ denote the binary functions sum and prod. Recall that we introduced the infix notation in Chapter 4 to distinguish the contact relation in a contact algebra from the contact relation in a logical axiomatization.

From previous work we know that the models of the theory *RCC* are definably equivalent to BCAs (as defined in Definition 4.23) and vice versa:

Theorem 8.1. *[DW04] Any model of RCC is definably equivalent to a BCA removed of its null element.*

The algebraic representation of RCC models as BCAs is particular helpful in separating the mereological substructure from the topological substructure of RCC models (compare Chapter 4): while the Boolean algebra captures the structure of the mereological relation of parthood, the topological relation of contact is axiomatized in BCAs by C0 − C4 and C-Ext. While we are primarily interested in the first-order theory *RCC*, we will exploit the more compact and mathematically more elegant axiomatization of the BCAs.

8.1.3 The RCC interprets substructures of models of $CODI_{\updownarrow}$

Now we want to examine how the theory *RCC* can be seen as an extension of our multidimensional mereotopology with up- and downwards closures, $CODI_{\updownarrow}$. Instead of showing an equivalence between two first-order theories, we utilize the metatheoretic results from Theorem 8.1 and Theorem 7.6 and simply provide a mapping between the two Boolean structures.

In order to restrict $CODI_{\updownarrow}$ to models of equidimensional mereotopology we cannot—somewhat counter-intuitively—completely prohibit entities of lower dimensions. Otherwise, 'external connection' EC in the RCC, a special case of superficial contact SC, must have an empty extension by SC-T4. This would reduce the mereotopology to a pure mereology with overlap as only contact relation. Instead, we base the mapping to BCAs on the set $\mathbf{B}_{\mathrm{u}}$ of entities of maximum dimension as defined in Theorem 7.6[2]. Recall that a maximum dimension is guaranteed to exist by D-A7, which must be satisfied because $CODI_{\updownarrow}$ is an extension of $DI_{\mathrm{linear-bounded}}$ (compare Lemma 7.44). Note that there exists a unique zero entity in $\mathbf{B}_{\mathrm{u}}$. We extract BCAs from the models of $CODI_{\updownarrow}$ as follows.

(C-E3) $MaxDim(x) \wedge MaxDim(y) \rightarrow [x = y \leftrightarrow \forall z[MaxDim(z) \rightarrow (C(z,x) \leftrightarrow C(z,y))]]$
(extensionality of C among regions of maximal dimension)

Axiom Set 8.3: Extension axiom C-E3 of $CODI_{\updownarrow}$.

Theorem 8.2. *Let $\mathcal{M}$ be a model of $CODI_{\updownarrow} \cup$ C-E3.*
Then the structure $\mathcal{M}_{\mathrm{u}} = (\langle \mathbf{B}_{\mathrm{u}}, \times, +, ', \mathrm{ze}, \mathrm{u}\rangle, \mathbf{C})$ with

1. *$\langle \mathbf{B}_{\mathrm{u}}, \times, +, ', \mathrm{ze}, \mathrm{u}\rangle$ defined as in Theorem 7.6(1) and*
2. *for all $x, y \in \mathbf{B}_{\mathrm{u}}$, $x\mathbf{C}y \Leftrightarrow \langle x, y\rangle \in \mathbf{C}_{\mathcal{M}}$*

is a BCA.

Proof. Assume $\mathcal{M}$ is a model of $CODI_{\updownarrow} \cup$ C-E3.
By Theorem 7.6(2), $\langle \mathbf{B}_{\mathrm{u}}, \times, +, ', \mathrm{ze}, \mathrm{u}\rangle$ is a Boolean algebra. For the structure $\mathcal{M}_{\mathrm{u}}$ to be a BCA, it remains to show that the defined contact relation $\mathbf{C}$ satisfies C0 – C4 and C-Ext (compare Definition 4.23).

(C0): $0 \neg \mathbf{C} x$.
Follows directly from C-T4.

(C1): $x \neq 0 \rightarrow x\mathbf{C}x$.
Follows directly from C-T2.

(C2): $x\mathbf{C}y \leftrightarrow y\mathbf{C}x$.
Follows directly from C-T3.

(C3): $x\mathbf{C}y \wedge y \leq z \rightarrow x\mathbf{C}z$.
We define the partial order for all $x, y \in \mathbf{B}_{\mathrm{u}}$ as in the proof of Theorem 7.6(2) as

$$x \leq y \Leftrightarrow \langle x, y\rangle \in \mathbf{P}_{\mathcal{M}} \text{ or } x \in \mathbf{ZEX}_{\mathcal{M}}.$$

Then, $\leq$ defines the partial order of the Boolean algebra $\langle \mathbf{B}_{\mathrm{u}}, \times, +, ', \mathrm{ze}, \mathrm{u}\rangle$. Because of C-T5 and EP-D, we conclude

$$\mathbf{P}(y,x) \wedge \mathbf{C}(z,y) \rightarrow \mathbf{C}(z,x)$$

for all $x, y \in \mathbf{B}_{\mathrm{u}}$ so that C3 is valid.

[2] We can use this method to construct a set of RCC models from a given model of $CODI_{\updownarrow}$ by recursively iterating through the dimensions, starting from highest to lowest. At each stage, we can construct an RCC model from the entities of maximal dimension and afterwards delete the entities of maximal dimension from the $CODI_{\updownarrow}$ model. Then use the entities of next-highest dimension, which automatically become the new entities of maximal dimension, to construct another RCC model. This results in a stack of RCC models, one for each dimension of an $CODI_{\updownarrow}$ model.

(C4): $x\mathbf{C}(y+z) \rightarrow x\mathbf{C}y \vee x\mathbf{C}z$.

Assume $x\mathbf{C}(y+z)$ for arbitrary $x, y, z \in \mathbf{B}_\mathrm{u}$.

Then by Theorem 7.6(1), $\mathbf{C}(x, y+z)$ so that by C-D there must exist a $w \in \mathbf{B}_\mathrm{u}$ such that $\mathbf{Cont}(w, x)$ and $\mathbf{Cont}(w, y+z)]$. By Corollary 7.4, $\mathbf{Cont}(w, y+z)]$ entails

$$\mathbf{Cont}(w, y) \vee \mathbf{Cont}(w, z) \ \vee [w = (w \cdot y) + (w - y) \wedge \mathbf{Cont}(w \cdot y, y) \wedge \mathbf{Cont}(w - y, z)]$$

That means some entity contained in w is contained in either y or z, hence by C-T5, either $\mathbf{C}(x, y)$ or $\mathbf{C}(x, z)$. Thus C4 follows immediately.

(C-Ext): $\forall z(z\mathbf{C}x \leftrightarrow z\mathbf{C}y) \leftrightarrow x = y$.

Is explicitly posited as C-E3 because $b \in \mathbf{B}_\mathrm{u} = \mathbf{MaxDim}_\mathcal{M}$.

Hence the structure $\mathcal{M}_\mathrm{u} = (\langle \mathbf{B}_\mathrm{u}, \mathrm{u}, \mathrm{ze}, \times, +, ' \rangle, \mathbf{C})$ is an BCA. □

In other words, for any model $\mathcal{M}$ of $CODI_\updownarrow \cup$ C-E3, the substructure $\mathcal{M}_\mathrm{u}$ is a BCA. Note that C-E3 postulates extensionality of C among all entities of maximal dimension, while the scope of the general axiom for C-extensionality (C-E2) is over all entities, regardless of their dimension. It is not difficult to see that C-E3 is much stronger than C-E2, that is:

Lemma 8.1. *$CODI_\updownarrow \cup$ C-E2 $\not\models$ C-E3*

Proof. Consider a model $\mathcal{M}$ with domain $\mathbf{M}$ consisting of three entities of highest dimension x, y, and $\mathrm{u} = x + y$, and three entities of lowest dimension $p, q, p+q \in \mathbf{MinDim}_\mathcal{M}$, and $\mathrm{ze} \in \mathbf{ZEX}_\mathcal{M}$. Let the containment relation be defined as

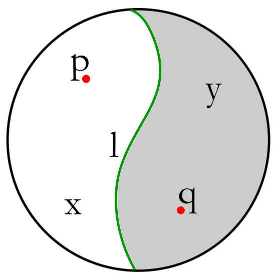
$$\mathbf{Cont}_\mathcal{M} = \{\langle x, \mathrm{u}\rangle, \langle y, \mathrm{u}\rangle, \langle l, x\rangle, \langle l, y\rangle, \langle l, \mathrm{u}\rangle, \langle p, x\rangle, \langle p, \mathrm{u}\rangle, \langle q, y\rangle, \langle q, \mathrm{u}\rangle, \langle p+q, \mathrm{u}\rangle\} \cup \{\langle x, x\rangle \, : \, x \in \mathbf{M} \setminus \{\mathrm{ze}\}\}.$$

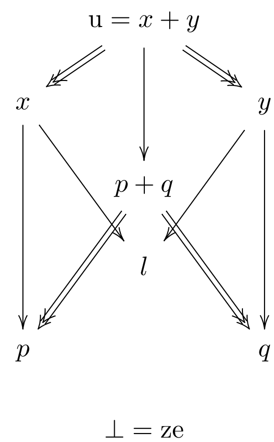
It can be easily verified that this is a model of $CODI_\updownarrow \cup$ C-E2, i.e., that the contact relation C is extensional among the seven entities of the model. See Figure 8.2 for a depiction of the model, the containment relations and the extension of C in $\mathcal{M}$ and in $\mathcal{M}_\mathrm{u}$. □

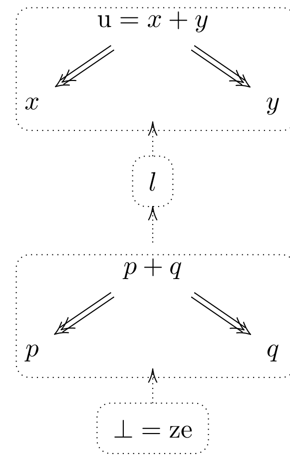
8.1.4 The difficulty of interpreting RCC models in *CODI* theories

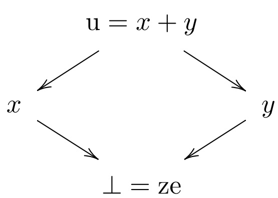
We have shown that every model of $CODI_\updownarrow \cup$ C-E3 has a substructure that is a BCA and thus a model of *RCC*. We are also interested in the reverse direction: is every BCA a substructure of a model of $CODI_\updownarrow \cup$ C-E3? In order to show this, we must show how we can extend an arbitrary given BCA so that is satisfies all axioms of $CODI_\updownarrow$ (note that C-E3 is satisfies trivially, we only introduced it because it is satisfied by all BCAs). The difficulty lies in the fact that a BCA may have entities in external contact *EC*, but in order to define a corresponding model that preserves this contact relation as *SC*, we have to introduce new entities of lower dimension because any kind of contact in *CODI* requires an actual shared entity to exist.

Notice that the RCC theory cannot distinguish whether three entities that are pairwise in external contact, are actually in contact at one or more points common to all three entities or whether they are only in pairwise external contact. Consider Figure 8.3 for an example of three different situations that are all identical with respect to their extension of C in RCC. Likewise, we cannot distinguish similar situations involving more than three entities. This lies in the nature of equidimensional mereotopology









C	u	x	y	ze
u	•	•	•	-
x	•	•	•	-
y	•	•	•	-
ze	-	-	-	-

C	u	x	y	l	p+q	p	q	ze
u	•	•	•	•	•	•	•	-
x	•	•	•	•	•	•	-	-
y	•	•	•	•	•	-	•	-
l	•	•	•	•	-	-	-	-
p+q	•	•	•	-	•	•	•	-
p	•	•	-	-	•	•	-	-
q	•	-	•	-	•	-	•	-
ze	-	-	-	-	-	-	-	-

Figure 8.2: A spatial configuration and its model $\mathcal{M}$ of *CODI*$_\updownarrow$ that satisfies C-E2 (C-extensionality) but does not satisfy C-E3 (C-extensionality among entities of highest dimension). The top row gives from left to right a pictorial view of the model, the containment relations (non-equidimensional containment is depicted as single lines, parthood as double lines), and the ordering by relative dimension (highest dimension at the top; each dashed box contains entities of equal dimensions, while parthood within a dimension is indicated by double lines). The middle row shows the Boolean lattice of the corresponding parthood relation in $\mathcal{M}_\mathrm{u}$ as defined by Theorem 7.6. In the bottom row, we have the non-extensional contact relation in $\mathcal{M}_\mathrm{u}$ on the left and the extensional contact relation in $\mathcal{M}$ on the right.

and its language that is strictly weaker than that of *CODI*, and will make the construction of a model of *CODI* from a BCA much easier. We can simply introduce a single entity of lowest dimension for each pair of entities in external contact in the BCA.

However, we have no guarantee that the resulting model of *CODI*$_\updownarrow$ correctly captures the intended structure. This is due to the fact that the RCC abstracts away much more information from the intended structure than *CODI*$_\updownarrow$. Once we have abstracted away additional knowledge—such as the relative dimensions of superficial contact—we cannot retrieve this information from the RCC model. In this light, *CODI*$_\updownarrow$ is a less abstract representation of the intended structures compared to the RCC, that is, *CODI* has a more expressive nonlogical language than *RCC*. For any given structure in the class of intended structures $\mathbb{M}$, the *CODI*$_\updownarrow$ model stores knowledge about how the entities of *all* dimension are topologically and mereologically related to one another. An RCC model of the same structure can only store knowledge about how the entities of a *single* dimension are topologically and mereologically related to one another. Whether there is a natural, "minimal", way to extend every *RCC* model to a model of *CODI*$_\updownarrow$ remains an open question (**Question 3**). One way to extend an *RCC* model, as

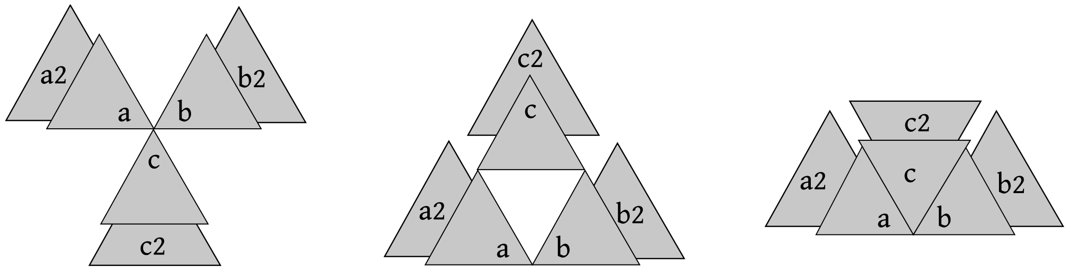


Figure 8.3: Three spatial configurations that are identical models of *RCC*. Only the atomic entities are labelled, the universal is the entire gray-shaded region (including its outer boundary). In the configuration on the left, a, b, and c are externally connected in a single point, while in the middle configuration a, b, and c are only pairwise externally connected (a with b, a with c, and b with c). In the right configuration, a, b, and c share a point, but each of a, b, and c share pairwise more points not shared with the third. In this sense, in the left configuration, the entire set of points (only a single point) that creates the external contact between a, b, and c is common to all, while in the right configuration only a subset of the points that create the external contact is common to all. In the middle configuration, the external contact is strictly pairwise, that is the subset of points common to all three of a, b, and c is empty.

suggested by Michael Winter, is via its topological representation from [DW05a]. Clusters of entities, which are maximal sets of pairwise connected entities very similar to the ultrafilters used in the Stone representation theorem, represent points in the constructed topological space. We could use those points as lower-dimensional entities we need to introduce in a corresponding model of $CODI_{\updownarrow}$.

8.1.5 Summary

In this section we have shown that the entities of maximal dimensions of a model of $CODI_{\updownarrow}$ form a Boolean Contact Algebra if the model satisfies C-E3, i.e., if the entities of maximal dimension are extensional with respect to the contact relation. Since it is well-known that Boolean Contact Algebras are isomorphic to models of the RCC, we immediately know that the RCC definably interprets the theory $CODI_{\updownarrow} \cup$ C-E3. Similar results could be obtained for all entities of some dimension in a model of $CODI_{\updownarrow} \cup$ C-E3 because the entities of any particular dimension form a Boolean lattice. Through this relative interpretation, we confirmed that $CODI_{\updownarrow}$ is indeed a mereotopology that differs from the RCC in that it may contain entities of various dimensions in a single model and can describe spatial relations between entities of different dimensions, which is not possible in the RCC. In that sense the RCC is a more abstract spatial model than $CODI_{\updownarrow}$, which limits our ability to reconstruct a model of $CODI_{\updownarrow}$ from a given model of the RCC. Any model of the RCC gives rise to a set of models of $CODI_{\updownarrow}$.

Notice that even though we did not discuss boundaries in $CODI_{\updownarrow}$, the extracted substructure of a model of $CODI_{\updownarrow} \cup$ C-E3 that is a RCC model may have entities in external contact. This is only the case because we restrict the substructure to the entities of highest dimension of the model of $CODI_{\updownarrow} \cup$ C-E3; for those we can define boundaries and thus also use the relations *EC* (external contact) and *TPP* and *NTPP* (tangential and nontangential proper parthood) without changing their intended meaning. In that sense, the RCC models constructed from models of $CODI_{\updownarrow} \cup$ C-E3 as in Theorem 8.2 are true RCC-8 models and not just RCC-5 models, i.e., the resulting models are not restricted to the base relations $\neg C$, PP, PP^{-1}, PO, and $=$. In general, the distinction between tangential and nontangential parthood is not definable in models of $CODI_{\updownarrow}$ as we will discuss in more detail in Chapter 9—but this is because tangential containment is not definable for entities of nonmaximal dimension.

8.2 Multidimensional mereotopology: The INCH Calculus

Apart from our work here, only two multidimensional mereotopology are fully axiomatized: the work by Galton [Gal96] and the INCH Calculus by Gotts [Got96]. Neither has been studied in much detail. Galton's theory is based on the notion of boundary, which is used to establish a dimensionality ordering which arises from the treatment of boundaries as entities of next-lowest dimension. Because we have no means yet to define boundaries, we cannot formally relate our theories from the *CODI* hierarchy to Galton's work.

Gott's work, on the other hand, is based on a single primitive relation *INCH*, which is used to define a concept of relative dimensionality *GED* similar to our relation $\geq_{\text{dim}}$. In some sense, $\mathit{INCH}(x, y)$ is a multidimensional topological relation. It is a dimensionally restricted form of contact: the second argument y must share something with the first argument x that is of the dimension of y. This is not too different from our approach in *CODI* where containment *Cont* is a multidimensional mereological relation, though it is the most general such relation (CD-A1 does not restrict containment to certain pairs of entities, but instead only captures a property that all pairs of entities in containment relation satisfy). Naturally, the question arises how the theories in the *CODI* hierarchy relate to the INCH Calculus. To understand this relationship, we will integrate the INCH Calculus into the *CODI* hierarchy. More precisely, we find an extension of the INCH Calculus that is definably equivalent to a theory in the *CODI* hierarchy. This will not only make the relationship between the INCH Calculus and the *CODI* hierarchy clear, but at the same time it will help us better understand ontological assumptions that are implicit in the INCH Calculus and, to a lesser degree, in the *CODI* theories.

The section is structured as following. First, we introduce the INCH Calculus and prove some properties about it. We also give axioms that map the primitive relation *INCH* of the INCH Calculus to the primitive relations of *CODI* and vice versa. As it turns out, the combination of those mappings prove a property, called I-A7′, that is not provable from the INCH Calculus alone. Thus, the intuitive mappings of the INCH Calculus to *CODI* actually nonconservatively extend the original INCH Calculus. However, one would expect I-A7′ to hold in the INCH Calculus. Therefore, we extend its original axiomatization $\mathit{INCH}_{\text{original}}$ by this property to obtain the theory $\mathit{INCH}_{\text{calculus}}$. In Subsection 8.2.3 we show that the so-extended INCH Calculus together with the additional axiom I-E1 definably interprets CODI^0. In Subsection 8.2.4 we then show that the theory $\mathit{CODI}_{\updownarrow}$, the theory *CODI* extended by mereological closure operations, together with a new axiom C-E4 definably interprets the corrected INCH Calculus. Finally, Subsection 8.2.5 identifies two more properties, I-E2 and I-E3, that are necessary to prove all axioms of $\mathit{CODI}_{\updownarrow} \cup$ C-E4 from $\mathit{INCH}_{\text{calculus}}$. Hence, the theories $\mathit{CODI}_{\updownarrow} \cup$ C-E4 and $\mathit{INCH}_{\text{calculus}} \cup$ {I-E1 – I-E3} are definably equivalent. In the summary section, we will discuss how $\mathit{INCH}_{\text{calculus}}$ and $\mathit{CODI}_{\updownarrow}$ differ in their ontological assumptions, which manifest themselves in I-E1 – I-E3 and C-A4.

8.2.1 The original INCH Calculus: $\mathit{INCH}_{\text{original}}$

The INCH Calculus is a multidimensional mereotopology whose axiomatization revolves around the only primitive relation $\mathit{INCH}(x, y)$ with the intended interpretation of 'x includes a chunk (a part) of y'. We define

$$\mathit{INCH}_{\text{original}} = \{\text{I-D1} - \text{I-D9},\ \text{I-A1} - \text{I-A10}\}$$

to denote Gott's original axiomatization [Got96]. We maintain the original numbering of definitions and axioms.

(I-D1) $CS(x,y) \leftrightarrow \forall z[INCH(x,z) \rightarrow INCH(y,z)]$ (x is a constituent of y)

(I-D2) $OV(x,y) \leftrightarrow INCH(x,y) \wedge INCH(y,x)$ (overlap)

(I-D3) $CO(x,y) \leftrightarrow \exists z\,[\neg ZEX_I(z) \wedge CS(z,x) \wedge CS(z,y)]$ (connection)

(I-D4) $CH(x,y) \leftrightarrow INCH(x,y) \wedge \forall z\,[OV(x,z) \rightarrow OV(y,z)]$
(x is a chunk (equidimensional part) of y)

(I-D5) $EL(x,y) \leftrightarrow CS(x,y) \wedge \neg INCH(x,y)$ (x is a (lower-dimensional) element of y)

(I-D6) $ZEX_I(x) \leftrightarrow \neg INCH(x,x)$ (zero entity)

(I-D7) $GED(x,y) \leftrightarrow ZEX_I(y) \vee \exists z\,[INCH(x,z) \wedge INCH(z,y)]$ (greater or equal dimension)

(I-D8) $ED(x,y) \leftrightarrow GED(x,y) \wedge GED(y,x)$ (equal dimension)

(I-D9) $GD(x,y) \leftrightarrow GED(x,y) \wedge \neg GED(y,x)$ (greater dimension)

(I-A1) $x = y \leftrightarrow \forall z\,[INCH(x,z) \leftrightarrow INCH(y,z)]$ (extensionality)

(I-A2) $x = y \leftrightarrow \forall z\,[INCH(z,x) \leftrightarrow INCH(z,y)]$ (extensionality)

(I-A3) $INCH(x,y) \rightarrow INCH(x,x)$ (*INCH* reflexive)

(I-A4) $GED(x,y) \vee GED(y,x)$ (dimensional comparability)

(I-A5) $GED(x,y) \wedge GED(y,z) \rightarrow GED(x,z)$ (*GED* transitive)

(I-A6) $INCH(x,y) \wedge INCH(y,z) \wedge INCH(z,x) \rightarrow INCH(y,x)$ (*INCH* transitive)

(I-A7) $INCH(x,y) \rightarrow \exists z\,[CS(z,x) \wedge OV(z,y)]$
($INCH(x,y)$ requires a constituent of x to overlap with y)

(I-A8) $CH(x,y) \rightarrow CS(x,y)$ (a chunk is a constituent)

(I-A9) $ED(x,y) \rightarrow \exists z \forall w\,[INCH(z,w) \leftrightarrow INCH(x,w) \vee INCH(y,w)]$
(a sum $z = x + y$ exists for equidimensional entities x and y)

(I-A10) $ED(x,y) \rightarrow \exists z \forall w\,[INCH(z,w) \leftrightarrow \exists v\,[INCH(v,w) \wedge CH(v,x) \wedge \neg OV(v,y)]]$
(a difference $z = x - y$ exists for equidimensional entities x and y)

Axiom Set 8.4: Axioms I-A1 – I-A10 and definitions I-D1 – I-D9 of the theory $INCH_{\text{calculus}}$.

In addition to the theorems I-T1 to I-T4 already proved in [Got96], we prove I-T5 to I-T13, which will come in handy throughout the section. I-T5 confirms that a chunk of an entity is an equidimensional constituent thereof; I-T6 shows that the zero entity is a constituent of every entity; I-T7 shows that no nonzero entity can be a constituent of the zero entity; I-T8 confirms a requirement for an entity to *INCH* another; I-T9 shows that *INCH* is symmetric for equidimensional entities; I-T10 shows that if all chunks of an entity x are constituents of another entity y, then the entity x itself is a constituent of y; I-T11 proves that constituency can be defined in terms of constituents; I-T12 shows that 'being a chunk' requires monotonicity with respect to overlap; and I-T13 proves transitivity of 'being a chunk'.

(I-T1) $INCH(x,y) \rightarrow GED(x,y)$ (Theorem 1 of [Got96])

(I-T2) $OV(x,y) \rightarrow ED(x,y)$ (Theorem 2 of [Got96])

(I-T3) $CS(x,y) \leftrightarrow EL(x,y) \vee CH(x,y)$ (Theorem 3 of [Got96], depends on I-A8)

(I-T4) $x = y \leftrightarrow \forall z[OV(z,x) \leftrightarrow OV(z,y)]$ (Theorem 4 of [Got96], extensionality of OV)

(I-T5) $CH(x,y) \leftrightarrow CS(x,y) \wedge ED(x,y) \wedge \neg ZEX(x)$ (a chunk is an equidimensional constituent)

(I-T6) $ZEX_I(x) \rightarrow CS(x,y)$ (the zero entity is a constituent of every entity)

(I-T7) $ZEX_I(x) \wedge \neg ZEX_I(y) \rightarrow \neg CS(y,x)$ (no nonzero entity is a constituent of the zero entity)

(I-T8) $\exists z[CS(z,x) \wedge CH(z,y)] \rightarrow INCH(x,y)$

(if x has a chunk of y as constituent, then x includes a chunk of y, i.e., $INCH(x,y)$)

(I-T9) $ED(x,y) \wedge INCH(x,y) \rightarrow INCH(y,x)$

(for entities x,y of equal dimension, $INCH(x,y)$ implies $INCH(y,x)$)

(I-T10) $\forall z[CH(z,x) \rightarrow CS(z,y)] \rightarrow CS(x,y)$

(if every chunk of x is a constituent of y, then x is a constituent of y)

(I-T11) $\forall z[CS(z,x) \rightarrow CS(z,y)] \leftrightarrow CS(x,y)$

(x is a constituent of y iff every constituent of x is a constituent of y)

(I-T12) $CH(x,y) \rightarrow \forall z[OV(x,z) \rightarrow OV(y,z)]$

(any entity z that a chunk x of y overlaps, y must also overlap)

(I-T13) $CH(x,y) \wedge CH(y,z) \rightarrow CH(x,z)$ (CH transitive)

Lemma 8.2. $INCH_{\text{original}} \models$ *{I-T1 – I-T13}*

The objective of this section is to find the theory in the *CODI* hierarchy that is definably equivalent to the INCH Calculus. Showing that such as theory, let us call it T for now, is definably equivalent to $INCH_{\text{original}}$ involves two separate proofs: (1) we express the relation $INCH$, which is the only primitive of $INCH_{\text{original}}$, in terms of the primitive and defined relations of T in a so-called mapping axiom I-M1. Ideally, we can show that all axioms of the INCH Calculus are provable from T together with the mapping axiom:

$$T \cup \text{I-M1} \models INCH_{\text{original}}$$

That would establish that the INCH Calculus is interpreted by T. For the reverse direction, we express the primitive relations ZEX, $<_{\text{dim}}$, and $Cont$ of T, a theory in the *CODI* hierarchy and thus defined by the primitives in the nonlogical language thereof, in terms of $INCH$ and its defined relations from $INCH_{\text{original}}$ resulting in three mapping axioms I-M1′, I-M2′, and I-M3′. Ideally, we then want to show that all axioms of T are provable from $INCH_{\text{original}}$ together with these mapping axioms:

$$INCH_{\text{original}} \cup \{\text{I-M1', I-M2', I-M3'}\} \models T$$

That would show that T is interpreted by $INCH_{\text{original}}$.

As it turns out, we cannot quite achieve the objective in its ideal form. $INCH_{\text{original}}$ entails some sentences that are not entailed by any of our *CODI* theories, while even the least restrictive *CODI* theories entail sentences that are not entailed by $INCH_{\text{original}}$. Hence, we nonconservatively extend either theory by the missing axioms (without changing their nonlogical languages), and establish mutual interpretability and thus definable equivalence between the two extended theories. This helps us to understand the relationship between $INCH_{\text{original}}$ and the theories in the *CODI* hierarchy.

As candidate mappings between *CODI* and $INCH_{\text{original}}$ we have extracted I-M1 (expressing the only primitive relation *INCH* of $INCH_{\text{original}}$ in terms of relations of *CODI*) and I-M1′ – I-M3′ (expressing the primitive relations of *CODI* in terms of relations of $INCH_{\text{original}}$) from the textual description of the intended models of the INCH Calculus.

(I-M1) $INCH(x,y) \leftrightarrow \exists z\,[Cont(z,x) \land Cont(z,y) \land z =_{\text{dim}} y]$ (mapping of *INCH*)

Axiom Set 8.5: Mapping axiom I-M1 from *CODI* theories to *INCH* theories.

(I-M1′) $Cont(x,y) \leftrightarrow CS(x,y) \land \neg ZEX_I(x)$ (mapping of *Cont*)

(I-M2′) $ZEX(x) \leftrightarrow ZEX_I(x)$ (mapping of *ZEX*)

(I-M3′) $x <_{\text{dim}} y \leftrightarrow GED(y,x) \land \neg GED(x,y)$ (mapping of $<_{\text{dim}}$)

Axiom Set 8.6: Mapping axioms I-M1′ – I-M3′ from *INCH* theories to *CODI* theories.

8.2.2 The corrected INCH Calculus: $INCH_{\text{calculus}}$

Our attempts to find a theory definably equivalent to $INCH_{\text{original}}$ revealed that the following sentence, which should intuitively hold in the INCH Calculus according the verbal description of its relations, is not a theorem of the original axiomatization:

(I-A7′) $INCH(x,y) \rightarrow \exists z[CS(z,x) \land CH(z,y)]$
($INCH(x,y)$ requires x to include some chunk of y as constituent)

Axiom Set 8.7: Axiom I-A7′ extending $INCH_{\text{original}}$ to the theory $INCH_{\text{calculus}}$.

I-A7′ is slightly stronger than the original axiom I-A7. If we replaced I-A7 by I-A7′ in $INCH_{\text{original}}$, then I-A7 is entailed. The sentence I-A7′ can be derived by substitutions of the mapping axioms I-M1, I-M1′, I-M2′, and I-M3′. First notice that I-M1′ and I-M2′ have the consequence:

(I-M1′′) $\neg ZEX(x) \rightarrow [CS(x,y) \leftrightarrow Cont(x,y)]$ (theorem of I-M1′, I-M2′)

When we substitute the mappings I-M3′, and I-M1′′ into I-M1, we get the following bidirectional implications which should always hold in the INCH Calculus according to the verbal descriptions of the defined relations *CS*, *GED*, and *ZEX* in [Got96]:

$$\begin{aligned}
INCH(x,y) &\leftrightarrow \exists z[Cont(z,x) \land Cont(z,y) \land z =_{\text{dim}} y] && \text{(I-M1)}\\
&\leftrightarrow \exists z[Cont(z,x) \land Cont(z,y) \land \neg ZEX(z) \land z =_{\text{dim}} y] && \text{(C-A4)}\\
&\leftrightarrow \exists z[CS(z,x) \land CS(z,y) \land \neg ZEX(z) \land z =_{\text{dim}} y] && \text{(I-M1'')}\\
&\leftrightarrow \exists z[\neg ZEX(z) \land CS(z,x) \land CS(z,y) \land z \leq_{\text{dim}} y \land y \leq_{\text{dim}} z] && \text{(D-D3,4)}\\
&\leftrightarrow \exists z[\neg ZEX(z) \land CS(z,x) \land CS(z,y) \land GED(y,z) \land GED(z,y)] && \text{(I-M3')}\\
&\leftrightarrow \exists z[\neg ZEX(z) \land CS(z,x) \land CS(z,y) \land ED(y,z)] && \text{(I-D8)}\\
&\leftrightarrow \exists z[CS(z,x) \land CH(z,y)] && \text{(I-T5)}
\end{aligned}$$

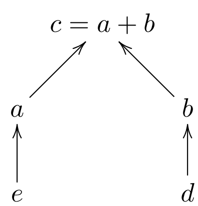


Figure 8.4: The mereological relation *CH* of a model of $INCH_{\text{original}}$ that violates I-A7′. We have $CH(x, y)$ if and only if there is a directed path of length ≥ 0 from x to y in the structure.

While we proved that the direction $\leftarrow$ of the last line is a theorem of $INCH_{\text{original}}$ (I-T8), the direction $\rightarrow$ of the last line, captured by I-A7′, is not valid in $INCH_{\text{original}}$.

Lemma 8.3. $INCH_{\text{original}} \not\models$ *I-A7′*

Proof. A counterexample has been automatically generated by the automated model finder Paradox3, see inch/consistency/inch_original_notI-PA7.clif.

Consider a model $\mathcal{M}$ of $INCH_{\text{original}}$ with domain $\mathbf{M} = \{a, b, c, d, e, \text{ze}\}$. The model is completely specified by the extension of *INCH*:

$$\begin{aligned}\mathbf{INCH}_{\mathcal{M}} =& \{\langle x, x\rangle \mid x \in \mathbf{M} \setminus \{\text{ze}\}\} \cup \\ & \{\langle a, b\rangle, \langle a, c\rangle, \langle a, e\rangle, \langle b, a\rangle, \langle b, c\rangle, \langle b, d\rangle, \langle c, a\rangle, \langle c, b\rangle, \langle c, d\rangle, \langle c, e\rangle, \langle d, b\rangle, \langle d, c\rangle\}\end{aligned}$$

In particular we have

$$\begin{aligned}\mathbf{ED}_{\mathcal{M}} =& \{\langle x, y\rangle \mid x \in \mathbf{M} \setminus \{\text{ze}\}\} \cup \{\langle \text{ze}, \text{ze}\rangle\} \\ \mathbf{CH}_{\mathcal{M}} =& \{\langle a, a\rangle, \langle a, c\rangle, \langle b, b\rangle, \langle b, c\rangle, \langle c, c\rangle, \langle d, b\rangle, \langle d, c\rangle, \langle d, d\rangle \langle e, a\rangle, \langle e, c\rangle, \langle e, e\rangle\} \\ \mathbf{CS}_{\mathcal{M}} =& \mathbf{CH}_{\mathcal{M}} \cup \{\langle \text{ze}, x\rangle \mid x \in \mathbf{M} \setminus \{\text{ze}\}\}\end{aligned}$$

In other words, all entities except for ze are of equal dimension and the *CH* relation is thus equivalent to the *CS* except that ze is a constituent of any no-zero entity, but ze is not a chunk of any entity. The extension of the *CH* relation is displayed in Figure 8.4.

It is easy to verify that this is a model of $INCH_{\text{original}}$. However, we have

$$\langle b, a\rangle \in \mathbf{INCH}_{\mathcal{M}} \text{ and for all } z \in \mathbf{M}, \big(\langle z, a\rangle \in \mathbf{CS}_{\mathcal{M}} \text{ or } \langle z, b\rangle \notin \mathbf{CH}_{\mathcal{M}}\big).$$

Hence

$$\mathcal{M} \models \mathbf{INCH}(b, a) \wedge \neg\exists z[\mathbf{CS}(z, a) \wedge \mathbf{CH}(z, b)],$$

so that I-A7′ is clearly not satisfied in $\mathcal{M}$. The only chunks of b are b and d, but neither b nor d is a constituent of a. □

To rule out unintended models of the INCH Calculus that violate I-A7′, we extend $INCH_{\text{original}}$ by I-A7′ and work from now on with the theory

$$INCH_{\text{calculus}} = INCH_{\text{original}} \cup \text{I-A7}'$$

instead. This extension seems particularly reasonable given that I-A7′ captures an intuitive property that the verbal description of the relation $INCH(x, y)$—'x includes a chunk of y'—suggests. This new axiom I-A7′ strengthens I-A7.

Lemma 8.4. $INCH_{\text{calculus}} \setminus$ *{I-A7}* $\models$ *I-A7*

Proof. Consider the following logical derivation:

$$
\begin{aligned}
INCH(x, y) &\to \exists z[CS(z, x) \wedge CH(z, y)] && \text{I-A7}' \\
&\to \exists z[CS(z, x) \wedge CH(z, y) \wedge INCH(z, y)] && \text{I-D4} \\
&\to \exists z[CS(z, x) \wedge CH(z, y) \wedge INCH(z, z)] && \text{I-A3} \\
&\to \exists z[CS(z, x) \wedge CH(z, y) \wedge OV(z, z)] && \text{I-D2} \\
&\to \exists z[CS(z, x) \wedge \wedge OV(y, z)] && \text{I-D4} \\
&\to \exists z[CS(z, x) \wedge OV(z, y)] && \text{I-D2}
\end{aligned}
$$

The last line is I-A7. □

From $INCH_{\text{calculus}}$ we can prove the property I-T14, which is not provable from $INCH_{\text{original}}$.

(I-T14) $OV(x, y) \to \exists z[CH(z, x) \wedge CH(z, y)]$ (overlapping entities share a chunk)

Lemma 8.5. $INCH_{\text{calculus}} \models$ *I-T14*

For the remainder of the chapter we will exclusively work with the corrected version $INCH_{\text{calculus}}$ of the INCH Calculus.

8.2.3 $INCH_{\text{calculus}} \cup$ I-E1 interprets $CODI^0$

In order to find a theory in the language of *CODI* that is definably equivalent to $INCH_{\text{calculus}}$, we start off with exploring which axioms of *CODI* are provable and which are not provable from $INCH_{\text{calculus}}$ together with the mapping axioms I-M1′, I-M2′ and I-M3′. Thereby we identify a subtheory of $CODI^0$ interpretable by $INCH_{\text{calculus}}$. At the same time we want to identify the subset of axioms of $INCH_{\text{calculus}}$ that are sufficient to prove all the axioms of $CODI^0$ that are provable from $INCH_{\text{calculus}}$. For this purpose we define two subtheories of $INCH_{\text{calculus}}$, namely the theories

$$INCH_{\text{weak}} = \{\text{I-A1} - \text{I-A5, I-D1} - \text{I-D6}\}$$

and

$$INCH_{\text{weak}-\text{closed}} = INCH_{\text{weak}} \cup \{\text{I-A9, I-A10}\},$$

and attempt to prove the axioms of $CODI^0$ from them. Recall that $CODI^0$ is defined by the axioms of CO_{basic} and of DI_{linear} together with CD-A1 and Z-A1.

In the presence of the mapping axioms I-M1′ to I-M3′ all the axioms of basic containment, CO_{basic}, are provable in $INCH_{\text{weak}}$:

Lemma 8.6. $INCH_{\text{weak}} \cup$ *{I-M1′ – I-M3′}* $\models CO_{\text{basic}}$

However, in the presence of the definition of contact (C-D), C-A5 is not provable even in the stronger theory $INCH_{\text{weak}-\text{closed}}$, we can automatically generate a counterexample to C-A5.

Lemma 8.7. $INCH_{\text{weak-closed}} \cup$ *{I-M1′ – I-M3′, C-D}* $\not\models$ *C-A5*

Proof. Counterexample provided in inch/consistency/inch_calculus_notC-A5.clif. ☐

The axioms of the theory of linear relative dimension, DI_{linear}, are all provable with the exception of D-A6. Again, we use the mapping axioms as well as the definitions D-D1 to D-D6.

Lemma 8.8. $INCH_{\text{weak}} \cup$ *{I-M1′ – I-M3′}* $\cup$ *{D-D1 – D-D6}* $\models$ *{D-A1 – D-A5}*

D-A6 is not provable even in the complete INCH Calculus $INCH_{\text{calculus}}$.

Lemma 8.9. $INCH_{\text{calculus}} \cup$ *{I-M1′ – I-M3′}* $\cup$ *{D-D1 – D-D6}* $\not\models$ *D-A6*

Proof. All counterexamples have an infinite domain and must be constructed manually. They consist of an infinite chain of entities with successively lower dimension but each having a dimension greater than that of the zero entity. ☐

CD-A1 is also provable in $INCH_{\text{weak}}$ with I-M1′ and I-M3′.

Lemma 8.10. $INCH_{\text{weak}} \cup$ *{I-M1′ – I-M3′}* $\cup$ *{D-D1 – D-D6}* $\models$ *CD-A*

Finally, Z-A1 is provable in $INCH_{\text{weak-closed}}$ with the mapping axioms, but is not provable in the weaker theory $INCH_{\text{weak}}$. The automated proof require I-A10 (existence of differences for equidimensional entities).

Lemma 8.11. $INCH_{\text{weak-closed}} \cup$ *{I-M1′ – I-M3′}* $\models$ *Z-A1.*

Lemma 8.12. $INCH_{\text{weak}} \cup$ *{I-M1′ – I-M3′}* $\not\models$ *Z-A1*

Proof. Counterexample provided in inch/theorems/inch_weak_Z-A1.clif. ☐

That means all the axioms of $CODI^0$ except for D-A6 are provable from the theory $INCH_{\text{weak-closed}}$. Except for Z-A1, they are all provable in the less restricted theory $INCH_{\text{weak}}$. That means $INCH_{\text{weak}}$ is an extension of $CODI_{\text{unbounded}}$.

Arguably, a minimal dimension should also exist for models of the INCH Calculus. Therefore, we extend $INCH_{\text{calculus}}$ by I-E1, which will not only ease the mapping between the INCH Calculus and $CODI^0$ but also contributes to a more complete axiomatization of the INCH Calculus. As Gotts [Got96] intended the INCH Calculus to formalize common-sense topology, it is only naturally to assume a minimal dimension. Equally, we could argue for axioms forcing the dimensions to be a discrete ordering (similar to D-A8 and D-A9). But this exceeds our primary intention of establishing a mapping between the INCH Calculus and one of our theories.

(I-E1) $\exists x[\neg ZEX_I(x) \land \forall y(\neg ZEX_(y) \rightarrow GED(y,x))]$
(a nonzero entity of minimal dimension must exist)

Axiom Set 8.8: Extension axiom I-E1 of $INCH_{\text{calculus}}$.

Then we can prove D-D6 from $INCH_{\text{weak}} \cup$ I-E1.

Lemma 8.13. $INCH_{\text{weak}} \cup$ *I-E1* $\cup$ *{I-M1′ – I-M3′}* $\cup$ *{D-D1 – D-D6}* $\models$ *D-A6*

Summarily, we further obtain the following two corollaries.

Corollary 8.1. $INCH_{\text{weak}} \cup$ *I-E1* $\cup$ *{I-M1′ – I-M3′}* $\models$ *CODI*

Proof. Follows immediately from Lemmas 8.6, 8.8, 8.10, and 8.13. □

Corollary 8.2. $INCH_{\text{weak-closed}} \cup$ *I-E1* $\cup$ *{I-M1′ – I-M3′}* $\models CODI^0$

Proof. Follows immediately from Lemmas 8.6, 8.8, 8.10, 8.11, and 8.13. □

Thus, in the *CODI* hierarchy, $CODI^0$ is the weakest theory of interest for a representation of the INCH Calculus extended by I-E1. Note that the axioms I-A6, I-A7, and I-A8 are not necessary to prove any of the axioms of $CODI^0$. We will explore those axioms in more detail later.

8.2.4 $CODI_{\updownarrow} \cup$ C-E4 interprets the INCH Calculus

Unfortunately, the theory $CODI^0$ is not strong enough to interpret the INCH Calculus given the mapping axiom I-M1 for *INCH*. There exist sentences that are not entailed by $CODI^0$, but whose translations are theorems of $INCH_{\text{calculus}}$. Before we identify such sentences, we first identify the subset of axioms of $INCH_{\text{calculus}}$ that are provable from $CODI^0$ given I-M1.

The axioms I-A3, I-A6, and I-A7 of the INCH Calculus become immediately provable in the theory $CODI^0$ in the presence of the definitions I-D1 to I-D5 together with the mapping I-M1.

Lemma 8.14. $CODI^0 \cup$ *{I-D1 – I-D9, I-M1}* $\models$ *{I-A3, I-A6, I-A7}*

Furthermore, the relation *ZEX* in $CODI^0$ maps to the relation ZEX_I of the INCH Calculus. Because of this interchangeability of *ZEX* and ZEX_I, we subsequently denote both by *ZEX* unless the distinction is relevant.

(I-M2) $ZEX_I(x) \leftrightarrow ZEX(x)$	(mapping of ZEX_I)
(I-M3) $GED(y, x) \leftrightarrow x \leq_{\text{dim}} y$	(mapping of *GED*)

Axiom Set 8.9: Mapping axioms I-M2 and I-M3 from *CODI* theories to *INCH* theories.

Lemma 8.15. $CODI^0 \cup$ *{I-D1 – I-D9, I-M1}* $\models$ *I-M2*

Surprisingly, I-M3 (equivalent to I-M3′) is not provable.

Lemma 8.16. $CODI^0 \cup$ *{I-D1 – I-D9, I-M1, I-M2}* $\not\models$ *I-M3*

Proof. The following computation proves the direction $GED(y, x) \rightarrow x \leq_{\text{dim}} y$:

$$
\begin{aligned}
& GED(y, x) \\
\Leftrightarrow\ & ZEX_I(x) \vee \exists z[INCH(y, z) \wedge INCH(z, x)] && \text{(I-D7)} \\
\Leftrightarrow\ & ZEX(x) \vee \exists z[\exists v(Cont(v, y) \wedge P(v, z)) \wedge \exists w(Cont(w, z) \wedge P(w, x))] && \text{(I-M1, I-M2)} \\
\Rightarrow\ & ZEX(x) \vee \exists z[y \geq_{\text{dim}} z \wedge z \geq_{\text{dim}} x] && \text{(EP-D, CD-A1)} \\
\Rightarrow\ & ZEX(x) \vee y \geq_{\text{dim}} x && \text{(D-A3)} \\
\Rightarrow\ & y \geq_{\text{dim}} x && \text{(D-A5)}
\end{aligned}
$$

Now consider a model of $CODI^0$ that contains two entities that are not in contact and thus do not include a chunk of each other, but are still dimensionally comparable. Such model is a counterexample to I-M3, it violates the direction $GED(y,x) \leftarrow x \leq_{\dim} y$. See inch/theorems/codi_linear_I-M3.clif for the counterexample. □

Thus, I-A4 and I-A5 are also not directly provable from $CODI^0$.

To establish a correspondence between GED and $\leq_{\dim}$ we require the additional axiom C-E4. It essentially requires two dimensionally comparable entities to have parts that are contained in a common entity.

(C-E4) $x \leq_{\dim} y \rightarrow \big[ZEX(x) \vee \exists z,v,w[P(v,x) \wedge Cont(v,z) \wedge P(w,z) \wedge Cont(w,y)]\big]$
(manifestation of relative dimension through a common entity z)

Axiom Set 8.10: Extension axiom C-E4 of *CODI*.

Lemma 8.17. $CODI^0 \cup$ *C-E4* $\cup$ *{I-D1 – I-D9, I-M1, I-M2}* $\models$ *I-M3*

Proof. We already showed $GED(y,x) \rightarrow x \leq_{\dim} y$ in Lemma 8.16.

The missing direction $GED(y,x) \leftarrow x \leq_{\dim} y$ can be proved automatically. □

With the help of I-M3 we can prove I-E1, I-A4, and I-A5.

Lemma 8.18. $CODI^0 \cup$ *{I-D1 – I-D9, I-M1}* $\models$ *I-E1*

Lemma 8.19. $CODI^0 \cup$ *C-E4* $\cup$ *{I-D1 – I-D9, I-M1 – I-M3}* $\models$ *{I-A4, I-A5}*

Given the mappings I-M1, I-M2, and I-M3 in $CODI^0 \cup$ C-E4 the other defined dimension relations *ED* and *GD* of the INCH Calculus correspond to our dimension predicates $=_{\dim}$ and $>_{\dim}$, respectively (I-M4, I-M5). We further verify that partial overlap *PO* is equivalent to *OV* in the INCH Calculus (I-M6).

(I-M4) $ED(x,y) \leftrightarrow x =_{\dim} y$ (mapping of *ED*)

(I-M5) $GD(x,y) \leftrightarrow y <_{\dim} x$ (mapping of *GD*)

(I-M6) $OV(x,y) \leftrightarrow PO(x,y)$ (mapping of *OV*)

Axiom Set 8.11: Mapping axioms I-M4 – I-M6 from *CODI* theories to *INCH* theories.

Lemma 8.20. $CODI^0 \cup$ *C-E4* $\cup$ *{I-D1 – I-D9, I-M1 – I-M3}* $\models$ *{I-M4, I-M5, I-M6}*

Other mappings, in particular those for *CS* and *CH*, are not yet provable from the theory $CODI^0$ extended by C-E4 and the mappings I-M1 to I-M6. *CS* and *CH* rely on extensionality of *INCH* (I-A1 and I-A2) and extensionality of *OV* (I-T4) which are so far not guaranteed in $CODI^0$. To prove extensionality of *INCH* (I-A1, I-A2) from $CODI^0$ we have to further extend the theory by closures under intersection and differences (Int-A1 to Int-A4, Dif-A1 to Dif-A4), effectively using the theory $CODI_{\downarrow} = CODI^0 \cup$ {Int-A1 – Int-A4, Dif-A1 – Dif-A4, Inc-D, Con-D} that we defined in Chapter 7.

From this more restricted theory together with C-E4 and the mapping I-M1, we can prove the mappings I-M7 and I-M8 as well as extensionality of *INCH*.

(I-M7)	$CS(x,y) \leftrightarrow Cont(x,y) \vee ZEX(x)$	(mapping of CS)
(I-M8)	$CH(x,y) \leftrightarrow P(x,y)$	(mapping of CH)

Axiom Set 8.12: Mapping axioms I-M7 and I-M8 from *CODI* theories to *INCH* theories.

Lemma 8.21. *$CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M6} $\models$ I-M7*

Proof. An automatic proof of the direction $Cont(x,y) \vee ZEX(x) \rightarrow CS(x,y)$ is provided.

We were unable to automatically prove the direction $CS(x,y) \rightarrow Cont(x,y) \vee ZEX(x)$. But we can simplify the proof by considering the following equivalences:

$$
\begin{aligned}
& CS(x,y) \rightarrow Cont(x,y) \vee ZEX(x) \\
\Leftrightarrow & \forall z[INCH(x,z) \rightarrow INCH(y,z)] \rightarrow Cont(x,y) \vee ZEX(x) && \text{(I-D1)} \\
\Leftrightarrow & \neg ZEX(x) \wedge \neg Cont(x,y) \\
& \rightarrow \neg\forall z[INCH(x,z) \rightarrow INCH(y,z)] && \text{(contrapositive)} \\
\Leftrightarrow & \neg ZEX(x) \wedge \neg Cont(x,y) \rightarrow \exists z[INCH(x,z) \wedge \neg INCH(y,z)] \\
\Leftrightarrow & \neg ZEX(x) \wedge \neg Cont(x,y) \rightarrow \exists z[\exists v[Cont(v,x) \wedge P(v,z)] \wedge \neg\exists w[Cont(w,y) \wedge P(w,z)]] && \text{(I-M1)} \\
\Leftrightarrow & \neg ZEX(x) \wedge \neg Cont(x,y) \rightarrow \exists z,v[Cont(v,x) \wedge P(v,z) \wedge \forall w[P(w,z) \rightarrow \neg Cont(w,y)]]
\end{aligned}
$$

For any choice of z, if the formula $\forall w[P(w,z) \rightarrow \neg Cont(w,y)]$ is satisfied, this formula is also satisfied for any v such that $P(v,z)$. Thus, it suffices to prove the special case $z := v$ $(*)$:

$$
\begin{aligned}
& \neg ZEX(x) \wedge \neg Cont(x,y) \rightarrow \exists z,v[Cont(v,x) \wedge P(v,z) \wedge \forall w[P(w,z) \rightarrow \neg Cont(w,y)]] \\
\Leftarrow & \neg ZEX(x) \wedge \neg Cont(x,y) \rightarrow \exists z[Cont(z,x) \wedge \forall w[P(w,z) \rightarrow \neg Cont(w,y)]] && (*) \\
\Leftarrow & \neg ZEX(x) \wedge \neg Cont(x,y) \rightarrow \exists z[P(z,x) \wedge z \cdot y <_{\dim} z] && \text{(EP-D, Int-A3)}
\end{aligned}
$$

The last step follows from $P(z,x) \rightarrow Cont(z,x)$ (EP-D) together with the sentence

$$\forall y,z\big[z \cdot y <_{\dim} z \rightarrow \forall w[Cont(w,y) \rightarrow \neg P(w,z)]\big].$$

If the consequent $P\forall w[Cont(w,y) \rightarrow \neg P(w,z)]$ were false, we would have $z \cdot y =_{\dim} z$ by Int-A3, contradicting the antecedent.

The last sentence itself is identical to EP-E3 (a theorem of $CODI_{\downarrow}$) is is thereby valid in $CODI_{\downarrow}$. Hence, $CS(x,y) \rightarrow Cont9x,y) \vee ZEX(x)$ is valid in $CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M6}. ☐

Lemma 8.22. *$CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M6} $\models$ I-M8*

Proof. An automatic proof of the direction $CH(x,y) \rightarrow P(x,y)$ is provided.

The proof of the direction $CH(x,y) \leftarrow P(x,y)$ requires strong supplementation (EP-E2) or its consequence PO-E1. The following equivalences manually prove the latter direction:

$$CH(x,y) \rightarrow P(x,y) \qquad \text{(I-M8R)}$$
$$\Leftrightarrow INCH(x,y) \wedge \forall z[OV(x,z) \rightarrow OV(y,z)] \rightarrow P(x,y) \qquad \text{(I-D4)}$$
$$\Leftrightarrow INCH(x,y) \wedge \forall z[PO(x,z) \rightarrow PO(y,z)] \rightarrow P(x,y) \qquad \text{(I-M6)}$$
$$\Leftrightarrow \exists z[Cont(z,x) \wedge P(z,y)] \wedge \forall z[PO(x,z) \rightarrow PO(y,z)] \rightarrow P(x,y) \qquad \text{(I-M1)}$$
$$\Leftarrow \neg ZEX(x) \wedge \neg ZEX(y) \wedge \forall z[PO(x,z) \rightarrow PO(y,z)] \rightarrow P(x,y) \qquad \text{(C-A1)}$$

The last sentence is valid by PO-E1 as shown in the proof of PO-E1 from $CODI_{\downarrow}$ on page 7.11.

□

Though I-A8 ($CH(x,y) \rightarrow CS(x,y)$) was not provable in $CODI^0$, it now easily follows from Lemmas 8.21 and 8.22 in $CODI_{\downarrow}$: It amounts to proving

$$\forall x,y[P(x,y) \rightarrow Cont(x,y)].$$

Lemma 8.23. *$CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8} $\models$ I-A8*

Proof. Consider the following implications:

$$CH(x,y) \rightarrow P(x,y) \qquad \text{(I-M8)}$$
$$\rightarrow Cont(x,y) \qquad \text{(EP-D)}$$
$$\rightarrow CS(x,y) \qquad \text{(I-M7)}$$

□

Finally, I-M7 and I-M8 also let us prove the mappings I-M9 and I-M10 for *EL* and *CO*. The latter requires contact, *C*, to be defined as in C-D.

(I-M9) $EL(x,y) \leftrightarrow (Cont(x,y) \wedge x <_{\dim} y) \vee ZEX(x)$ (mapping of *EL*)

(I-M10) $CO(x,y) \leftrightarrow C(x,y)$ (mapping of *CO*)

Axiom Set 8.13: Mapping axioms I-M9 and I-M10 from *CODI* theories to *INCH* theories.

Lemma 8.24. *$CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8} $\models$ {I-M9, I-M10}*

Summarily, all mappings I-M2 – I-M10 have thus been proved from the mapping I-M1 alone in $CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9:

Corollary 8.3. *$CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1} $\models$ {I-M2 – I-M10}*

Proof. Follows from the Lemmas 8.15, 8.17, 8.20, 8.21, 8.22, and 8.24. □

The mappings I-M1 to I-M10 completely define all relations of the INCH Calculus in terms of $CODI_{\downarrow} \cup$ C-E4. With regards to the axioms of the INCH Calculus, it remains to prove I-A1, I-A2, I-A9, and I-A10. We start by proving extensionality of *INCH*, i.e., I-A1 and I-A2.

Lemma 8.25. *$CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8} $\models$ I-A1*

Proof. The direction $x = y \rightarrow \forall z[INCH(x, z) \leftrightarrow INCH(y, z)]$ is trivial. For the converse direction, consider the following equivalence:

$$\forall z[INCH(x,z) \leftrightarrow INCH(y,z)] \rightarrow x = y$$
$$\Leftrightarrow \forall z[\exists v(Cont(v,x) \wedge P(v,z)) \leftrightarrow \exists w(Cont(w,y) \wedge P(w,z))] \rightarrow x = y \quad \text{(I-M1)}$$

Case (I): $x, y \in \mathbf{ZEX}_{\mathcal{M}}$.
Then by D-A4 we immediately obtain the desired consequence $x = y$.

Case (II): $y \in \mathbf{ZEX}_{\mathcal{M}}$ and $x \notin \mathbf{ZEX}_{\mathcal{M}}$.
The choice of $z := x$ violates the antecedent because $\mathbf{Cont}(x, x) \wedge \mathbf{P}(x, x)$ makes the left side of the biconditional true, but $\forall w[\neg\mathbf{Cont}(w, y)]$ always makes the right side of the biconditional false.

Case (III): $y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $x \in \mathbf{ZEX}_{\mathcal{M}}$.
Analogue to the previous case of $ZEX(y)$.

Case (IV): $x, y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\langle y, x\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$.
Consider the choice $z := x$. Then there exists a $v \in \mathbf{M}$ so that the left side of the biconditional is true, namely $v := x$, which results in $bf\ Cont(x, x) \wedge \mathbf{P}(x, x)$ which is trivially true by $x \notin \mathbf{ZEX}_{\mathcal{M}}$. But the right side of the biconditional is always falsified because $\forall w[\mathbf{P}(w, x) \rightarrow \neg\mathbf{Cont}(w, y)]$.

Case (V): $x, y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\langle x, y\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$.
Analogue to Case (IV).

Case (VI): $x, y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\langle x, y\rangle \in (=_{\mathbf{dim}})_{\mathcal{M}}$.
By EP-T2 it suffices to prove that

$$\forall z\big[\exists v[\mathbf{Cont}(v,x) \wedge \mathbf{P}(v,z)] \rightarrow \exists w[\mathbf{Cont}(w,y) \wedge \mathbf{P}(w,z)]\big] \rightarrow \mathbf{P}(x,y) \quad (*)$$

Suppose the antecedent of $(*)$ is satisfied for arbitrary $x, y \in \mathbf{M}$ of a model of $CODI_{\downarrow} \cup$ C-E4 but its consequent is not, i.e., , $\langle x, y\rangle \notin P$. Because $x, y \notin \mathbf{M}_{\mathcal{M}}$ and $\langle x, y\rangle \in (=_{\mathbf{dim}})_{\mathcal{M}}$, we must have $\langle x, y\rangle \notin \mathbf{Cont}_{\mathcal{M}}$. Then by EP-E3, there exists a $z \in \mathbf{M} \setminus \mathbf{ZEX}_{\mathcal{M}}$ such that $bf P(z, x) \wedge z \cdot y <_{\mathbf{dim}} z$. Consider the choice $v := z$ in the antecedent of $(*)$, which amounts to

$$\mathbf{Cont}(z,x) \wedge \mathbf{P}(z,z) \rightarrow \exists w[\mathbf{Cont}(w,y) \wedge \mathbf{P}(w,z)].$$

While $\mathbf{Cont}(z, x)$ and $\mathbf{P}(z, z)$ are trivially satisfied for this z, the consequent cannot be satisfied: if such a $w \in \mathbf{M}$ exists then $\mathbf{Cont}(w, z \cdot y)$. Because $\langle w, z\rangle \in \mathbf{P}_{\mathcal{M}}$ and $\langle z \cdot y, z\rangle \in (\leq_{\mathbf{dim}})_{\mathcal{M}}$ we must have $\langle z \cdot y, z\rangle \in (=_{\mathbf{dim}})_{\mathcal{M}}$ by Int-A2. This contradicts our earlier assumption of $\langle z \cdot y, z\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$ and thus $\langle z \cdot y, z\rangle \notin (=_{\mathbf{dim}})_{\mathcal{M}}$ by D-D2.

The cases (I)–(VI) cover all possible relative dimensions between x and y. Thereby we proved I-A1. □

The proof of I-A2 is very similar:

Lemma 8.26. *$CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8} $\models$ I-A2*

Proof. The direction $x = y \rightarrow \forall z[INCH(z,x) \leftrightarrow INCH(z,y)]$ is trivial. For the converse direction, consider the following equivalence:

$$\begin{aligned} &\forall z[INCH(z,x) \leftrightarrow INCH(z,y)] \rightarrow x = y \\ \Leftrightarrow &\forall z[\exists v(Cont(v,z) \wedge P(v,x)) \leftrightarrow \exists w(Cont(w,z) \wedge P(w,y))] \rightarrow x = y \qquad \text{(I-M1)} \end{aligned}$$

Case (I): $x, y \in \mathbf{ZEX}_{\mathcal{M}}$.
Then by D-A4 we immediately obtain the desired consequence $x = y$.

Case (II): $x \in \mathbf{ZEX}_{\mathcal{M}}$ and $y \notin \mathbf{ZEX}_{\mathcal{M}}$.
Then for all $v \in \mathbf{m}$ we must have $\neg\mathbf{P}(v,x)$ which always falsifies the left side of the biconditional. For the biconditional to be true, for any $z \in \mathbf{M} \setminus \mathbf{ZEX}_{\mathcal{M}}$ we cannot have $\mathbf{P}(z,y)$ because we already have $\mathbf{Cont}(z,z)$. Hence $y \in \mathbf{ZEX}_{\mathcal{M}}$, a contradiction to our assumption.

Case (III): $X \notin \mathbf{ZEX}_{\mathcal{M}}$ and $y \in \mathbf{ZEX}_{\mathcal{M}}$.
Analogue to Case (II) where $x \in \mathbf{ZEX}_{\mathcal{M}}$.

Case (IV): $x, y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\langle x, y\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$.
Then we can choose $z := x$ and $v := x$, which results in $\mathbf{Cont}(x,x)$ and $\mathbf{P}(x,x)$, which are trivially true, but for all $w \in \mathbf{M}$ with $\mathbf{P}(w,y)$ we have $\neg\mathbf{Cont}(w,x)$. Therefore the antecedent is falsified.

Case (V): $x, y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\langle y, x\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$.
Analogue to Case (IV).

Case (VI): $x, y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\langle x, y\rangle \in (=_{\mathbf{dim}})_{\mathcal{M}}$.
By EP-T2 it suffices to prove that

$$\forall z\big[\exists v[\mathbf{Cont}(v,z) \wedge \mathbf{P}(v,x)] \rightarrow \exists w[\mathbf{Cont}(w,z) \wedge \mathbf{P}(w,y)]\big] \rightarrow \mathbf{P}(x,y) \qquad (*)$$

Suppose the antecedent of $(*)$ is satisfied but its consequent is not, i.e., , $\neg\mathbf{P}(x,y)$. Because $x, y \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\langle x, y\rangle \in (=_{\mathbf{dim}})_{\mathcal{M}}$ by our assumption, we must have $\neg\mathbf{Cont}(x,y)$. Then by EP-E3, there exists a $(z \in \mathbf{M} \setminus \mathbf{ZEX}_{\mathcal{M}}$ such that $\langle z, x\rangle \in \mathbf{P}_{\mathcal{M}}$ and $\langle z \cdot y, z\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$. Consider the choice $v := z$ in the antecedent of $(*)$, which amounts to

$$\mathbf{Cont}(z,z) \wedge \mathbf{P}(z,x)) \rightarrow \exists w(\mathbf{Cont}(w,z) \wedge \mathbf{P}(w,y)).$$

While $\mathbf{Cont}(z,z)$ and $\mathbf{P}(z,x)$ are trivially satisfied for this z, the consequent cannot be satisfied: note that $z{=_{\mathbf{dim}}}x{=_{\mathbf{dim}}}y{=_{\mathbf{dim}}}w$ and by $\mathbf{Cont}(w, z \cdot y)$ we further have $z \cdot y{=_{\mathbf{dim}}}w{=_{\mathbf{dim}}}z$, which contradicts our earlier assumption $\langle z \cdot y, z\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$.

The cases (I)–(VI) cover all possible relative dimensions between x and y. Thereby we proved I-A2. ☐

Next, we prove the closure under differences for equidimensional entities (I-A10).

Lemma 8.27. *$CODI_{\downarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8} $\vDash$ I-A10*

Proof. We have the following equivalence:

$$\begin{aligned} &ED(x,y) \rightarrow \exists z \forall w\big[INCH(z,w) \leftrightarrow \exists v\,[INCH(v,w) \wedge CH(v,x) \wedge \neg OV(v,y)]\,\big] \\ \Leftrightarrow &x =_{\dim} y \rightarrow \exists z \forall w\big[INCH(z,w) \leftrightarrow \exists v[INCH(v,w) \wedge P(v,x) \wedge \neg PO(v,y)]\big] \end{aligned}$$

by substituting ED, CH, and OV using the mapping axioms I-M4, I-M8, and I-M6. The latter sentence is satisfied if $z := x - y$ satisfies it:

$$x =_{\dim} y \rightarrow \forall w\big[INCH(x-y, w) \leftrightarrow \exists v[INCH(v, w) \wedge P(v, x) \wedge \neg PO(v, y)]\big]$$

We can split this further into two sentences, which we will prove separately:

(a) $x =_{\dim} y \wedge INCH(x-y, w) \rightarrow \exists v[INCH(v, w) \wedge P(v, x) \wedge \neg PO(v, y)]$

(b) $x =_{\dim} y \wedge \exists v[INCH(v, w) \wedge P(v, x) \wedge \neg PO(v, y)] \rightarrow INCH(x-y, w)$

Direction (a):

$$\begin{aligned} & x =_{\dim} y \wedge INCH(x-y, w) \rightarrow \exists v[INCH(v, w) \wedge P(v, x) \wedge \neg PO(v, y)] \\ \Leftrightarrow & x =_{\dim} y \wedge Cont(w, x-y) \rightarrow \exists v\big[\exists z[Cont(z, v) \wedge P(z, w)] \wedge P(v, x) \wedge \neg PO(v, y)\big] \end{aligned}$$

Choose $v := x - y$ and $z := w$, then this amounts to showing

$$x =_{\dim} y \wedge Cont(w, x-y) \rightarrow Cont(w, x-y) \wedge P(w, w)) \wedge P(x-y, x) \wedge \neg PO(x-y, y)]$$

Assume the antecedent to be true, then the consequent is also true:

- $Cont(w, x-y)$ by the antecedent;
- $P(w, w)$ trivially with $Cont(w, x-y)$ implying $\neg ZEX(w)$;
- $P(x-y, x)$ by $Cont(w, x-y)$ implying $\neg ZEX(x-y)$ together with Dif-T1;
- $\neg PO(x-y, y)$ by Dif-T3.

Hence sentence (a) is valid.

Direction (b):

$$\begin{aligned} & x =_{\dim} y \wedge \exists v[INCH(v, w) \wedge P(v, x) \wedge \neg PO(v, y)] \rightarrow INCH(x-y, w) \\ \Leftrightarrow & x =_{\dim} y \wedge \exists v[\exists z(Cont(z, v) \wedge P(z, w)) \wedge P(v, x) \wedge \neg PO(v, y)] \rightarrow \exists z[Cont(z, x-y) \wedge P(z, w)] \\ \Leftarrow & x =_{\dim} y \wedge \exists v[P(w \cdot v, w) \wedge P(v, x) \wedge \neg PO(v, y)] \rightarrow \exists z[Cont(z, x-y) \wedge P(z, w)] \end{aligned}$$

Assume the antecedent is true. From $P(v, x) \wedge \neg PO(v, y)$ we obtain $\neg PO(v, x \cdot y)$ and further by Dif-T5 $P(v, x-y)$. Then also $Cont(w \cdot v, x-y)$ by transitivity of $Cont$. Moreover, $P(w \cdot v, w)$ by the antecedent. Hence $z := w \cdot v$ satisfies the consequent.

The two directions (a) and (b) together imply the property I-A10. □

Finally, we tackle closure under sums for equidimensional entities. This is not provable from $CODI_{\downarrow} \cup$ C-E4, but requires an explicit extension that enforces closures under sums. We use $CODI_{\updownarrow}$ for that purpose. Notice that even though $CODI_{\updownarrow}$ also defines sum for two entities that are not of equal dimension, this sum is always one of the entities (compare Sum-A2 on page 136).

Lemma 8.28. *$CODI_{\updownarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8} $\models$ I-A9*

Proof. We have the following equivalence

$$\begin{aligned}
&ED(x,y) \rightarrow \exists z \forall w \,[INCH(z,w) \leftrightarrow INCH(x,w) \vee INCH(y,w)] \\
\Leftarrow\, & x =_{\text{dim}} y \rightarrow \forall w [INCH(x+y,w) \leftrightarrow INCH(x,w) \vee INCH(y,w)] \\
\Leftrightarrow\, & x =_{\text{dim}} y \rightarrow \forall w \big[\exists z[Cont(z,x+y) \wedge P(z,w)] \leftrightarrow \exists z[P(z,w) \wedge (Cont(z,x) \vee Cont(z,y)]\big]
\end{aligned}$$

We can split this further into two sentences, which we will prove separately:

(a) $x =_{\text{dim}} y \wedge \exists z[Cont(z,x+y) \wedge P(z,w)] \rightarrow \exists v[P(v,w) \wedge (Cont(v,x) \vee Cont(z,y))]$

(b) $x =_{\text{dim}} y \wedge Cont(z,x) \rightarrow Cont(z,x+y)$

Direction (a): The sentence is satisfied for all $w \in \mathbf{M}$ if the following is satisfied for all $z \in \mathbf{M}$:

$$x \boldsymbol{=_{\text{dim}}} y \wedge \mathbf{Cont}(z,x+y) \rightarrow \exists v \big[\mathbf{P}(v,z) \wedge [\mathbf{Cont}(v,x) \vee \mathbf{Cont}(z,y)]\big],$$

which is equivalent to

$$x \boldsymbol{=_{\text{dim}}} y \wedge \mathbf{Cont}(z,x+y) \wedge \neg\exists v[\mathbf{P}(v,z) \wedge \mathbf{Cont}(v,x)] \rightarrow \exists v[\mathbf{P}(v,z) \wedge \mathbf{Cont}(z,y)].$$

If either $\mathbf{Cont}(z,x)$ or $\mathbf{Cont}(z,y)$, this is trivially satisfied by choosing $v := z$ for the first or second occurrence, respectively. Now assume that $\neg\mathbf{Cont}(z,x)$ and $\neg\mathbf{Cont}(z,y)$. Then by Sum-T9, there must exist a $v \in \mathbf{M}$ such that $\mathbf{P}(v,z)$ and $\mathbf{Cont}(v,y)$, which satisfies our consequent. Hence direction (a) is also satisfied.

Direction (b): $x =_{\text{dim}} y \wedge Cont(z,x) \rightarrow Cont(z,x+y)$ follows directly from Sum-A3.

The two directions (a) and (b) together entail I-A9. □

Thus, the theory $CODI_{\downarrow} \cup$ C-E4 definably interprets the corrected INCH Calculus, $INCH_{\text{calculus}}$, amended by I-E1.

Corollary 8.4. *$CODI_{\updownarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1} $\models INCH_{\text{calculus}} \cup$ I-E1*

Proof. Follows immediately from the Lemmas 8.14, 8.18, 8.19, 8.23, 8.25, 8.26, 8.27, and 8.28 together with Corollary 8.3. □

This interpretability of the INCH Calculus by $CODI_{\updownarrow} \cup$ C-E4 should convince the reader that $CODI_{\updownarrow}$ is in fact a multidimensional mereotopology with well-defined closure operations. We essentially used the INCH Calculus, an independently developed theory, to cross-verify our axiomatization $CODI_{\updownarrow}$. Of course, any omission or error in the axiomatization of the INCH Calculus could potentially pose also a problem for our axiomatization. Nevertheless, this kind of relative interpretation helps us to increase our confidence in either axiomatization. An immediate consequence is that $CODI^0$ is indeed a weak multidimensional mereotopology that lacks mereological closures, since we already established the relationship between $CODI_{\updownarrow}$ and $CODI^0$ in Chapter 7.

However, the reverse question is still unanswered: Does $INCH_{\text{calculus}} \cup$ I-E1 definably interpret $CODI_{\updownarrow} \cup$ C-E4 as well? We will tackle this question in the next subsection.

8.2.5 $INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} and $CODI_{\updownarrow} \cup$ C-E4 are definably equivalent

We already finished a significant portion of the work necessary to prove that the INCH Calculus extended by I-E1 interprets $CODI_{\updownarrow} \cup$ C-E4 by showing in Section 8.2.3 that the INCH Calculus interprets $CODI^0$, a subtheory of $INCH_{\text{calculus}} \cup$ I-E1 (see Corollary 8.2 on page 177). Hence, we also have the following corollary:

Corollary 8.5. *$INCH_{\text{calculus}} \cup$ I-E1 $\cup$ {I-M1′ – I-M3′} $\models CODI^0$*

What remains to be shown is that the other axioms, in particular those enforcing closures under intersections (Int-A1 – Int-A4), differences (Dif-A1 – Dif-A4), sums (Sum-A1 – Sum-A4), and universals (U-A1) are also provable from $INCH_{\text{calculus}} \cup$ I-E1 for suitable mappings of $\cdot$, $-$, and $+$. We use I-M4 – I-M6′ as mappings.

(I-M4′) $$x + y = \begin{cases} x & \text{if } GD(x,y) \\ y & \text{if } GD(y,x) \\ z : \forall w[INCH(z,w) \leftrightarrow INCH(x,w) \vee INCH(y,w)] & \text{if } ED(x,y) \end{cases}$$

(I-M5′) $$x \cdot y = z : CS(z,x) \wedge CS(z,y) \wedge \forall w[CS(w,x) \wedge CS(w,y) \rightarrow GED(z,w)] \wedge \forall w[ED(w,z) \wedge CS(w,x) \wedge CS(w,y) \rightarrow CS(w,z)]$$

(I-M6′) $$x - y = \begin{cases} x & \text{if } GD(x, x\cdot y) \\ z : \forall w\big[INCH(z,w) \leftrightarrow \exists v[INCH(v,w) \wedge CH(v,x) \wedge \neg OV(v, x\cdot y)]\big] & \text{if } ED(x, x\cdot y) \end{cases}$$

Axiom Set 8.14: Mapping axioms I-M4′ – I-M6′ from *INCH* theories to *CODI* theories.

We have to prove that these mappings specify functions, i.e., for every pair x, y there is a unique z and that the specified functions satisfy the axioms Int-A1 – Int-A4, Dif-A1 – Dif-A4, and Sum-A1 – Sum-A4. Crucial to this proof is that the mappings I-M4′ to I-M6′ are really defined functions, i.e., that they do not force additional entities to exist. Only mappings that do not nonconservatively extend the underlying theory $INCH_{\text{calculus}}$ are adequate mappings, otherwise they would be axioms restricting the models of $INCH_{\text{calculus}}$ in some way.

Lemma 8.29. *$INCH_{\text{calculus}} \cup$ I-E1 $\cup$ {I-M1′ – I-M6′} defines + as a total function.*

Proof. We show that such a $z = x + y$ always exists and is uniquely defined for all pairs x, y. We distinguish the following three cases.

Case (I): Assume $\langle x, y\rangle \in \mathbf{ED}_{\mathcal{M}}$.

Then, by I-A9, an entity $z \in \mathbf{M}$ that satisfies the condition

$$\forall w[INCH(z,w) \leftrightarrow INCH(x,w) \vee INCH(y,w)]$$

must exist. If two distinct $z_1, z_2 \in \mathbf{M}$ would satisfy this condition, then

$$\forall w[INCH(z_1,w) \leftrightarrow INCH(z_2,w)],$$

which by I-A1 implies $z_1 = z_2$. Hence the sum $z = x + y$ is uniquely defined.

Case (II): Assume $\langle x, y\rangle \in \mathbf{GD}_{\mathcal{M}}$.
Then $x + y = x$ is always uniquely defined.

Case (III): Assume $\langle y, x\rangle \in \mathbf{GD}_{\mathcal{M}}$.
Then $x + y = y$ is always uniquely defined.

The following logical equivalences show that those three cases are all possible cases:

$$\forall x, y[GED(x, y) \vee GED(y, x)] \quad \text{(I-A4)}$$
$$\Leftrightarrow \forall x, y\big[[GED(x, y) \wedge GED(y, x)] \vee [GED(x, y) \wedge \neg GED(y, x)] \vee [\neg GED(x, y) \wedge GED(y, x)]\big]$$
$$\Leftrightarrow \forall x, y\big[ED(x, y) \vee \vee[GED(x, y) \wedge \neg GED(y, x)] \vee [\neg GED(x, y) \wedge GED(y, x)]\big] \quad \text{(I-D8)}$$
$$\Leftrightarrow \forall x, y[ED(x, y) \vee GD(x, y) \vee GD(y, x)] \quad \text{(I-D9)}$$

The three cases (I) to (III) are exhaustive for every pair x, y. Hence, I-M4′ defines + as a total function.
□

But the INCH Calculus in not closed under intersections in the same way $CODI_{\updownarrow}$ is closed. Though Gotts [Got96] claims that the intersection $prod(x, y)$ is definable as $diff(x, diff(x, y))$ this is only true if x and y are of equal dimension, i.e., $ED(x, y)$, because otherwise the difference is not required to exist in the INCH Calculus. To enforce closure under intersections, we add I-E2.

(I-E2) $CO(x, y) \rightarrow \exists z\big[CS(z, x) \wedge CS(z, y) \wedge \forall w[CS(w, x) \wedge CS(w, y) \rightarrow GED(z, w)] \wedge \forall w[CS(w, x) \wedge CS(w, y) \wedge ED(w, z) \rightarrow CS(w, z)]\big]$
(for two connected entities x, y, a maximal shared constituent of maximal dimension exists)

Axiom Set 8.15: Extension axiom I-E2 of $INCH_{\text{calculus}}$.

Lemma 8.30. *$INCH_{\text{calculus}} \cup$ I-E1 $\not\models$ I-E2*

Proof. By the definition of CO, we have

$$CO(x, y) \rightarrow \exists z[\neg ZEX_I(z) \wedge CS(z, x) \wedge CS(z, y)]$$

as a theorem of $INCH_{\text{calculus}} \cup$ I-E1. We will construct counterexamples to either of the other two conditions separately.

First, we construct a model of $INCH_{\text{calculus}} \cup$ I-E1 that contains two connected entities $a, b \in \mathbf{M}$ with $\langle a, b\rangle \in \mathbf{CO}_{\mathcal{M}}$, whose set of shared constituents does not contain an entity of greatest dimension, i.e., no $z \in \mathbf{M}$ exists with $\mathbf{CS}(z, a)$, $\mathbf{CS}(z, b)$ and for all $w \in \mathbf{M}$, $\mathbf{CS}(w, a) \wedge \mathbf{CS}(w, b) \rightarrow \mathbf{GED}(z, w)]$. Obviously, this can only happen in an infinite model, more precisely, in a model with an infinite number of constituents of both x and y with increasing dimensions. In other words, x and y must be of infinite dimension and must share infinite constituents, none of which has a dimension higher than all others, but x and y do not share a chunk.
More formally, we can define the model $\mathcal{M}$ as following. $\mathcal{M}$ has the domain of entities

$$\mathbf{M} = \{a, b, d, \text{ze}, c_0, c_1, c_2, \ldots, c_\infty\}$$

with the extension of the only primitive relation *INCH* defined as

$$\begin{aligned}\mathbf{INCH}_{\mathcal{M}} =& \{\langle a, c_i\rangle, \langle a, c_0\rangle, \langle a, d\rangle, \langle b, c_i\rangle, \langle b, c_0\rangle, \langle b, d\rangle, \langle c_i, c_0\rangle, \langle d, a\rangle, \langle d, b\rangle, \langle d, c_i\rangle\}\\ &\cup \{\langle x, x\rangle \mid x \in \mathbf{M} \setminus \{\text{ze}\}\} \cup \{\langle c_i, c_j\rangle \mid i \geq j > 0\}.\end{aligned}$$

The extensions of the defined relations ZEX_I and *GED* are

$$\mathbf{ZEX_I}_{\mathcal{M}} = \{\text{ze}\}$$

and

$$\begin{aligned}\mathbf{GED}_{\mathcal{M}} =& \{\langle c_i, c_0\rangle, \langle c_i, \text{ze}\rangle, \langle c_0, \text{ze}\rangle\} \cup \{\langle x, y\rangle \mid x \in \{a, b, d\} \text{ and } y \in \mathbf{M}\}\\ &\cup \{\langle x, x\rangle \mid x \in \mathbf{M}\} \cup \{\langle c_i, c_j\rangle \mid i \geq j > 0\}.\end{aligned}$$

We can further determine the extensions of *CS* and *CH* as

$$\begin{aligned}\mathbf{CS}_{\mathcal{M}} =& \{\langle a, d\rangle, \langle b, d\rangle, \langle c_i, a\rangle, \langle c_i, b\rangle, \langle c_i, d\rangle, \langle c_0, a\rangle, \langle c_0, b\rangle, \langle c_0, c_i\rangle, \langle c_0, d\rangle\}\\ &\cup \{\langle x, x\rangle \mid x \in \mathbf{M}\} \cup \{\langle \text{ze}, x\rangle \mid x \in \mathbf{M}\} \cup \{\langle c_j, c_i\rangle \mid i \geq j > 0\}\end{aligned}$$

and

$$\mathbf{CH}_{\mathcal{M}} = \{\langle a, d\rangle, \langle b, d\rangle\} \cup \{\langle x, x\rangle \mid x \in \mathbf{M} \setminus \{\text{ze}\}\}$$
$$\mathbf{CO}_{\mathcal{M}} = \{\langle x, y\rangle \mid x, y \in \mathbf{M} \setminus \{\text{ze}\}\}$$

From the specification of $\mathbf{INCH}_{\mathcal{M}}$ it is easy to see that $\mathcal{M}$ satisfies I-A1, I-A2, and I-A3. From the specification of $\mathbf{GED}_{\mathcal{M}}$ we can also verify that I-A4 and I-A5 are satisfied. Because the extension of *INCH* is antitransitive, i.e., if $\mathbf{INCH}(x, y)$ and $\mathbf{INCH}(y, z)$ then we never have $\mathbf{INCH}(z, x)$, I-A6 is also satisfied. I-A7′ is always satisfied if we choose the existentially quantified variable in I-A7′ to be $z := y$. I-A8 is satisfied because $\mathbf{CH}_{\mathcal{M}} \subseteq \mathbf{CS}_{\mathcal{M}}$. Moreover, I-A9 and I-A10 are trivially satisfied, because b, d, c_1 are the only distinct entities of equal dimension, we have $d + c_1 = b$, $b - d = c_1$ and $b - c_1 = d$. Finally, $x := c_0$ satisfies I-E1: $c_0 \notin \mathbf{ZEX}_{\mathcal{M}}$ and for any $y \notin \mathbf{ZEX}_{\mathcal{M}}$ we have $\langle y, c_0\rangle \in \mathbf{GED}_{\mathcal{M}}$.

In the final part of the proof, we construct a model of $INCH_{\text{calculus}} \cup$ I-E1 that contains two connected entities $a, b \in \mathbf{M}$ with $\langle a, b\rangle \in \mathbf{CO}_{\mathcal{M}}$ for which no common constituent $z \in \mathbf{M}$ exists such that all entities of the same dimension that are constituents of either are also constituents of z. Thereby, the constructed model will violate the second condition in the consequent of I-E2. More precisely, in this model no $z \in \mathbf{M}$ exists with $\mathbf{CS}(z, a)$, $\mathbf{CS}(z, b)$ and such that for all $w \in \mathbf{M}$, $\mathbf{CS}(w, a) \wedge \mathbf{CS}(w, b) \wedge \mathbf{ED}(w, z) \rightarrow \mathbf{CS}(w, z)]$. The model is similar to the previous one except that instead of a infinite chain of entities c_i of decreasing dimension we use an infinite chain of increasingly bigger, nested entities of equal dimension as the constituents of both a and b. The model must contain two additional distinct elements $a', b' \in \mathbf{M}$ with

$$\langle a', a\rangle, \langle a', d\rangle, \langle b', b\rangle, \langle b', d\rangle \in \mathbf{CH}_{\mathcal{M}} \text{ but}$$
$$\langle a', a'\rangle, \langle a', b\rangle, \langle a', b'\rangle, \langle a', c_0\rangle, \langle a', c_i\rangle, \langle a', \text{ze}\rangle\langle b', a'\rangle, \langle b', b\rangle, \langle b', b'\rangle, \langle b', c_0\rangle, \langle b', \text{ze}\rangle, \langle b', c_i\rangle \notin \mathbf{CH}_{\mathcal{M}} \text{ and}$$
$$\langle x, a'\rangle, \langle x, b'\rangle \notin \mathbf{CH}_{\mathcal{M}} \text{ for all } x \in \mathbf{M} \setminus \{a', b'\}$$

to ensure that I-A1 and I-A2 are satisfied. The extension of *INCH* and *GED* are defined as follows:

$$\mathbf{INCH}_{\mathcal{M}} = \{\langle x,y\rangle \mid x,y \in \mathbf{M} \setminus \{a',b',\text{ze}\}\} \cup \{\langle a,a'\rangle, \langle a',a\rangle, \langle a',d\rangle, \langle b,b'\rangle, \langle b',b\rangle, \langle b',d\rangle, \langle d,a'\rangle, \langle d,b'\rangle\},$$
$$\mathbf{GED}_{\mathcal{M}} = \{\langle x,y\rangle \mid x \in \mathbf{M} \setminus \{\text{ze}\} \text{ and } y \in \mathbf{M}\} \cup \langle \text{ze},\text{ze}\rangle.$$

Even though any pair c_i, c_j has a sum (the larger entity of the two), no maximal entity exists that is a constituent of both a and b and of which all other c_i are constituents of. □

This verifies that we really need both of the additional conditions imposed by I-E2 to guarantee intersections to exist for any two entities. But with the help of I-E2, it is easily provable that the intersection $x \cdot y$ as defined in I-M5′ is indeed a total function.

Lemma 8.31. *$INCH_{\text{calculus}} \cup$ {I-E1, I-E2} $\cup$ {I-M1′ – I-M6′} defines $\cdot$ as a total function.*

Proof. We must show that $x \cdot y$ is uniquely defined for all pairs x, y. We distinguish the following two cases for arbitrary $x, y \in \mathbf{M}$ of a model $\mathcal{M}$ of $INCH_{\text{calculus}} \cup$ {I-E1, I-E2}.

Case (I): Assume $\langle x,y\rangle \notin \mathbf{CO}_{\mathcal{M}}$.

Then there exists no $z \in \mathbf{M} \setminus \mathbf{ZEX}_{\mathcal{M}}$ such that $\mathbf{CS}(z,x)$ and $\mathbf{CS}(z,y)$. Hence the unique $z \in \mathbf{M}_{\mathcal{M}}$ that exists by Corollary 8.5 and D-A4 satisfies the definition of I-M5′. By I-T6 we have $\mathbf{CS}(z,x)$, $\mathbf{CS}(z,y)$, and $\mathbf{CS}(z,z)$. Because for any other $w \in \mathbf{M}$, $\mathbf{CS}(w,x) \wedge \mathbf{CS}(w,y)$ cannot hold, the remaining two conditions are trivially satisfied.

Case (II): Assume $\langle x,y\rangle \in \mathbf{CO}_{\mathcal{M}}$.

By I-E2, a $z \in \mathbf{M}$ must exist that satisfies the condition for $z = x \cdot y$. Suppose two distinct such z exists, call them z_1 and z_2. Then, we have $\mathbf{CS}(z_1,x) \wedge \mathbf{CS}(z_1,y)$ and $\mathbf{CS}(z_2,x) \wedge \mathbf{CS}(z_2,y)$. Thus, we must have $\mathbf{GED}(z_1,z_2)$ as well as $\mathbf{GED}(z_2,z_1)$, which by I-D8 results in $\mathbf{ED}(z_1,z_2)$. Hence, the antecedent of the last condition of I-E2 can be invoked, so that $\mathbf{CS}(z_1,z_2)$ and $\mathbf{CS}(z_2,z_1)$ must hold. With the mapping I-M1′ we obtain $\mathbf{Cont}(z_1,z_2)$ and $\mathbf{Cont}(z_2,z_1)$, which together imply $z_1 = z_2$ by Corollary 8.5 and C-A2. However, this contradicts our initial assumption that z_1 and z_2 are distinct.

For all pairs $x, y \in \mathbf{M}$ either $\langle x,y\rangle \in \mathbf{CO}_{\mathcal{M}}$ or not; hence the two cases are exhaustive and $x \cdot y$ is uniquely defined for all pairs $x, y \in \mathbf{M}$. □

Lemma 8.32. *$INCH_{\text{calculus}} \cup$ {I-E1,I-E2} $\cup$ {I-M1′ – I-M6′} defines $-$ as a total function.*

Proof. We must show that $x - y$ is uniquely defined for arbitrary pairs $x, y \in \mathbf{M}$ of a model $\mathcal{M}$ of $INCH_{\text{calculus}} \cup$ {I-E1, I-E2}.

Case (I): $\mathbf{ED}(x, x \cdot y)$ and $\mathbf{ED}(y,x)$.

Then, by I-A10, an entity $z \in \mathbf{M}$ that satisfies the condition

$$\forall w\big[\mathbf{INCH}(z,w) \leftrightarrow \exists v[\mathbf{INCH}(v,w) \wedge \mathbf{CH}(v,x) \wedge \neg\mathbf{OV}(v,y)]\big]$$

must exist. If two distinct z_1, z_2 would satisfy this condition, then

$$\forall w[\mathbf{INCH}(z_1,w) \leftrightarrow \mathbf{INCH}(z_2,w)],$$

which by I-A1 implies $z_1 = z_2$. Hence the difference $z = x - y$ is uniquely defined.

Case (II): $\mathbf{ED}(x, x \cdot y)$ and $\mathbf{GD}(y, x)$.

Then by Lemma 8.31 the entity $x \cdot y$ is uniquely defined. Then again, by I-A10, an entity $z \in \mathbf{M}$ that satisfies the condition

$$\forall w\big[\mathbf{INCH}(z, w) \leftrightarrow \exists v[\mathbf{INCH}(v, w) \wedge \mathbf{CH}(v, x) \wedge \neg\mathbf{OV}(v, x \cdot y)]\big]$$

must exist. If two distinct z_1, z_2 would satisfy this condition, then

$$\forall w[\mathbf{INCH}(z_1, w) \leftrightarrow \mathbf{INCH}(z_2, w)],$$

which by I-A1 implies $z_1 = z_2$. Hence the difference $z = x - y$ is uniquely defined as $z = x - (x \cdot y)$.

Case (III): $\mathbf{GD}(x, x \cdot y)$.

I-M6′ uniquely defines $x - y = x$.

Finally, we show that these three cases are the only possible ones for arbitrary pairs $x, y \in \mathbf{M}$. Recall that

$$\forall x, y[ED(x, y) \vee GD(x, y) \vee GD(y, x)]$$

is a theorem of $INCH_{\text{original}}$ (compare proof of Lemma 8.29), which also applies to $y := x \cdot y$. By I-M5′, there exists a uniquely defined $x \cdot y$ for any pair of $x, y \in \mathbf{M}$. This intersection has the property $\mathbf{CS}(x \cdot y, x)$. The latter implies $\mathbf{Cont}(x \cdot y, x)$ or $(x \cdot y) \in \mathbf{ZEX}_{\mathcal{M}}$ by Corollary 8.5. If $(x \cdot y) \in \mathbf{ZEX}_{\mathcal{M}}$ then $\neg\mathbf{GD}(x \cdot y, x)$. Now suppose $(x \cdot y) \notin \mathbf{ZEX}_{\mathcal{M}}$, that is, $\mathbf{Cont}(x \cdot y, x)$. Then $x \cdot y \leq_{\mathbf{dim}} x$ must hold (CD-A1), which in turn requires $\mathbf{GED}(x, x \cdot y)$, thus preventing $\mathbf{GD}(x \cdot y, x)$ by I-D9. Hence, $\mathbf{GD}(x, x \cdot y)$ and $\mathbf{ED}(x, x \cdot y)$ are the only two possibilities, which we covered.

Notice that I-M5′ requires $\mathbf{CS}(x \cdot y, y)$ and hence $\mathbf{GED}(y, x \cdot y)$. If we have $\mathbf{ED}(x, x \cdot y)$, we can immediately deduce $\mathbf{GED}(y, x)$ and thus either $\mathbf{ED}(y, x)$ or $\mathbf{GD}(y, x)$ must hold. Hence the cases (1) and (2) cover all possibilities when $\mathbf{ED}(x, x \cdot y)$.

Consequently, the function $x - y$ is uniquely defined for any pair $x, y \in \mathbf{M}$. □

Lemma 8.33. *$INCH_{\text{calculus}}$ ∪ {I-E1, I-E2} ∪ {I-M1′ – I-M6′} ∪ {EP-D, EPP-D, PO-D} ⊨ {Int-A1 – Int-A4}*

Proof. **(Int-A1)** $\neg C(x, y) \leftrightarrow ZEX(x \cdot y)$.

Consider the following logical equivalences:

$$\begin{aligned} & ZEX(x \cdot y) & \\ \Leftrightarrow & \forall z[\neg ZEX(z) \rightarrow \neg CS(z, x) \vee \neg CS(z, y)] & \text{(I-M5}'\text{)} \\ \Leftrightarrow & \forall z[\neg Cont(z, x) \vee \neg Cont(z, y)] & \text{(I-M1}'\text{)} \\ \Leftrightarrow & \neg C(x, y) & \text{(C-D)} \end{aligned}$$

(Int-A2) $\neg ZEX(x \cdot y) \rightarrow Cont(x \cdot y, x)$.

I-M5′ implies $\forall x, y[CS(x \cdot y, x)]$. Then the assumption $\forall x, y[\neg ZEX(x \cdot y)]$ immediately implies $\forall x, y[Cont(x \cdot y, x)]$ by I-M1′.

(Int-A3) $Cont(v, x) \wedge Cont(v, y) \rightarrow v \leq_{\text{dim}} x \cdot y$

Assume $Cont(v, x)$ and $Cont(v, y)$ for arbitrary $x, y, v \in \mathbf{M}$.

Then $\mathbf{CS}(v,x)$ and $\mathbf{CS}(v,y)$ by I-M1′ and thus $\mathbf{GED}(x\cdot y, v)$ by I-M5′, which translates to $v \leq_{\mathbf{dim}} x\cdot y$ by I-M3′, D-D3, and I-D8.

(Int-A4) $Cont(v,x) \wedge Cont(v,y) \wedge v =_{\text{dim}} x \cdot y \leftrightarrow P(v, x \cdot y)$.

Note that if $v \in \mathbf{ZEX}_{\mathcal{M}}$, then Int-A4 is trivially satisfied. For $v \notin \mathbf{ZEX}_{\mathcal{M}}$ consider the following computation:

$$\begin{aligned} & Cont(v,x) \wedge Cont(v,y) \wedge v =_{\text{dim}} x \cdot y \leftrightarrow P(v, x \cdot y) \\ \Leftrightarrow & Cont(v,x) \wedge Cont(v,y) \wedge v =_{\text{dim}} x \cdot y \leftrightarrow Cont(v, x \cdot y) \wedge v =_{\text{dim}} x \cdot y \qquad \text{(EP-D)} \\ \Leftrightarrow & CS(v,x) \wedge CS(v,y) \wedge ED(v, x \cdot y) \leftrightarrow CS(v, x \cdot y) \wedge ED(v, x \cdot y) \end{aligned}$$

The last step follows from I-M1′, I-M3′, D-D2, and I-D8. Both directions of the biconditional in the last line follow directly from I-M5′.

□

Lemma 8.34. *$INCH_{\text{calculus}} \cup$ {I-E1, I-E2} $\cup$ {I-M1′ – I-M6′} $\cup$ {EP-D, EPP-D, PO-D} $\models$ {Dif-A1 – Dif-A4}*

Proof. Note that we prove Dif-A3a last, to make use of Dif-A3b and Dif-A4 in its proof.

(Dif-A1) $\neg ZEX(x-y) \rightarrow x - y =_{\text{dim}} x$.

Assume $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$. We distinguish two cases.

Case (I): Assume $\langle x, x\cdot y\rangle \in \mathbf{GD}_{\mathcal{M}}$.

Then $x - y = x$ by I-M6′ and thus $x - y =_{\mathbf{dim}} x$.

Case (II): Assume $\langle x, x\cdot y\rangle \in \mathbf{ED}_{\mathcal{M}}$.

Subcase (II.a): Assume $\langle x, x\cdot y\rangle \in \mathbf{ED}_{\mathcal{M}}$ and $x \in \mathbf{ZEX}_{\mathbf{I}\mathcal{M}}$.

Then for all $v \in \mathbf{M}$, $\neg\mathbf{CH}(v,x)$ and thus for all $v \in \mathbf{M}$, $\neg\mathbf{INCH}(x-y, v)]$ by I-M6′. Hence $\neg\mathbf{INCH}(x-y, x-y)$, thus $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$ by I-M2′ and D-A4. We conclude $x - y =_{\mathbf{dim}} x$.

Subcase (II.b): Assume $\langle x, x\cdot y\rangle \in \mathbf{ED}_{\mathcal{M}}$, $x \notin \mathbf{ZEX}_{\mathbf{I}\mathcal{M}}$, and $\langle x, x\cdot y\rangle \in \mathbf{CH}_{\mathcal{M}}$.

Then by I-D4, I-A3, I-D6 $(x\cdot y) \notin \mathbf{ZEX}_{\mathbf{I}\mathcal{M}}$.

Then by I-T12 and I-D4, for all $z \in \mathbf{M}$,

$$\mathbf{CH}(z,x) \rightarrow \mathbf{OV}(z, x\cdot y)].$$

Then the left-hand side of the biconditional in I-M6′ will never be satisfied for any $w \in \mathbf{M}$. Hence for all $w \in \mathbf{M}$, $\neg\mathbf{INCH}(x-y, w)$ so that $(x-y) \in \mathbf{ZEX}_{\mathbf{I}\mathcal{M}}$ by I-A3, contrary to our initial assumption that $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$. Hence this case is irrelevant.

Subcase (II.c): Assume $\langle x, x\cdot y\rangle \in \mathbf{eD}_{\mathcal{M}}$, $x \notin \mathbf{ZEX}_{\mathbf{I}\mathcal{M}}$, and $\langle x, x\cdot y\rangle \notin \mathbf{CH}_{\mathcal{M}}$.

From the assumption $\neg\mathbf{CH}(x, x\cdot y)$ we obtain

$$\neg\mathbf{INCH}(x, x\cdot y) \text{ or there exists a } v \in \mathbf{M} \text{ such that } [\mathbf{OV}(x,v) \wedge \neg\mathbf{OV}(x\cdot y, v)]$$

using I-D4. We distinguish two subcases depending on whether the last of the two conditions is satisfied.

Subcase (II.c.i): Assume that for all $v \in \mathbf{M}$, $\mathbf{OV}(x,v) \rightarrow \mathbf{OV}(x \cdot y, v)]$.

Then $\neg\mathbf{INCH}(x, x \cdot y)$ must hold by the previous equation and thus by I-A7′, no $z \in \mathbf{M}$ exists such that

$$\mathbf{CS}(z,x) \wedge \mathbf{CH}(z, x \cdot y)].$$

Note that $(x \cdot y) \notin \mathbf{ZEX}_{\mathcal{M}}$ because $\mathbf{ED}(x, x \cdot y)$ and $x \notin \mathbf{ZEX}_{\mathcal{M}}$.

Then, $\mathbf{CS}(x \cdot y, x)$ and $\mathbf{CH}(x \cdot y, x \cdot y)$. Hence some $z \in \mathbf{M}$ exists such that

$$\mathbf{CS}(z,x) \wedge \mathbf{CH}(z, x \cdot y),$$

a contradiction to our earlier statement that no such z may exist. Thus, this case cannot occur.

Subcase (II.c.ii): Assume there exists a $v \in \mathbf{M}$ such that $\mathbf{OV}(x,v)$ and $\neg\mathbf{OV}(x \cdot y, v)$.

Consider the following logical derivation:

$$\begin{aligned}
&\exists v[OV(x,v) \wedge \neg OV(x \cdot y, v)] && \\
\Leftrightarrow &\exists v[INCH(v,x) \wedge \neg OV(x \cdot y, v)] && \text{(I-D2)}\\
\Rightarrow &\exists v\big[\exists u[CS(u,v) \wedge CH(u,x)] \wedge \neg OV(x \cdot y, v)\big] && \text{(I-A7}'\text{)}\\
\Rightarrow &\exists v\big[\exists u[CH(u,v) \wedge CH(u,x)] \wedge \neg OV(x \cdot y, v)\big] && \text{(I-T2, I-T5)}\\
\Rightarrow &\exists u[CH(u,x) \wedge \neg OV(u, x \cdot y)] && \text{(I-T12)}\\
\Rightarrow &\exists u[INCH(u,x) \wedge \neg OV(u, x \cdot y)] && \text{(I-D4)}\\
\Rightarrow &\exists u[INCH(u,x) \wedge CH(x,x) \wedge \neg OV(u, x \cdot y)] && \text{(I-A3)}\\
\Rightarrow &INCH(x - y, x) && \text{(I-M6}'\text{)}\\
\Rightarrow &GED(x - y, x) && \text{(I-T1)}
\end{aligned}$$

That is, we conclude that $\mathbf{GED}(x-y,x)$ from our assumption that there exists a $v \in \mathbf{M}$ such that $\mathbf{OV}(x,v)$ and $\neg\mathbf{OV}(x \cdot y, v)$.

Now suppose $\mathbf{GD}(x-y,x)$. For any $w \in \mathbf{M}$ with $\mathbf{CH}(w, x-y)$ we have $\mathbf{INCH}(x-y, w)$ by I-D4 and I-T2 and thus there exists a $v \in \mathbf{M}$ such that

$$\mathbf{INCH}(v,w) \wedge \mathbf{CH}(v,x) \wedge \neg\mathbf{OV}(v, x \cdot y) \tag{8.34.1}$$

according to I-M6′. However, for any such v with $\mathbf{CH}(v,x)$ we have

$$v =_{\mathbf{dim}} x <_{\mathbf{dim}} x - y =_{\mathbf{dim}} w \tag{8.34.2}$$

by our supposition $\mathbf{GD}(x-y,x)$ and I-T5, I-D8, I-D9, and I-M3′. At the same time $\mathbf{INCH}(v,w)$ requires $\mathbf{GED}(v,w)$, which is equivalent to

$$v \geq_{\mathbf{dim}} w \tag{8.34.3}$$

Conditions (8.34.2) and (8.34.3) contradict another, consequently no v that satisfies Equation (8.34.1) can exist for any w. Hence, our supposition $\mathbf{GD}(x-y,x)$ was false, and we must have

$$\mathbf{GED}(x-y,x) \wedge \neg\mathbf{GD}(x-y,x)$$

and thus $\mathbf{ED}(x-y,x)$. Therefore, $x-y =_{\text{dim}} x$, our desired conclusion, follows by I-M3′ and D-D2.

Clearly, (II.c.i) and (II.c.ii) are exhaustive subcases of (II.c) because the assumption of (II.c.ii) is the negation of the assumption of (II.c.i).

The three subcases (II.a)–(II.c) are exhaustive subcases of (II).

The cases (I) and (II) are exhaustive for any pair x, y; the argument is the same as in the proof of Lemma 8.32. Hence Dif-A1 is valid.

(Dif-A2) $y <_{\text{dim}} x \rightarrow x - y = x$.

Assume $y \mathbf{<_{dim}} x$ for arbitrary $x, y \in \mathbf{M}$.

We also have $(x \cdot y) in \mathbf{ZEX}_{\mathcal{M}}$ or $\mathbf{Cont}(x \cdot y, y)$ by Int-A2 (which we just proved in Lemma 8.33) so that $x \cdot y \mathbf{\leq_{dim}} y \mathbf{<_{dim}} x$. We obtain $\mathbf{GD}(x, x \cdot y)$ by I-M3′ and I-D9, so that $x - y = x$ by I-M6′. Hence, Dif-A2 is satisfied for all $x, y \in \mathbf{M}$.

(Dif-A3b) $x \leq_{\text{dim}} y \rightarrow [Cont(z, x-y) \rightarrow Cont(z, x)]$.

We distinguish the following cases.

Case (I): Assume $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$.

Then no z exists such that $\mathbf{Cont}(z, x-y)$, hence Dif-D3b holds trivially.

Case (II): Assume $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{GD}(x, x \cdot y)$.

Then by I-M6′ $x = x - y$ and hence for all $z \in \mathbf{M}$, $\mathbf{Cont}(z, x-y)$ implies $\mathbf{Cont}(z, x)$ as desired.

Case (III): Assume $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{ED}(x, x \cdot y)$.

Consider the following logical derivation

$$\begin{aligned}
&Cont(z, x-y) \\
\Rightarrow\, &CS(z, x-y) && \text{(I-M1}'\text{)} \\
\Rightarrow\, &\forall w[INCH(z,w) \rightarrow INCH(x-y,w)] && \text{(I-D1)} \\
\Rightarrow\, &\forall w\big[INCH(z,w) \rightarrow \exists v[INCH(v,w) \wedge CH(v,x)]\big] && \text{(I-M6}'\text{)} \\
\Rightarrow\, &\forall w\big[INCH(z,w) \rightarrow \exists v[\exists u[CS(u,v) \wedge CH(u,w)] \wedge CH(v,x)]\big] && \text{(I-T8)} \\
\Rightarrow\, &\forall w\big[INCH(z,w) \rightarrow \exists u[CS(u,x) \wedge CH(u,w)]\big] && \text{(I-M1}'\text{,C-A3)} \\
\Rightarrow\, &\forall w[INCH(z,w) \rightarrow INCH(x,w)] && \text{(I-PA7}'\text{)} \\
\Rightarrow\, &CS(z,x) && \text{(I-D1)} \\
\Rightarrow\, &Cont(z,x) \vee ZEX(z) && \text{(I-M1}'\text{)} \\
\Rightarrow\, &Cont(z,x)
\end{aligned}$$

The last step follows from the first line: $Cont(z, x-y)$ entails $\neg ZEX(z)$.

Then for any $z \in \mathbf{M}$, if $\mathbf{Cont}(z, x-y)$ then $\mathbf{Cont}(z, x)$ as desired.

As argued before, the cases (I) to (III) are exhaustive for any pair $x, y \in \mathbf{M}$. Hence Dif-A3b is satisfied.

(Dif-A3c) $x \leq_{\dim} y \rightarrow [P(z, x-y) \rightarrow z \cdot y <_{\dim} z]$.

We distinguish three cases.

Case (I): Assume $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$.

Then $\forall z[\neg \mathbf{P}(z, x-y)]$ and thus Dif-A3c trivially holds.

Case (II): Assume $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{ED}(x, x \cdot y)$.

Consider the following logical derivation:

$$\begin{array}{ll}
\quad P(z,x-y) \wedge z \cdot y \not<_{\dim} z & \\
\Rightarrow P(z,x-y) \wedge z \cdot y \geq_{\dim} z \wedge \mathit{Cont}(z \cdot y, z) & \text{(Int-A2)} \\
\Rightarrow P(z,x-y) \wedge z \cdot y =_{\dim} z \wedge \mathit{Cont}(z \cdot y, z) \wedge \mathit{Cont}(z \cdot y, y) & \text{(CD-A1)} \\
\Rightarrow z =_{\dim} x-y \wedge \mathit{Cont}(z, x-y) \wedge z \cdot y =_{\dim} z \wedge \mathit{Cont}(z \cdot y, z) \wedge \mathit{Cont}(z \cdot y, y) & \text{(EP-D)} \\
\Rightarrow \mathit{ED}(z,x-y) \wedge \mathit{CS}(z,x-y) \wedge \mathit{ED}(z \cdot y, z) \wedge \mathit{CS}(z \cdot y, z) \wedge \mathit{CS}(z \cdot y, y) & \text{(I-M1}',\text{I-M3}') \\
\Rightarrow \mathit{CH}(z,x-y) \wedge \mathit{CH}(z \cdot y, z) \wedge \mathit{CS}(z \cdot y, y) & \text{(I-T5)} \\
\Rightarrow \mathit{CH}(z,x-y) \wedge \exists w[\mathit{CH}(w,z) \wedge \mathit{CS}(w,y)] & \text{(I-D4,I-A7}') \\
\Rightarrow \exists w[\mathit{CH}(w,z) \wedge \mathit{CH}(w,x-y) \wedge \mathit{CS}(w,y)] & \text{(C-A3)} \\
\Rightarrow \exists w[\mathit{CH}(w,z) \wedge \mathit{CH}(w,x-y) \wedge \mathit{CH}(w,x) \wedge \mathit{CS}(w,y)] & \text{(Dif-A3b)} \\
\Rightarrow \exists w[\mathit{CH}(w,z) \wedge \mathit{CH}(w,x-y) \wedge \mathit{CH}(w,x \cdot y)] & \text{(I-M5}') \\
\Rightarrow \exists w[\mathit{INCH}(x-y,w) \wedge \mathit{CH}(w,x) \wedge \mathit{CH}(w,x \cdot y)] & \text{(I-D2,I-D4)} \\
\Rightarrow \exists w[\exists v[\mathit{INCH}(v,w) \wedge \mathit{CH}(v,x) \wedge \neg \mathit{OV}(v,x \cdot y)] \wedge \mathit{CH}(w,x) \wedge \mathit{CH}(w,x \cdot y)] & \text{(I-M6}') \\
\Rightarrow \exists w[\exists v[\mathit{OV}(v,w) \wedge \mathit{CH}(v,x) \wedge \neg \mathit{OV}(v,x \cdot y)] \wedge \mathit{CH}(w,x) \wedge \mathit{CH}(w,x \cdot y)] & \text{(I-T9)} \\
\Rightarrow \exists w\big[\exists v[\exists u[\mathit{CH}(u,v) \wedge \mathit{CH}(u,w)] \wedge \mathit{CH}(v,x) \wedge \neg \mathit{OV}(v,x \cdot y)] \wedge \mathit{CH}(w,x \cdot y)\big] & \text{(I-T14)} \\
\Rightarrow \exists w\big[\exists v[\mathit{CH}(v,w) \wedge \neg \mathit{OV}(v,x \cdot y)] \wedge \mathit{CH}(w,x) \wedge \mathit{CH}(w,x \cdot y)\big] & \text{(I-T13)} \\
\Rightarrow \exists w\big[\exists v[\mathit{CH}(v,w) \wedge \neg \mathit{OV}(v,x \cdot y)] \wedge \forall v[\mathit{CH}(v,w) \rightarrow \mathit{OV}(w,x \cdot y)]\big] & \text{(I-D4)}
\end{array}$$

The last line is self-contradictory, hence no entity $z \in \mathbf{M} \backslash \mathbf{ZEX}_{\mathcal{M}}$ with $\mathbf{ED}(x, x \cdot y)$, $\mathbf{P}(z, x-y)$ and $z \cdot y \boldsymbol{\not<_{\dim}} z$ can exist. Thus for all $z \in \mathbf{M} \setminus \mathbf{ZEX}_{\mathcal{M}}$ with $\mathbf{ED}(x, x \cdot y)$ and $\mathbf{P}(z, x-y)$, Dif-A3c must hold.

Case (III): Assume $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{GD}(x, x \cdot y)$.

Consider the following logical derivation:

$$\begin{array}{ll}
\quad \exists z[P(z,x-y) \wedge z \cdot y \not<_{\dim} z] & \\
\Rightarrow \exists z[P(z,x-y) \wedge z \cdot y =_{\dim} z] & \text{(Int-A2,CD-A1)} \\
\Rightarrow \exists z\big[P(z,x-y) \wedge \exists u[P(u,z) \wedge P(u,y)]\big] & \text{(Int-A4,PO-D)} \\
\Rightarrow \exists z\big[P(z,x) \wedge \exists u[P(u,z) \wedge P(u,y)]\big] & (x = x-y \text{ by I-M6}') \\
\Rightarrow \exists u[P(u,x) \wedge P(u,y)] & \text{(C-A3,D-T3)} \\
\Rightarrow x \cdot y =_{\dim} x & \text{(Int-T7)} \\
\Rightarrow \mathit{ED}(x \cdot y, x) & \text{(I-M3}')
\end{array}$$

Therefore, for arbitrary $x, y \in \mathbf{M}$ with $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{GD}(x, x \cdot y)$ there cannot exist a $z \in \mathbf{M}$ such that $\mathbf{P}(z, x-y)$ and $z \cdot y \not\leq_{\mathbf{dim}} z$, because we can then derive a contradiction in $\mathbf{ED}(x \cdot y, x)$. Hence for all $z \in \mathbf{M} \setminus \mathbf{ZEX}_{\mathcal{M}}$ with $\mathbf{GD}(x, x \cdot y)$ and $\mathbf{P}(z, x-y)$, Dif-A3 must hold.

As argued before, the three cases (I)–(III) are exhaustive for all pairs $x, y \in \mathbf{M}$.

(Dif-A4) $\mathit{ZEX}(x-y) \leftrightarrow \mathit{ZEX}(x) \vee \mathit{Cont}(x,y)$. We prove the two directions of the biconditional individually.

Direction (a): $\mathit{ZEX}(x-y) \rightarrow \mathit{ZEX}(x) \vee \mathit{Cont}(x,y)$.

We distinguish two cases:

Case (a.i): Assume $\mathbf{ED}(x, x \cdot y)$.

Assume $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$ and $x \notin \mathbf{ZEX}_{\mathcal{M}}$.

We will show that $x \cdot y = x$ and thus, because fro, $\forall x, y[\neg \mathit{ZEX} x \cdot y \rightarrow \mathit{Cont}(x \cdot y, y)]$ (Int-A2) we immediately obtain $\mathit{Cont}(x,y)$ because we already have $(x \cdot y) \notin \mathbf{ZEX}_{\mathcal{M}}$ because $x \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{ED}(x, x \cdot y)$).

To prove $x \cdot y = x$, it suffices to prove:

$$\mathbf{CH}(x \cdot y, x) \wedge \mathbf{CH}(x, x \cdot y).$$

$\mathbf{CH}(x \cdot y, x)$ is trivial, considering that $x \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{ED}(x, x \cdot y)$ imply $(x \cdot y) \notin \mathbf{ZEX}_{\mathcal{M}}$, which in turn implies $\mathit{Cont}(x \cdot y, x)$ by Int-A2. With $\mathit{ED}(x, x \cdot y)$, we obtain $\mathit{CH}(x \cdot y, x)$ by I-M1′ and I-T5.

To prove $\mathbf{CH}(x, x \cdot y)$, we use the definition of CH (I-D4) and prove instead

$$\mathbf{INCH}(x, x \cdot y) \wedge \forall z[\mathbf{OV}(z, x) \rightarrow \mathbf{OV}(z, x \cdot y)].$$

From $\mathbf{CH}(x \cdot y, x)$, which we just proved, we obtain $\mathbf{INCH}(x, x \cdot y)$ by I-D2 and I-D4. Notice that

$$\forall z[\mathbf{CH}(z, x) \rightarrow \mathbf{OV}(z, x \cdot y)]. \tag{8.34.4}$$

suffices to prove

$$\forall z[\mathbf{OV}(z, x) \rightarrow \mathbf{OV}(z, x \cdot y)] \tag{8.34.5}$$

by the following logical derivation:

$$\begin{aligned}
& OV(z,x) && \\
\Rightarrow & \exists u[CH(u,z) \wedge CH(u,x)] && \text{(I-T14)} \\
\Rightarrow & \exists u[CH(u,z) \wedge OV(u, x \cdot y)] && \text{(Equation 8.34.4)} \\
\Rightarrow & OV(z, x \cdot y)] && \text{(I-D2,I-A7}'\text{)}
\end{aligned}$$

Finally, consider the following derivation:

$$\begin{aligned}
& \mathit{ZEX}(x-y) && \\
\Rightarrow & \forall w[\neg \mathit{INCH}(x-y, w)] && \text{(I-A3, I-D6,I-M2}'\text{)} \\
\Leftrightarrow & \forall w \neg \exists v[\mathit{INCH}(v,w) \wedge \mathit{CH}(v,x) \wedge \neg OV(v, x \cdot y)] && \text{(I-M6}'\text{)}
\end{aligned}$$

$$\Leftrightarrow \forall w, v[INCH(v,w) \wedge CH(v,x) \rightarrow OV(v, x \cdot y)]$$
$$\Rightarrow \forall v[CH(v,x) \rightarrow OV(v, x \cdot y)] \qquad \text{(choose } w = v\text{)}$$

Because we assumed $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$, $\mathbf{CH}(z,x) \rightarrow \mathbf{OV}(z, x \cdot y)]$ holds for all $z \in \mathbf{M}$ and thus Equation 8.34.5 is satisfied for all $z \in \mathbf{M}$. Hence $\mathbf{CH}(x, x \cdot y)$ and thus $x = x \cdot y$. Consequently, we have $\mathbf{Cont}(x,y)$ as desired.

Case (a.ii): Assume $\mathbf{GD}(x, x \cdot y)$.

Assume $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$.

Then the assumption $\mathbf{GD}(x, x \cdot y)$ lets us deduce $x - y = x$ by I-M6′ and thus $x \in \mathbf{ZEX}_{\mathcal{M}}$ so that the desired sentence is satisfied for any x, y with $\mathbf{GD}(x, x \cdot y)$ and $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$.

As argued before, the two cases (a.i) and (a.ii) exhaustively cover arbitrary pairs of entities x, y.

Direction (b): $ZEX(x-y) \leftarrow ZEX(x) \vee Cont(x,y)$.

If $x \in \mathbf{ZEX}_{\mathcal{M}}$, then for all $v \in \mathbf{M}$, $\neg\mathbf{CH}(v,x)$ by I-T5. Hence, by I-M6′ no w exists such that $\mathbf{INCH}(x-y, w)$, so that $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$ immediately follows.

If $\mathbf{Cont}(x,y)$, then $x \cdot y = x$ and thus $\mathbf{ED}(x, x \cdot y)$. Then I-M6′ is equivalent to

$$\forall w[\mathbf{INCH}(x-y, w) \leftrightarrow \exists v[bf\,INCH(v,w) \wedge \mathbf{CH}(v,x) \wedge \neg\mathbf{OV}(v,x)]$$

Because $CH(v,x) \rightarrow OV(v,x)$, we conclude for all $w \in \mathbf{M}$, $\neg\mathbf{INCH}(x-y, w)$. Hence $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$ as desired.

The two direction (a) and (b) together entail Dif-A4.

(Dif-A3a) $x \leq_{\dim} y \rightarrow [Cont(z,x) \wedge z \cdot y <_{\dim} z \rightarrow Cont(z, x-y)]$.

We distinguish the following cases.

Case (I): Assume $(x-y) \in \mathbf{ZEX}_{\mathcal{M}}$.

By Dif-A4, $x \in \mathbf{ZEX}_{\mathcal{M}}$ or $\mathbf{Cont}(x,y)$. If $x \in \mathbf{ZEX}_{\mathcal{M}}$, then for all $z \in \mathbf{M}$, $\neg\mathbf{Cont}(z,x)$ and thus Dif-A3a holds trivially.

If $\mathbf{Cont}(x,y)$ then $\mathbf{Cont}(z,x)$ results in $\mathbf{Cont}(z,y)$ and thus $z \cdot y = z$ and $z \cdot y \not<_{\mathbf{dim}} z$. Hence Dif-A3a holds trivially again.

Case (II): Assume $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{GD}(x, x \cdot y)$.

Then by I-M6′ we have $x = x - y$ and therefore for all $z \in \mathbf{M}$, $\mathbf{Cont}(z,x) \rightarrow \mathbf{Cont}(z, x-y)$. Hence Dif-A3a holds in this case.

Case (III): $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$ and $\mathbf{ED}(x, x \cdot y)$.

Assume z to be an arbitrary entity with $\mathbf{Cont}(z,x)$ (implying $z \notin \mathbf{ZEX}_{\mathcal{M}}$) and $z \cdot y <_{\mathbf{dim}} z$. Because of our assumption $(x-y) \notin \mathbf{ZEX}_{\mathcal{M}}$, we have $x - y =_{\mathbf{dim}} x$ by Dif-A1. Because Dif-A3b requires $\mathbf{Cont}(x-y, x)$, we can thus assume $\mathbf{CH}(x-y, x)$ by I-M1′.

If for all $z \in \mathbf{M}$, $\mathbf{CH}(v,x)$ implies $\mathbf{CS}(v, x-y)$, we are done: then $\mathbf{CS}(x, x-y)$ by I-T10 and thus $x = x - y$, satisfying Dif-A3a trivially.

Now let us suppose there exists a $z \in \mathbf{M}$ such that $\mathbf{CH}(z,x)$ and not $\mathbf{CS}(z, x-y)$].

Consider the following logical derivation:

$$\begin{aligned}
&\exists z[CH(z,x) \wedge \neg CS(z, x-y)] \\
\Leftrightarrow&\exists z\big[CH(z,x) \wedge \exists w[INCH(z,w) \wedge \neg INCH(x-y,w)]\big] && \text{(I-D1)} \\
\Leftrightarrow&\exists w\big[\neg INCH(x-y,w) \wedge \exists z[INCH(z,w) \wedge CH(z,x)]\big] \\
\Leftrightarrow&\exists w\big[\neg\exists v[INCH(v,w) \wedge CH(v,x) \wedge \neg OV(v,x\cdot y)] \wedge \exists z[INCH(z,w) \wedge CH(z,x)]\big] && \text{(I-M6}'\text{)} \\
\Leftrightarrow&\exists w\big[\forall v[INCH(v,w) \wedge CH(v,x) \rightarrow OV(v,x\cdot y)] \wedge \exists z[INCH(z,w) \wedge CH(z,x)]\big] \\
\Rightarrow&\exists z[CH(z,x) \wedge OV(z,x\cdot y)] && (v := z) \\
\Rightarrow&\exists z[CH(z,x) \wedge INCH(x\cdot y,z)] && \text{(I-D2)} \\
\Rightarrow&\exists z\big[CH(z,x) \wedge \exists v[CH(v,z) \wedge CS(v,x\cdot y)]\big] && \text{(I-A7}'\text{)} \\
\Rightarrow&\exists z\big[CH(z,x) \wedge \exists v[CH(v,z) \wedge CS(v,y)]\big] && \text{(I-M5}'\text{)} \\
\Rightarrow&\exists z[CH(z,x) \wedge CS(z,y)] && \text{(I-T13)} \\
\Rightarrow&\exists z[CH(z,x) \wedge CS(z,z\cdot y)] && \text{(I-M5}'\text{)} \\
\Rightarrow&\exists z[CH(z,x) \wedge z\cdot y =_{\text{dim}} z] && \text{(I-M5}'\text{)}
\end{aligned}$$

Hence $z \cdot y \not<_{\mathbf{dim}} z$ and thus Dif-A3a is satisfied in this case as well.

As argued before, the cases (I)–(III) are exhaustive for any pair $x, y \in \mathbf{M}$. Hence Dif-A3a is valid.

Altogether, all of Dif-A1 – Dif-A4 are satisfied in the a model of $INCH_{\text{calculus}} \cup$ {I-E1, I-E2} using the mappings I-M1′ – I-M6′ and the definitions EP-D, EPP-D, PO-D. □

Lemma 8.35. *$INCH_{\text{calculus}} \cup$ {I-E1, I-E2} $\cup$ {I-M1′ – I-M6′} $\cup$ {EP-D, EPP-D, PO-D} $\models$ {Sum-A1 – Sum-A4}*

Proof. **(Sum-A1)** $x + y = y + x$.

We distinguish three cases.

Case (I): Assume $\mathbf{GD}(x, y)$.

Then by I-M4′ we have $x + y = x = y + x$.

Case (II): $\mathbf{GD}(y, x)$.

Then by I-M4′ we have $x + y = y = y + x$.

Case (III): $\mathbf{ED}(x, y)$.

Then by I-M4′ we have

$$\forall w[\mathbf{INCH}(x + y, w) \leftrightarrow \mathbf{INCH}(x, w) \vee \mathbf{INCH}(y, w)]$$

and

$$\forall w[\mathbf{INCH}(y + x, w) \leftrightarrow \mathbf{INCH}(y, w) \vee \mathbf{INCH}(x, w)],$$

so that

$$\forall w[\mathbf{INCH}(x + y, w) \leftrightarrow \mathbf{INCH}(y + x, w)],$$

and hence $x + y = y + x$ by I-A1.

The three cases (I) to (III) are clearly exhaustive for arbitrary pairs $x, y \in \mathbf{M}$.

(Sum-A2) $x <_{\dim} y \to x + y = y$.

From $x <_{\dim} y$ we obtain $\mathbf{GD}(y, x)$ by I-M3′ and I-D9. Then by I-M4′ $x + y = y$.

(Sum-A3) $x \leq_{\dim} y \wedge Cont(z, y) \to Cont(z, x + y)$.

Assume $x \leq_{\mathbf{dim}} y$.

Then either $\mathbf{GD}(y, x)$ or $\mathbf{ED}(x, y)$ by I-M3′. We consider these two cases separately.

Case (I): Assume $\mathbf{GD}(y, x)$.

Then by I-M4′ $x + y = y$ and thus for all $z \in \mathbf{M}$, $\mathbf{Cont}(z, y)$ implies $\mathbf{Cont}(z, x + y)$.

Case (II): $\mathbf{ED}(x, y)$.

Consider the following logical derivation:

$$ED(x, y) \to \forall w[INCH(x + y, w) \leftrightarrow INCH(x, w) \vee INCH(y, w)] \qquad \text{(I-M4′)}$$

$$\Rightarrow ED(x, y) \to \forall w[INCH(y, w) \to INCH(x + y, w)]$$

$$\Rightarrow ED(x, y) \to CS(y, x + y) \qquad \text{(I-D1)}$$

$$\Rightarrow ED(x, y) \to \forall w[CS(z, y) \to CS(z, x + y)] \qquad \text{(I-T11)}$$

$$\Rightarrow ED(x, y) \to \forall w[Cont(z, y) \to Cont(z, x + y)] \qquad \text{(I-M1′)}$$

so that Sum-A3 holds when $\mathbf{ED}(x, y)$.

The cases (I) and (II) are clearly exhaustive for arbitrary pairs x, y with $x \leq_{\dim} y$.

(Sum-A4) $Cont(z, x + y) \wedge \neg Cont(z, x) \to Cont(z - x, y)$.

We distinguish three cases.

Case (I): Assume $\mathbf{GD}(x, y)$.

Then by I-M4′ we have $x+y = x$, and thereby for all $z \in \mathbf{M}$, $\mathbf{Cont}(z, x+y)$ implies $\mathbf{Cont}(z, x)$ so that Sum-A4 is trivially satisfied.

Case (II): Assume $\mathbf{GD}(y, x)$.

Then by I-M4′ we have $x + y = y$.

We prove the contrapositive

$$\neg\mathbf{Cont}(z - x, y) \to \neg\mathbf{Cont}(z, x + y) \vee \mathbf{Cont}(z, x).$$

Note that $\mathbf{Cont}(w, z - x) \to \mathbf{Cont}(w, z)$ by Dif-A3b. Hence, if $\neg\mathbf{Cont}(z - x, y)$ then $\neg\mathbf{Cont}(z, y)$. But by $x + y = y$ this would mean that $\neg\mathbf{Cont}(z, x + y)$, thereby satisfying the contrapositive.

Case (III): Assume $\mathbf{ED}(x, y)$.

To prove Sum-A4, let us suppose its consequent is false, that is, we suppose $\neg\mathbf{Cont}(z - x, y)$ for arbitrary $x, y, z \in \mathbf{M}$ with $\mathbf{ED}(x, y)$. Recall that by I-D1, I-M1′,

$$\neg Cont(z - x, y) \leftrightarrow \exists w[INCH(z - x, w) \wedge \neg INCH(y, w)].$$

We distinguish two subcases.

Subcase (III.a): Assume $\mathbf{ED}(x, y)$ and $\mathbf{ED}(z, z \cdot x)$.

Assume $\mathbf{Cont}(z - x, y)$.

If $(z-x) \in \mathbf{ZEX}_{\mathcal{M}}$, then either $z \in \mathbf{ZEX}_{\mathcal{M}}$ or $\mathbf{Cont}(z, x)$. In the first case, $\neg\mathbf{Cont}(z, x+y)$ immediately follows by C-A4. Thus, in either case the antecedent of Sum-A4 is falsified, so that Sum-A4 holds.

Now assume $(z - x) \notin \mathbf{ZEX}_{\mathcal{M}}$ and consider the following logical derivation:

$$
\begin{aligned}
&\neg Cont(z-x,y) \wedge \neg ZEX(z-x) \\
\Rightarrow&\neg CS(z-x,y) && \text{(I-M1}'\text{)} \\
\Rightarrow&\exists w[CH(w,z-x) \wedge \neg CS(w,y)] && \text{(I-T10)} \\
\Rightarrow&\exists w[CH(w,z-x) \wedge \exists u[INCH(w,u) \wedge \neg INCH(y,u)]] && \text{(I-D1)} \\
\Rightarrow&\exists w[CH(w,z-x) \wedge \neg INCH(y,w)] && (w := u) \\
\Rightarrow&\exists w[\neg INCH(y,w) \wedge INCH(z-x,w)] && \text{(I-D4,I-T9)} \\
\Rightarrow&\exists w\big[\neg INCH(y,w) \wedge \exists v[INCH(v,w) \wedge CH(v,z) \wedge \neg OV(v,z\cdot x)]\big] && \text{(I-M6}'\text{)} \\
\Rightarrow&\exists v[\neg INCH(y,v) \wedge CH(v,z) \wedge \neg OV(v,z\cdot x)] && \text{(I-D4)} \\
\Rightarrow&\exists v[CH(v,z) \wedge \neg INCH(x,v) \wedge \neg INCH(y,v)] && (*) \\
\Rightarrow&\exists v[INCH(z,v) \wedge \neg INCH(x,v) \wedge \neg INCH(y,v)] && \text{(I-T9)} \\
\Rightarrow&\exists v[INCH(z,v) \wedge \neg INCH(x+y,v)] && \text{(I-M4}'\text{)} \\
\Rightarrow&\neg CS(z,x+y) && \text{(I-D1)} \\
\Rightarrow&\neg Cont(z,x+y) && \text{(I-M1}'\text{)}
\end{aligned}
$$

Where step $(*)$ follows from:

$$
\begin{aligned}
&CH(v,z) \wedge \neg OV(v,z\cdot x) \\
\Rightarrow&CH(v,z) \wedge \neg\exists u[CH(u,v) \wedge CH(u,z\cdot x)] && \text{(I-T14)} \\
\Rightarrow&CH(v,z) \wedge \neg\exists u[CH(u,v) \wedge CH(u,z) \wedge CS(u,x)] && \text{(I-M5}'\text{)} \\
\Rightarrow&\forall u[CH(u,v) \rightarrow \neg CS(u,x)] \\
\Rightarrow&\neg INCH(x,v) && \text{(I-A7}'\text{)}
\end{aligned}
$$

Hence Sum-A4 is satisfied in this case.

Subcase (III.b): ED(x, y) and **GD**$(z, z \cdot x)$.

Consider the following computation:

$$
\begin{aligned}
&GD(z,z\cdot x) \wedge \neg Cont(z-x,y) \\
\Rightarrow&\forall w[CH(w,z) \rightarrow \neg INCH(w,x)] \wedge z-x=z \wedge \neg Cont(z-x,y) && \text{(I-M5}'\text{)} \\
\Rightarrow&\forall w[CH(w,z) \rightarrow \neg INCH(w,x)] \wedge \neg CS(z,y) && \text{(I-M1}'\text{)} \\
\Rightarrow&\forall w[CH(w,z) \rightarrow \neg INCH(w,x)] \wedge \exists w[CH(w,z) \wedge \neg INCH(w,y)] && (\text{I-T10}, ED(x,y)) \\
\Rightarrow&\exists w[CH(w,z) \wedge \neg INCH(w,y) \wedge \neg INCH(w,x)] \\
\Rightarrow&\exists w[INCH(w,z) \wedge \neg INCH(w,y) \wedge \neg INCH(w,x)] && \text{(I-D4,I-T9)} \\
\Rightarrow&\exists w[INCH(z,w) \wedge \neg INCH(y,w) \wedge \neg INCH(x,w)] && \text{(I-T9)} \\
\Rightarrow&\exists w[INCH(z,w) \wedge \neg INCH(x+y,w)] && \text{(I-M4}'\text{)}
\end{aligned}
$$

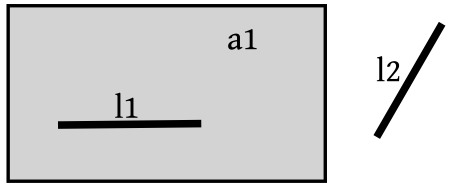


Figure 8.5: A model $\mathcal{M}$ of $INCH_{\text{calculus}} \cup$ {I-E1, I-E2} with domain $\mathbf{M} = \{a1, l1, l2, l1+l2, \text{ze}\}$. The model can be specified by its extension $\mathbf{INCH}_{\mathcal{M}} = \{\langle a1, a1\rangle, \langle a1, l1\rangle, \langle a1, l1+l2\rangle, \langle l1, l1\rangle, \langle l1, l1+l2\rangle, \langle l2, l2\rangle, \langle l2, l1+l2\rangle, \langle l1+l2, l1\rangle, \langle l1+l2, l2\rangle, \langle l1+l2, l1+l2\rangle\}$. With the mappings I-M1′ to I-M6′, we further have the defined extensions $\mathbf{Cont}_{\mathcal{M}} = \{\langle a1, a1\rangle, \langle l1, l1\rangle, \langle l1, l2\rangle, \langle l2, l2\rangle, \langle l1+l2, l1+l2\rangle\}$, $\mathbf{ZEX}_{\mathcal{M}} = \{\text{ze}\}$, and $(<_{\mathbf{dim}})_{\mathcal{M}} = \{\langle l1, a1\rangle, \langle l2, a1\rangle, \langle l1+l2, a1\rangle, \langle \text{ze}, a1\rangle, \langle \text{ze}, l1\rangle, \langle \text{ze}, l2\rangle, \langle \text{ze}, l1+l2\rangle\}$.

$$\Rightarrow \neg CS(z, x+y) \qquad \text{(I-D1)}$$
$$\Rightarrow \neg Cont(z, x+y) \qquad \text{(I-M1}'\text{)}$$

which falsifies the antecedent of Sum-A4, and hence Sum-A4 is satisfied in this case.

As argued before, (III.a) and (III.b) are exhaustive subcases of Case (III) for any triple x, y, z.

The three cases (I) to (III) are clearly exhaustive for any pair x, y.

□

However, as it turns out, U-A1 is not provable.

Lemma 8.36. *$INCH_{\text{calculus}} \cup$ {I-E1, I-E2} $\cup$ {I-M1′ – I-M6′} $\not\models$ U-A1*

Proof. A counterexample is provided in inch/consistency/inch_calculus_extended_notI-E3.clif; it is also specified in Figure 8.5. □

Therefore, we extend the INCH Calculus by an extra axiom I-E3, which guarantees an entity to exist of which every other is a constituent.

(I-E3) $\exists u \forall x[CS(u, x)]$	(an entity exists of which every entity is a constituent)

Axiom Set 8.16: Extension axiom I-E3 of $INCH_{\text{calculus}}$.

This is basically a translation of U-A1 into the nonlogical language of the INCH Calculus. Extending the INCH Calculus by this extra axiom, we are able to prove U-A1.

Lemma 8.37. *$INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} $\cup$ {I-M1′ – I-M6′} $\models$ U-A1*

Proof. Consider the following logical derivation:

$$\exists u \forall x[CS(x, u)] \qquad \text{(I-E3)}$$
$$\Rightarrow \exists u \forall x[\neg ZEX(x) \rightarrow Cont(x, u)] \qquad \text{(I-M1}'\text{)}$$

where the last line is U-A1. □

Summarily, Corollary 8.5 together with the Lemmas 8.29 to 8.37 show that all axioms of $CODI_{\updownarrow}$ are consequences of $INCH_{\text{calculus}} \cup$ {I-E1-I-E3} when we use I-M1′ – I-M6 as mapping axioms. The following corollary captures this.

Corollary 8.6. *$INCH_{\text{calculus}} \cup$ {I-E1-I-E3} $\cup$ {I-M1′ – I-M6′} $\cup$ {EP-D, EPP-D, PO-D} $\models CODI_{\updownarrow}$*

To show that the theories $CODI_{\updownarrow} \cup$ C-E4 and $INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} are definably equivalent, two final steps remain. First, we must prove C-E4 from $INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} and, second, we must show that the axioms I-E1 to I-E3 are consequences of $CODI_{\updownarrow} \cup$ C-E4 together with the mapping axioms. Only then we know that we do not have introduced restrictions that are not provable in $CODI_{\updownarrow}$.

Lemma 8.38. *$INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} $\cup$ {I-M1′ – I-M6′} $\cup$ {EP-D, EPP-D, PO-D} $\models$ C-E4*

Proof. Consider the following computation:

$$
\begin{aligned}
&GED(y,x) \rightarrow ZEX_I(x) \vee \exists z[INCH(y,z) \wedge INCH(z,x)] && \text{(I-D7)}\\
\Rightarrow\, &GED(y,x) \rightarrow ZEX(x) \vee \exists z[INCH(y,z) \wedge INCH(z,x)] && \text{(I-M2}'\text{)}\\
\Rightarrow\, &GED(y,x) \rightarrow ZEX(x) \vee \exists z\big[\exists w[CS(w,y) \wedge CH(w,z)] \wedge \exists v[CS(v,z) \wedge CH(v,x)]\big] && \text{(I-A7}'\text{)}\\
\Rightarrow\, &x \leq_{\text{dim}} y \rightarrow ZEX(x) \vee \exists z\big[\exists w[CS(w,y) \wedge CH(w,z)] \wedge \exists v[CS(v,z) \wedge CH(v,x)]\big] && \text{(I-M3}'\text{)}\\
\Rightarrow\, &x \leq_{\text{dim}} y \rightarrow \big[ZEX(x) \vee \exists z,v,w[P(v,x) \wedge Cont(v,z) \wedge P(w,z) \wedge Cont(w,y)]\big] && \text{(I-M1}'\text{,EP-D)}
\end{aligned}
$$

This proves C-E4. □

Lemma 8.39. *$CODI_{\updownarrow} \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M10} $\models$ {I-E1 – I-E3}*

Proof. **(I-E1):** $\exists x[\neg ZEX_I(x) \wedge \forall y(\neg ZEX(y) \rightarrow GED(y,x))]$.

Already proved in Lemma 8.18.

(I-E2): $CO(x,y) \rightarrow \exists z\big[CS(z,x) \wedge CS(z,y) \wedge \forall w[CS(w,x) \wedge CS(w,y) \rightarrow GED(z,w)] \wedge \forall w[CS(w,x) \wedge CS(w,y) \wedge ED(w,z) \rightarrow CS(w,z)]\big]$.

By the following axioms and theorems

- I-M1 to I-M10 (the mapping axioms),
- Int-A1: $C(x,y) \rightarrow \neg ZEX(x \cdot y)$, and
- I-T6: $ZEX_I(x) \rightarrow CS(x,y)$,

I-E2 is equivalent to (note that I-A2 requires $z \notin \mathbf{ZEX}_{\mathcal{M}}$):

$$
C(x,y) \rightarrow \exists z\big[Cont(z,x) \wedge Cont(z,y) \wedge \forall w[Cont(w,x) \wedge Cont(w,y) \rightarrow w \leq_{\text{dim}} z] \wedge \forall w[Cont(w,x) \wedge Cont(w,y) \wedge w =_{\text{dim}} z \rightarrow Cont(w,z)]\big]
$$

which is always satisfied for the choice $z := x \cdot y$ because

- $Cont(x \cdot y, x)$ and $Cont(x \cdot y, y)$ by Int-A2,
- $\forall w[Cont(w,x) \wedge Cont(w,y) \rightarrow w \leq_{\text{dim}} x \cdot y]$ by Int-A3, and
- $\forall w[Cont(w,x) \wedge Cont(w,y) \wedge w =_{\text{dim}} z \rightarrow Cont(w, x \cdot y)]$ by Int-A4.

(I-E3): $\exists u \forall x[\neg ZEX_I(x) \rightarrow INCH(u,x)]$.

Consider the following computation:

$$\begin{aligned} &\exists u\forall x[\neg ZEX(x) \rightarrow Cont(x,u)] && \text{(U-A1)}\\ \Rightarrow&\exists u\forall x[\neg ZEX_I(x) \rightarrow CS(x,u)] && \text{(I-M1, I-M2)}\\ \Rightarrow&\exists u\forall x[CS(x,u)] && \text{(I-T6)} \end{aligned}$$

This proves I-E3.

□

8.2.6 Summary

As a result of the proofs in this section, particularly in Subsections 8.2.4 and 8.2.5, it follows that $CODI_{\updownarrow} \cup$ C-E4 and $INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} are definably equivalent spatial theories.

Theorem 8.3. *$CODI_{\updownarrow} \cup$ C-E4 and $INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} are definably equivalent.*

Proof. Immediate from Corollaries 8.4 and 8.6 together with Lemmas 8.38 and 8.39. □

This theorem maps a slightly extended version of the INCH Calculus to a theory in the *CODI* hierarchy. Though neither $CODI_{\updownarrow}$ definably interprets $INCH_{\text{calculus}}$ nor $INCH_{\text{calculus}}$ definably interprets $CODI_{\updownarrow}$, the two theories are still closely related. Theorem 8.3 shows that $CODI_{\updownarrow}$ and $INCH_{\text{calculus}}$ only differ in the following ontological assumptions:

- $CODI_{\updownarrow}$ does not assume C-E4, which is an assumption of $INCH_{\text{calculus}}$. In other words, in models of $CODI_{\updownarrow}$ two entities may be dimensionally comparable without sharing a common entity. This has deeper consequences, in particular every model of the INCH Calculus must consist of a single connected component in the sense that all entities are indirectly connected.

- $INCH_{\text{calculus}}$ does not assume I-E1. In other words, models of $INCH_{\text{calculus}}$ may not contain a nonzero entity of lowest dimension. This is enforced by D-A6 in $CODI_{\updownarrow}$.

- $INCH_{\text{calculus}}$ does not assume I-E2. That is, models of $INCH_{\text{calculus}}$ may not be closed under intersections. This is particularly the case when the intersection would be of a lower dimension than both of the intersecting entities. In other words, two entities in a model of $INCH_{\text{calculus}}$ may have a zero intersection even though they are in superficial contact, i.e., they share a common lower-dimensional entity.

- $INCH_{\text{calculus}}$ does not assume I-E3, i.e., in a model of $INCH_{\text{calculus}}$ there may not exist an entity of which all other entities are constituents of.

Nevertheless, our result integrates the INCH Calculus into the *CODI* hierarchy. It shows that the theory $INCH_{\text{calculus}}$ is definable using only the primitive relations *Cont*, $<$, and *ZEX* from *CODI*. The relationship between $INCH_{\text{original}}$, $INCH_{\text{calculus}}$, and the *CODI* hierarchy is illustrated in Figure 8.6.

The integration greatly enhances our understanding of two theories. In our particular case, the integration helped to identify restrictions and shortcomings of the INCH Calculus. Particularly, proving the interpretation helped to discover the problem of the missing axiom I-A7′ and helped to identify that the INCH Calculus lacks the guarantee that a nonzero entity of minimal dimension exists (I-E1). I-E1 should probably be added to the INCH Calculus since it is difficult to conceive intended models in which

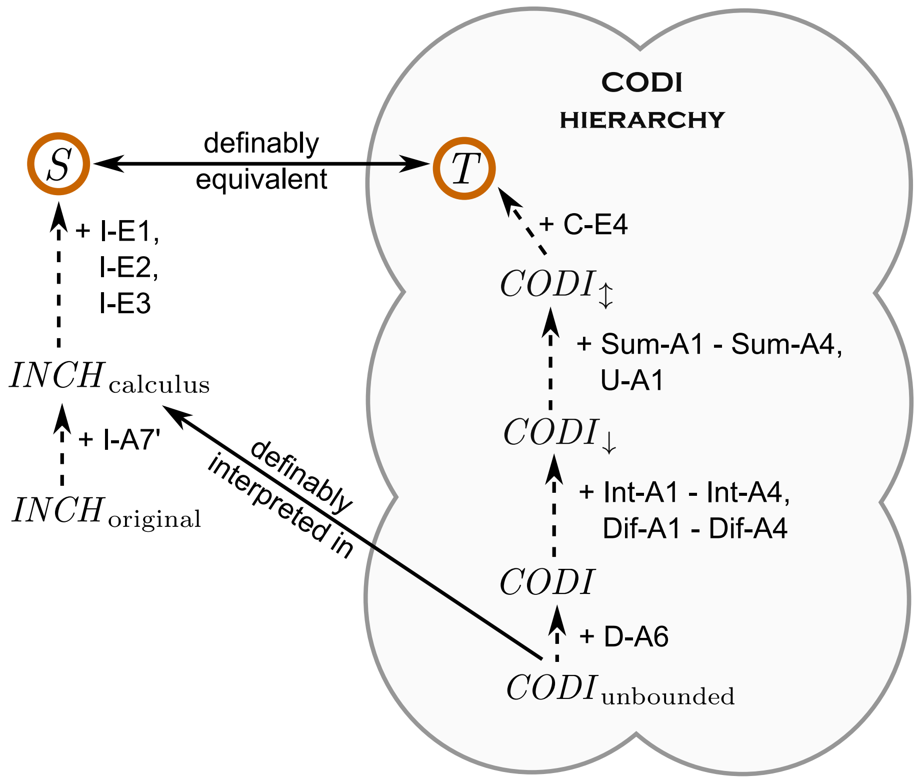


Figure 8.6: The relationship of the original INCH Calculus and the corrected INCH Calculus to the *CODI* hierarchy.

I-E1 would not hold. But this is a ontological problem, not a logical problem. Equally, whether or not to add I-E2 and I-E3 is a matter of ontological choice and the particular domain of interest. On the other side, C-E4 is an inherent limitation of the INCH Calculus, if we do not want to make this ontological assumption, we cannot use INCH Calculus as our theory of choice.

From a methodological point of view, this section has demonstrated that establishing relative interpretability between two theories can be extremely insightful even if the relationship between two theories (or a theory and a hierarchy of theories as in our case) is not a relative interpretation in either direction. By extending either of the two theories by axioms that are not provable in the theory itself, but whose translations are theorems of the other theory, we identified the differences between the two theories. As result, we identified nonconservative extensions of $INCH_{\text{original}}$ and of *CODI* that are definably equivalent. The structure of the section emphasizes that such an integration of two theories in different languages (or a theory and a hierarchy) requires an interactive approach and is usually not achieved in one shot. We started by proving axioms of *CODI* from the INCH Calculus in Section 8.2.3. The ones that were not provable required us to introduce an extension to $INCH_{\text{calculus}}$, namely I-E1. Then in Section 8.2.4 we proved the axioms of $INCH_{\text{calculus}}$ as well as I-E1 from *CODI*, which required us to introduce C-E4. Finally, though most of the axioms of $CODI_{\updownarrow}$ were provable from $INCH_{\text{calculus}} \cup$ I-E1, we still needed to introduce I-E2 and I-E3 to prove all axioms of $CODI_{\updownarrow}$ (Section 8.2.5). To complete the proof of definable equivalence, we had to ensure that the extensions I-E1 – I-E3 were indeed provable in $CODI_{\updownarrow}$.

This section corrects and extends the result from [HG11a; HG11b] in various ways. Firstly, it shows that the interpretation as stated in [HG11a; HG11b] is not a faithful interpretation since there exist sentences whose translations are consistent with $INCH_{\text{original}}$ but which contradict D-A6 in $CODI^0$. For this reason we have introduced I-E1.

Secondly, it became clear that the original mapping axioms I-M1 to I-M10 were used too liberally in [HG11a; HG11b]: they nonconservatively extend $CODI^0$. In other words, we only proved the interpretability of a nonconservative extension of $CODI^0$ in terms of the theory $INCH_{\text{original}}$ (which has been called $INCH_{\text{calculus}}$ in [HG11a; HG11b]). In this sense, the original result in [HG11a; HG11b]) is not wrong, but less meaningful because we did not make explicit which extension of $CODI^0$ definable interpretability had been proved for, because the extra axiom was hidden within the mapping axioms. Now, we have made the additional axiom explicit in the form of C-E4.

Finally, we significantly strengthen our original result [HG11a; HG11b]) by proving the interpretability of a theory in the language of *CODI* instead of maintaining I-A1, I-A2, and I-A8 to I-A10 as extensions axioms. Now, it is clear that those axioms are reflected in the axioms that extend $CODI^0$ to $CODI_{\updownarrow}$.

Chapter 9

Boundaries in multidimensional mereotopological space

Boundaries are a key concept within topology and mereotopology. All the different conceptions and aspects of boundaries, borders, and surfaces comprise a far too broad topic to be covered satisfactorily in a single chapter. Instead, this chapter continues our investigation of qualitative space in the spirit of the previous chapters by concentrating on boundaries as they occur in abstract space.

As we discussed before, the concept of a boundary is not definable in the language of *CODI*. More precisely, the relation $\Sigma \mathrm{MF}_1 \subseteq \Delta \mathrm{MF}_2$ that is definable in every intended structure is not definable in *CODI*. Hence, we must extend the nonlogical language. For that purpose, we introduce a primitive notion of *boundary-containment*, $BCont(x, y)$, that refines the containment relation and has the intended interpretation $\Sigma \mathrm{MF}_1 \subseteq \Delta \mathrm{MF}_2$ where MF_1 and MF_2 are the manifolds captured by x and y, respectively. This constructs the new hierarchy *CODIB*. In the *CODIB* hierarchy, the primitive relation *BCont* will enable us to capture condition (3) of the definition of our class of intended structures $\mathbb{M}$ (Definition 5.11) more adequately. For this to work as desired, we must restrict the class of intended structures $\mathbb{M}$ by an extra condition on the dimension ordering, resulting in the class $\mathbb{M}_{\mathrm{dense}}$ in Section 9.1.

We will show that two prevalent conceptions of boundaries in abstract space, namely (1) bodiless, *thin boundaries* and (2) bulky, *thick boundary regions* can be modelled in the *CODIB* theories. Intriguingly, both boundary conceptions can coexist in a single theory—we do not have to choose one over the other. We will start in Section 9.3 by defining thin boundaries, i.e., boundaries that are of a lower dimension than the entities that are bounded by them, and later proceed to thick boundaries, i.e., boundaries of the same dimension as the entities that they bound, in Section 9.4. among other uses, thick boundaries are suited to capture the abstract space regions occupied by physical (material) surfaces, independent of their granularity. Along the way we define interior and tangential containment and interior and tangential parthood, which are exhaustive and disjoint subrelations of containment and parthood, respectively, as we will show in Sections 9.3.4 and 9.4.1. boundary-containment and boundary parthood are then subsumed by tangential containment or parthood, respectively.

One of the key results of this chapter is the satisfiability proof for $CODIB_{\downarrow}$ (Theorem 9.2), which shows that every model in the class $\mathbb{M}_{\mathrm{dense}}$ is not only a model of the theory $CODI_{\downarrow}$, but also satisfies all axioms pertaining to the primitive notion of boundary-containment. The class $\mathbb{M}_{\mathrm{dense}}$ is introduced in Section 9.1 as a minor restriction of the class of intended structures, $\mathbb{M}$, from Chapter 5. This restriction lets us also define the notion of 'internal self-connectedness', an essential concept in the axiomatization of boundaries that refines self-connectedness as axiomatized earlier, in Section 9.2.

In Section 9.5 we define refinements of contact based on whether the interior or boundary of an entity is in contact to the interior or boundary of a second entity. The set of four resulting relations IO, IBC, IBC^{-1}, and BO exhaustively classify contact (Theorem 9.4), though their extensions are not necessarily disjoint. The chapter finishes with a proof (Theorem 9.5) that those four relations together with another five defined relations that express whether an entity's interior, boundary, or exterior overlaps the second entity's exterior and vice versa are capable of defining the manifold-equivalents of the nine topological intersections proposed by Egenhofer et al. [Ege89; Ege91; EF91; EH91].

To place our work in context, some words about how bodiless and bulky boundaries in abstract space can be used to capture the space occupied by material boundaries that reside in our conceptualization in physical space. In abstract space, we can simultaneously have bodiless and bulky boundaries as spatial entities of different dimensions. All purely fiat boundaries, i.e., boundaries that do not describe a physical discontinuity, seem to reside only in abstract space because of their artificial nature[1]. Once we link physical space to its underlying abstract space, we can also talk about physical, i.e., bona-fide boundaries. For example, material surfaces are probably best modelled as occupying regions of the same dimension as the bounded physical object. In other words, the abstract regions of physical boundaries are bulky. But at the same time, we can talk about physical interfaces: their underlying abstract regions are bodiless. Our distinction between abstract and physical space allows bodiless boundaries of touching or adjacent objects to coincide: we can have bodiless boundaries of two distinct physical objects that happen to occupy some shared region of space. Thus, the two competing notions of boundaries are compatible and not mutually inconsistent. If we talk about the space physical objects occupy, then their physical boundaries require boundaries in the underlying abstract model of space to exist, but not all abstract spatial boundaries coincide with some physical boundaries. In Chapter 11 we will present a theory that relates physical to abstract space. Though we will not explicitly model boundaries in the physical space, it is possible to do so by using the abstract boundaries introduced in the present chapter.

9.1 A restriction of the intended structures

The logical theories we propose in this chapter are based on an additional constraint being imposed upon the class of intended models $\mathbb{M}^m$, namely that there exists a composite manifold of every dimension $0 \leq n \leq m$. We define the restricted class of intended structures $\mathbb{M}_{\text{dense}}$ as follows.

Definition 9.1. *Let $\mathfrak{M}^m$ be a complex m-manifold. We say $\mathfrak{M}^m$ has a dense dimension ordering and write $\mathfrak{M}^m_{\text{dense}}$ if and only if for all n with $n \leq m$ there exists a manifold $\mathrm{MF}^n \in \mathfrak{M}^m$.*
The class of all complex manifolds with a dense dimension ordering is denoted as $\mathbb{M}_{\text{dense}}$.

This restricts the interpretation of the relation of relative dimension $<_{\text{dim}}$ more than previously in

[1] We rely on the distinction between fiat and bona-fide boundaries introduced by Smith and Varzi [Smi95; SV97; SV00]. In their characterization, fiat boundaries depend on a cognitive act whereas bona-fide exist in material space independently of any cognitive act. Usually, fiat boundaries are induced and agreed upon by laws or social norms. Fiat boundaries may still coincide with a bona-fide boundary. In fact, often a bona-fide boundary is specifically erected to make a fiat boundary physical. For example, countries mark their borders or chose to build a fence to demarcate their terrestrial borders and to discourage people from illegally crossing. Then the fiat boundary that demarcates two countries spatially coincides with a physical boundary created by the fence. This is the same as a country boundary coinciding with a river. Other times, fiat boundaries, such as between countries, may not be physically visible, but only enforced through border patrols. We call such a boundary fiat even though crossing it may have physical consequences.

$\mathbb{M}$ and in the models of *CODI*. The restriction shall ensure that the relation $\prec_{\text{dim}}$ is interpreted as

$$\langle d_1, d_2\rangle \in (\prec_{\mathbf{dim}})_{\mathcal{M}} \iff \dim(d_1) + 1 = \dim(d_2) \qquad \text{(Prec-CODIB)}$$

for all atomic or composite manifolds $d_1, d_2 \in \mathfrak{M}$. Within *CODI* we previously used the following weaker interpretation (compare Theorem 7.4 and Definition D-D6) of $\prec_{\text{dim}}$, which is implied by the intended interpretation of $<_{\text{dim}}$ and the subsequent logical definition of $\prec_{\text{dim}}$.

$$\langle d_1, d_2\rangle \in (\prec_{\mathbf{dim}})_{\mathcal{M}} \iff \dim(d_1) < \dim(d_2) \text{ and no } d_3 \in \mathfrak{M} \text{ exists}$$
$$\text{such that}\big(\dim(d_1) < \dim(d_3) < \dim(d_2)\big) \qquad \text{(Prec-CODI)}$$

This restriction cannot be expressed axiomatically in *CODI* itself, i.e., any model of *CODI* may be interpreted in this way, but may also be interpreted differently. However, many of the axioms and definitions we will introduce in this chapter only apply to models that satisfy the restricted interpretation of $\prec_{\text{dim}}$ from (Prec-CODIB). In other words, a model of *CODI* that has only interpretations that violate (Prec-CODIB) may not be extensible to a model of the theory *CODIB* presented in this chapter. This does not mean, all models of *CODIB* can only be interpreted in ways that satisfy (Prec-CODIB). In particular, the axiom BC-A3 is not valid for models which do not satisfy the interpretation (Prec-CODIB).

Such a restricted reading of $\prec_{\text{dim}}$ allows us to define a—previously undefinable—refined notion of self-connectedness. For completeness, we will introduce this notion at this point, even though it departs from the main objective of this chapter. It will become extremely useful later in the chapter.

9.2 Internal self-connectedness

We already defined a simple notion of self-connectedness, *Con*, in $CODI_{\downarrow}$ on page 127. Once we restrict the class of intended structures to those that contain an entity of each dimension, we can define a stronger notion of self-connectedness called internal self-connectedness [CV03], $ICon(x)$, meaning the interior of x is a single piece. This relation is also known as 'strong self-connectedness' $SSC(x)$ in equidimensional mereotopologies [compare BGM96; CV99a; CR08]. We say a connected entity x is internally self-connected if the intersection between every proper part y of x and the difference $x - y$ is of exactly one dimension lower than x itself (ICon-D). For this definition to work as expected we require (Prec-CODIB) to assure that the next-lowest dimension is actually the next lowest Euclidean dimension as expressed by $\prec_{\text{dim}}$.

For completeness, we also define a notion called uniform self-connectedness, which is a different strengthening of self-connectedness. A self-connected entity x is uniformly self-connected if all entities z in the intersection of a proper part y of x and the difference $x - y$ are contained in the intersection of highest dimension $y \cdot (x - y)$ (UCon-D). See Figure 9.1 for examples and counterexamples for the three different notions of self-connectedness.

Just as every minimal entity and the zero entity are self-connected, they are also internally self-connected and uniformly self-connected. This follows directly from *ICon* and *UCon* specializing *Con*. *ICon* and *UCon* are neither exhaustive nor disjoint subrelations of *Con*.

Because minimal entities have no proper parts, they are trivially internally and uniformly self-connected. This matches our understanding that those correspond to m-manifolds with boundaries, which

(ICon-D) $ICon(x) \leftrightarrow Con(x) \wedge \forall y[PP(y,x) \rightarrow y \cdot (x-y) \prec_{\dim} x]$ (internal self-connectedness)

(UCon-D) $UCon(x) \leftrightarrow Con(x) \wedge \forall y,z[PP(y,x) \wedge Cont(z,y) \wedge Cont(z,x-y) \rightarrow Cont(z,y \cdot (x-y))]$ (uniform self-connectedness)

Axiom Set 9.1: Definitions ICon-D and UCon-D of the *CODI* hierarchy.

intuitively must be internally and uniformly self-connected, in the intended class of structures $\mathbb{M}_{\text{dense}}$.

(ICon-T1) $Min(x) \vee ZEX(x) \rightarrow ICon(x)$ (minimal and zero entities are internally self-connected)

(UCon-T1) $Min(x) \vee ZEX(x) \rightarrow UCon(x)$ (minimal and zero entities are uniformly self-connected)

Lemma 9.1. $CODI_{\downarrow} \cup$ *{Con-D, ICon-D, UCon-D}* $\models$ *{ICon-T1, UCon-T1}*

Often, we want to assume that the universal entity—if it exists as in the extension $CODI_{\updownarrow}$ with {Con-D, ICon-D, UCon-D}—to be internally self-connected (ICon-E1).

(ICon-E1) $ICon(\mathrm{u})$ (the universal region must be internally self-connected)

Axiom Set 9.2: Extension axiom ICon-E1 of the theory $CODI_{\updownarrow}$.

Later in Chapter 11, we often need to express that the sum of two complementary parts is internally self-connected without directly referring to the sum, which may not always exist as such. To accommodate this need, we explicitly define a binary predicate called *strongly connected*—or abbreviated *s-connected*—$C_S(x,y)$, to capture this case. Instead of relying on the sum, we define s-connectedness analogous to ICon-D. Two entities are s-connected, $C_S(x,y)$, if they are superficially connected and share an entity z of the next-lower dimension than x and y, i.e., $z \prec_{\dim}, x, y$ (C_S-D). Observe that only entities x and y of equal dimension can be s-connected. For example, two 3D bodies are s-connected if they touch in a 2D surface, but not if they only touch in a line segment or in points.

In the presence of C_S-D, we can rewrite the definition of internal self-connectedness (ICon-D) to resemble the definition of self-connectedness (Con-D) by using s-connection C_S instead of general connection C.

(ICon-T2) $ICon(x) \leftrightarrow Con(x) \wedge \forall y[PP(y,x) \rightarrow C_S(y,x-y)]$

(alternative definition of internal self-connectedness)

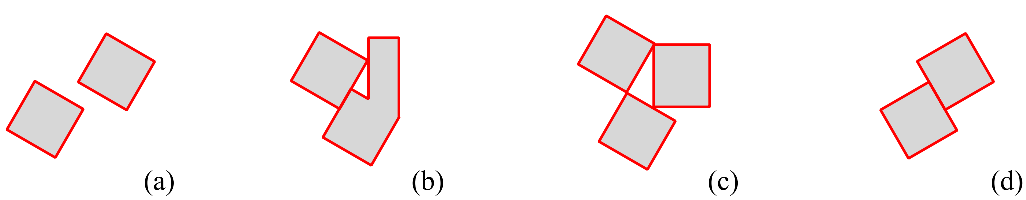


Figure 9.1: Examples of the different kinds of self-connectedness: (a) is not self-connected, (b) is internally but not uniformly self-connected, (c) is uniformly but not internally self-connected, and (d) is internally and uniformly self-connected.

(C_S-D) $C_S(x,y) \leftrightarrow SC(x,y) \wedge x =_{\text{dim}} y \wedge x \cdot y \prec_{\text{dim}} x$ (s-connectedness)

Axiom Set 9.3: Definition C_S-D of the *CODI* hierarchy.

We obtain the definitional module

$$CON = \{\text{Con-D, ICon-D, UCon-D, } C_S\text{-D}\},$$

which serves as a definitional extension of $CODI_{\downarrow}$ and any extension thereof. We will include those definitions in all theories of the *CODIB* hierarchy we are about to define in this chapter.

After this brief detour, we will now return to the main pursuit of this chapter.

9.3 Lower-dimensional boundaries

First we will capture so-called abstract, i.e., 'bodiless' spatial boundaries that are of a lower dimension than the entities they bound. For this reason we also call those abstract boundaries 'thin' boundaries. In the subsequent section, we show how abstract boundaries can be used to define 'bulky'—or 'thick'—boundaries that are of the same dimension as the entities they bound.

9.3.1 Boundary-containment

It is widely agreed upon that boundaries are 'dependent' entities in the sense that they cannot exist without their respective host, compare e.g., [Mas+03; Str88; Var08]. However, we can think of at least two different dependencies. The first one is dependency upon a single host; both abstract space and physical space allow such an interpretation. But abstract space also allows an interpretation in which a boundary is dependent upon two spatial regions meeting in a common boundary (what Stroll calls the 'interface' interpretation, compare [Str88]). We capture the dependency upon a single host, but have to be aware of its limitations. Effectively, we can only axiomatize to the extent of the second interpretation in that we rely on two entities meeting at a boundary. In other words, we have no way to conclusively state whether a lower-dimensional entity that is contained only in a single entity is in that entity's boundary or not unless we are explicitly told so. We may miss some entities that naturally make up the boundary of an entity, in particular 'outer' boundaries such as the border of a map. Consider the map in Figure 9.2 showing part of the eastern United States: From this map alone we do not know whether the Appalachians stretch beyond the boundaries of the United States. Likewise, for any isolated spatial entity, we cannot say with certainty whether a lower-dimensional contained entity is contained in its boundary or not, that is, we have no way of knowing whether it represents a closed or a bounded manifold.

We introduce the primitive relation of boundary-containment $BCont(x,y)$, a specialization at the intersection of containment $Cont(x,y)$ and incidence $Inc(x,y)$, to express that 'x is contained in the boundary of y'. While x must be of a lower dimension than y, we do not require that it is necessarily of the next lowest dimension. Consistent with the meaning of the *Cont* relation, x can be of any dimension lower than y. The intended meaning of *BCont* is not definable in the theory $CODI_{\downarrow}$—as mentioned earlier and demonstrated in Figures 9.3 and 9.4. Because $CODI'_{\downarrow}$ and $CODI_{\updownarrow}$ do not introduce new

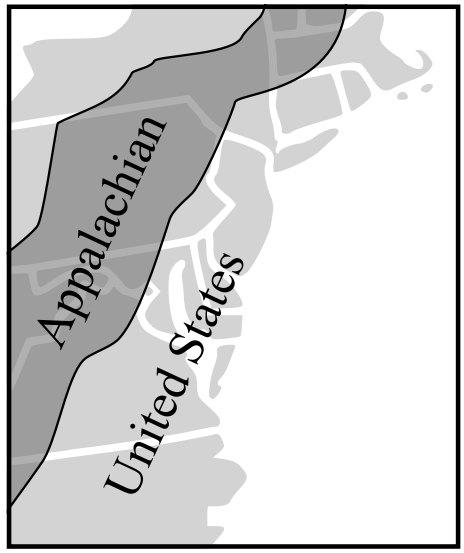


Figure 9.2: A map showing part of the east coast of the United States, including the Appalachians. From this map, we do not know whether the entire Appalachians are shown and whether part of the boundary of the United States coincides with the boundary of the Appalachians.

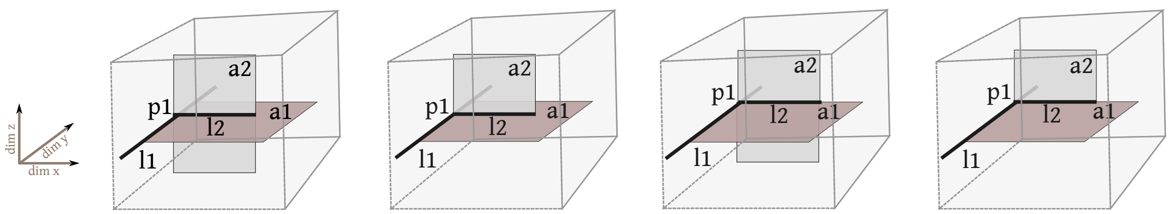


Figure 9.3: Four 3D spatial configurations in the class of intended structures whose representations are equivalent models of $CODI'_{\downarrow}$, i.e., $CODI'_{\downarrow}$ and thus $CODI_{\downarrow}$ cannot distinguish those spatial configurations. But the four are not equivalent in $CODIB_{\downarrow}$ ($CODI_{\downarrow}$ extended by an axiomatization of an additional primitive relation of boundary-containment), that is, they are distinct models of $CODIB_{\downarrow}$. They differ in their extensions of *BCont*: in the left model neither $\langle l2, a1\rangle \in \mathbf{BCont}_{\mathcal{M}}$ nor $\langle l2, a2\rangle \in \mathbf{BCont}_{\mathcal{M}}$, in the two models in the middle exactly one of them holds, and in the right model $\langle l2, a1\rangle, \langle l2, a2\rangle \in \mathbf{BCont}_{\mathcal{M}}$.

primitive relations over $CODI_{\downarrow}$, neither of them can satisfactorily define boundary-containment. For the purposes of distinguishing whether entities share interior or boundary points, we temporally work with $CODI_{\downarrow}$; later we reintroduce Sum′-A0 to Sum′-A5 and supplement the axioms for the ternary sum relation that force sums for entities that only share boundary points. This will allow us to logically capture the composite manifold resulting from the sum of two composite manifolds whose constituent manifolds only overlap in boundaries.

The five axioms BC-A1 – BC-A5 constrain the relation *BCont* and its interaction with all other relations previously introduced in *CODI*. First, boundary-containment specializes containment and incidence (BC-A1). Boundary-contained entities are guaranteed to exist in x when two entities x and y are in superficial contact and x has a "local" codimension of zero and is minimal (BC-A2). x has a local codimension of zero if and only of x y are both contained in a common entity v that is of the same dimension as x. The common entity v is the local space of interest. For example, a 1D minimal line segment l and a 2D area a that are in superficial contact and that are both contained in some greater 2D

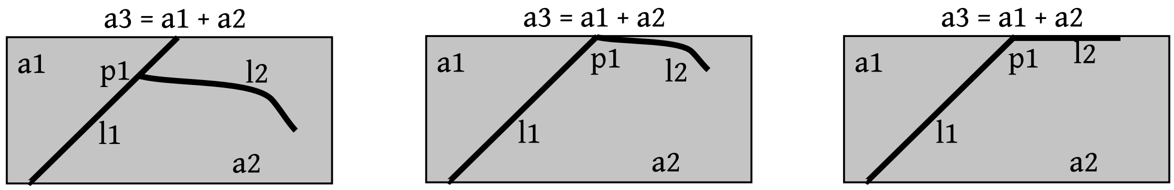


Figure 9.4: Three 2D spatial configurations in the class $\mathbb{M}_{\text{dense}}$ whose corresponding models in $CODI'_{\downarrow}$ are equivalent. Recall that the intended structures only allow manifolds in a composite manifold to share boundaries. So if $a3$ is the composite manifold composed of $a1$ and $a2$, $l1$ must be contained in the boundary of both $a1$ and $a2$ and $l1$ cannot be contained in the boundary of $a3$. Equally, $p1$ must be both in the boundary of $a1$ and $a2$, while it may or may not be in the boundary of $a3$. This problem cannot be avoided as long as we define boundaries as separating two entities: since $a3$ is the universal entity, it has no complement because it is not contained in anything larger. This problem causes also uncertainty about $l2$: it may or may not be in the boundary of $a2$.

(BC-A1) $BCont(x, y) \rightarrow Cont(x, y) \wedge Inc(x, y)$

(boundary-containment as special kind of containment and incidence)

(BC-A2) $SC(x, y) \wedge Min(x) \wedge Cont(z, x) \wedge Cont(z, y) \wedge P(x, v) \wedge Cont(y, v) \rightarrow BCont(z, x)$

(if x is a minimal entity and z is contained in the superficially connected entities x and y that are embedded in an entity v of the dimension of x, that is x has locally a codimension of zero, then z is contained in the boundary of x)

(BC-A3) $SC(x, y) \wedge P(x, v) \wedge P(y, v) \wedge Cont(z, x) \wedge Cont(z, y) \wedge z \prec_{\text{dim}} v \rightarrow \neg BCont(z, v)$

(any z contained in two superficially connected parts x and y of v that is of the next-lowest dimension of v is not in the boundary of v)

(BC-A4) $BCont(x, y) \wedge P(y, z) \wedge \forall v, w[P(v, z) \wedge \neg PO(v, y) \wedge P(w, x) \rightarrow \neg Cont(w, v)] \rightarrow BCont(x, z)$

(if x is boundary-contained in y, y is a part of z, and every part w of x cannot be contained in some part v of z that does not partially overlap y, then x is also boundary-contained in z)

(BC-A5) $BCont(x, y) \wedge Cont(z, x) \rightarrow BCont(z, y)$ ($BCont$ transitive with respect to $Cont$)

Axiom Set 9.4: Axioms BC-A1 – BC-A5 of the theory *CODIB*.

area requires the line segment l to touch the area a in its boundary. Note that in this example, the line segment may be touched by the area in its boundary (the endpoints of the line segment) or in its interior because the line segment can never have a local codimension of zero with respect to an area. As another example, two 2D areas in superficial contact share a boundary if they are contained in a (greater) 2D area (such as a plane) or if the entire space is 2D (then the entire space serves as local reference). But if the two 2D areas are only contained in a common 3D entity, we cannot say whether their boundaries are in contact or not—they could be connected in their interiors instead. For example, two half spheres (spheres removed of their northern hemisphere) may touch each other at their south poles only. BC-A2 does not apply to nonminimal entities as demonstrated by the example in Figure 9.5.

BC-A3 captures when something is definitely not contained in the boundary of v: any internal boundary shared by superficially connected parts of x and y of the next-lower dimension than v cannot be in the boundary of v, compare Figure 9.6. This axiom implicitly captures the restriction of the intended models of $\mathbb{M}$ to $\mathbb{M}_{\text{dense}}$. In a model in the class $(\mathbb{M} \setminus \mathbb{M}_{\text{dense}})$ BC-A3 does not hold. Also, BC-A3 may not hold for physical boundaries: a physical material object may have nonoverlapping parts

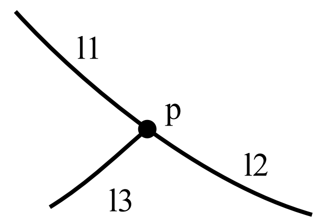


Figure 9.5: Example demonstrating that we cannot omit the requirement $Min(x)$ in the antecedent of BC-A2. In this configuration we have $\langle l1+l2, l3\rangle \in \mathbf{BCont}_{\mathcal{M}}$, $\langle l1+l2, l1+l2+l3\rangle \in \mathbf{P}_{\mathcal{M}}$, $\langle p, l1+l2\rangle, \langle p, l3\rangle \in \mathbf{Cont}_{\mathcal{M}}$, but $\langle p, l1+l2\rangle \notin \mathbf{BCont}_{\mathcal{M}}$. This is because $(l1+l2) \notin \mathbf{Min}_{\mathcal{M}}$.

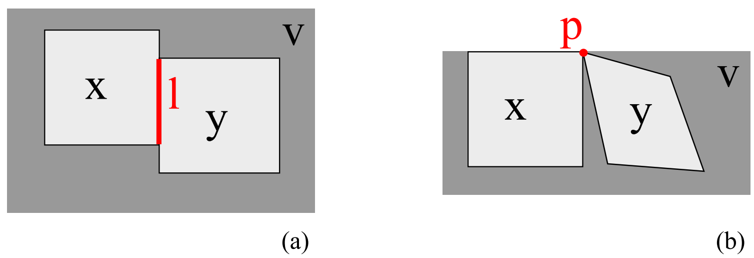


Figure 9.6: Spatial configurations that justify BC-A3. (a) illustrates why an internal boundary of $x+y$ of next-lowest dimension cannot be contained in the boundary of v, while (b) gives an example why anything of even lower dimension (no matter whether it superficially connects x and y) may still be contained in the boundary of v.

that are sharing a boundary that is an 'internal boundary' or a fissure or crack, compare Section 11.3.

BC-A4 states when an x boundary-contained in a part y of z is boundary-contained in z: when a part w of x is contained in a part v of z that does not overlap y.

BC-A5 ensures that boundary-containment is transitive with respect to containment: any z contained in some x, the latter being boundary-contained in y, is itself boundary-contained in y.

The extension of *CODI* by the axioms BC-A1 – BC-A5 ensures that every y that is boundary-contained in x is of a lower dimension than x and that *BCont* specializes *Cont* (BC-T1). Irreflexivity (BC-T2) and asymmetry (BC-T3) of *BCont* are also provable.

(BC-T1) $BCont(x, y) \rightarrow Cont(x, y) \wedge x <_{\text{dim}} y$

(*BCont* requires the contained entity to be a lower dimension than the container; 'thin' boundary)

(BC-T2) $\neg BCont(x, x)$ (*BCont* irreflexive)

(BC-T3) $BCont(x, y) \rightarrow \neg BCont(y, x)$ (*BCont* asymmetric)

Lemma 9.2. *CODI* $\cup$ *{BC-A1 – BC-A5}* $\models$ *{BC-T1 – BC-T3}*

Proof. To prove BC-T1 assume $\mathbf{BCont}(x, y)$ for some $x, y \in \mathbf{M}$.
By BC-A1 we obtain $\mathbf{Cont}(x, y)$ and $\mathbf{Inc}(x, y)$, the former requiring $x \leq_{\mathbf{dim}} y$ (CD-A1) and the latter implying $x <_{\mathbf{dim}} y$ or $y <_{\mathbf{dim}} x$ (Inc-T4). Hence, we must have $x <_{\mathbf{dim}} y$.

BC-T2 and BC-T3 follow directly from $x <_{\mathbf{dim}} y$ and the respective properties of $<_{\text{dim}}$ (D'-A1, D'-A2). □

In the presence of U-A1, BC-A2 also yields the property BC-T4.

(BC-T4) $SC(x,y) \wedge Min(x) \wedge MaxDim(x) \wedge Cont(z,x) \wedge Cont(z,y) \rightarrow BCont(z,x)$

(anything contained in two superficially connected entities x and y with x being of codimension zero, i.e., x being of maximal dimension, is in the boundary of x)

Lemma 9.3. *CODI* $\cup$ *{BC-A1 – BC-A5}* $\cup$ *U-A1* $\models$ *BC-T4*

Proof. Consider the following computation:

$$
\begin{aligned}
& SC(x,y) \wedge Min(x) \wedge MaxDim(x) \wedge Cont(z,x) \wedge Cont(z,y) && \\
\rightarrow & SC(x,y) \wedge Min(x) \wedge Cont(z,x) \wedge Cont(z,y) \wedge \neg ZEX(x) \wedge \neg ZEX(y) && \text{(C-A4)} \\
\rightarrow & SC(x,y) \wedge Min(x) \wedge Cont(z,x) \wedge Cont(z,y) \wedge Cont(x,\mathrm{u}) \wedge Cont(y,\mathrm{u}) && \text{U-A1} \\
\rightarrow & SC(x,y) \wedge Min(x) \wedge Cont(z,x) \wedge Cont(z,y) \wedge P(x,\mathrm{u}) \wedge Cont(y,\mathrm{u}) && \text{EP-D} \\
\rightarrow & BCont(z,x) && \text{(BC-A2)}
\end{aligned}
$$

□

Further properties, such as *BCont* being closed under intersections, differences, and sums also directly follow from BC-A1 – BC-A5, but are of lesser importance here.

Next, we want to distinguish *bounded entities*, i.e., those that have some entities contained in their boundary, from *closed entities* (CL-D), i.e., those that do not have entities in their boundary[2], as defined for m-manifolds in Section 5.2. For example, a sphere (the surface of a ball) is closed, but also a plane, which extends into the infinite, is called closed.

(CL-D) $Closed(x) \leftrightarrow \forall y[\neg BCont(y,x)]$ (a closed entity has nothing in its boundary)

Axiom Set 9.5: Definition CL-D of the theory *CODIB*.

We define the theory $CODIB = CODI \cup \{$BC-A1 – BC-A5, CL-D, ICon-D, UCon-D$\}$. In *CODIB* we can prove that all entities of minimal dimension are closed manifolds (BC-T5) and thus cannot boundary-contain any entities. This matches the understanding that 0-manifolds (points) have an empty manifold boundary and are therefore closed (compare Definition 5.4).

(BC-T5) $MinDim(x) \rightarrow Closed(x)$ (entities of minimal dimensions are closed)

Lemma 9.4. *CODIB* $\models$ *BC-T5*

Proof. Assume $x \in \mathbf{M}$ such that $x \in \mathbf{MinDim}_{\mathcal{M}}$.
Then no $y \in \mathbf{M}$ exists such that $y <_{\mathbf{dim}} x$ and thus by BC-T1, no $y \in \mathbf{M}$ exists such that $\mathbf{BCont}(y,x)$.
Hence $x \in \mathbf{Closed}_{\mathcal{M}}$. □

[2]Note that this is the manifold notion of 'closed', not the topological notion of 'closed'.

9.3.2 Interior containment

Now we can state when an entity contains another in its boundary. It is only logical to also want to express that an entity contains another in its interior. Note that $Cont(x, y)$ without $BCont(x, y)$ does not necessarily mean x is contained in the interior of y: it may be partly in the boundary of y and partly in the interior of y. We define interior containment, $ICont(x, y)$, meaning that 'x is *entirely* contained in the interior of y' in the theory *CODIB* as 'nothing contained in x is contained in the boundary of y' (IC-D).

(IC-D) $ICont(x, y) \leftrightarrow Cont(x, y) \wedge \forall z[Cont(z, x) \rightarrow \neg BCont(z, y)]$ (interior containment)

Axiom Set 9.6: Definition IC-D of the theory *CODIB*.

Interior containment is the multidimensional equivalent of nontangential parthood *NTPP* in equidimensional mereotopology as defined, e.g., through nontangential proper parthood, *NTPP*, for the RCC in Section 8.1.1.

The following properties are provable for interior containment. First, interior containment specializes containment (IC-T1). Moreover, interior and boundary-containment are disjoint, that is, nothing can be contained both in the interior and the boundary (IC-T2). A closed entity has nothing in its boundary and hence any entity contained in a closed entity must be contained in its interior (IC-T3), in particular, *ICont* is reflexive for closed entities (IC-T4). For bounded entities, *ICont* is antireflexive (IC-T5). Finally, *Cont* is monotone with respect to *ICont* (IC-T6): some x contained in y is contained in the interior of z if y is contained in the interior of z.

(IC-T1) $ICont(x, y) \rightarrow Cont(x, y)$ (*ICont* specializes *Cont*)

(IC-T2) $\neg ICont(x, y) \vee \neg BCont(x, y)$ (*ICont* and *BCont* disjoint)

(IC-T3) $Closed(x) \wedge Cont(y, x) \rightarrow ICont(y, x)$

(everything contained in a closed entity is contained in its interior)

(IC-T4) $Closed(x) \wedge \neg ZEX(x) \rightarrow ICont(x, x)$ (*ICont* reflexive for closed entities)

(IC-T5) $\neg Closed(x) \rightarrow \neg ICont(x, x)$ (*ICont* antireflexive for non-closed entities)

(IC-T6) $Cont(x, y) \wedge ICont(y, z) \rightarrow ICont(x, z)$ (*Cont* transitive with respect to *ICont*)

Lemma 9.5. *CODIB* $\cup$ *{IC-D}* $\models$ *{IC-T1 – IC-T6}*

9.3.3 Tangential containment

When x is contained in y, but neither entirely contained in the interior of y nor entirely contained in the boundary of y, then we say x is tangentially contained in y. TC-D defines $TCont(x, y)$ meaning 'x is tangentially contained in y' in the theory *CODIB*. Tangential containment is the multidimensional equivalent of tangential parthood *TP* known from equidimensional mereotopology.

Similarly to interior containment, tangential containment specializes containment (TC-T1); moreover, boundary-containment further specializes tangential containment (TC-T2). A closed entity has nothing in its boundary and thus nothing tangentially contained (TC-T3). In particular, *TCont* is antireflexive for closed entities (TC-T4). For non-closed entities, *TCont* is reflexive (TC-T5).

(TC-D) $TCont(x,y) \leftrightarrow Cont(x,y) \wedge \exists z[Cont(z,x) \wedge BCont(z,y)]$ (tangential containment)

Axiom Set 9.7: Definition TC-D of the theory *CODIB*.

(TC-T1) $TCont(x,y) \rightarrow Cont(x,y)$ (*TCont* specializes *Cont*)

(TC-T2) $BCont(x,y) \rightarrow TCont(x,y)$ (boundary-containment specializes tangential containment)

(TC-T3) $Closed(x) \rightarrow \forall y[\neg TCont(y,x)]$ (closed entities cannot tangentially contain anything)

(TC-T4) $Closed(x) \rightarrow \neg TCont(x,x)$ (*TCont* antireflexive for closed entities)

(TC-T5) $\neg Closed(x) \rightarrow TCont(x,x)$ (*TCont* reflexive for non-closed entities)

Lemma 9.6. *CODIB* $\cup$ *{TC-D}* $\models$ *{TC-T1 – TC-T5}*

While by BC-A1 parts (of equal dimension) cannot be contained in the lower-dimensional boundary of an entity, parts can be contained tangentially.

9.3.4 Exhaustiveness and disjointness of *ICont* and *TCont*

We can now prove that tangential containment and interior containment are jointly exhaustive and disjoint subrelations of containment in the theory *CODIB* $\cup$ {IC-D, TC-D}.

Theorem 9.1. *In a model* $\mathcal{M}$ *of CODIB* $\cup$ *{IC-D, TC-D},* $\mathbf{ICont}_{\mathcal{M}}$ *and* $\mathbf{TCont}_{\mathcal{M}}$ *partition* $\mathbf{Cont}_{\mathcal{M}}$*.*

Proof. From IC-T1 and TC-T1 we already know that *ICont* and *TCont* are specializations of *Cont*. For $\mathbf{ICont}_{\mathcal{M}}$ and $\mathbf{TCont}_{\mathcal{M}}$ to form a partition of $\mathbf{Cont}_{\mathcal{M}}$, we additionally must prove that the extensions of *ICont* and *TCont* are disjoint (BC-T6) and exhaustive subrelations of the extension of *Cont* (BC-T7).

(BC-T6) $\neg ICont(x,y) \vee \neg TCont(x,y)$ (*ICont* and *TCont* are disjoint)

(BC-T7) $Cont(x,y) \rightarrow ICont(x,y) \vee TCont(x,y)$

(*ICont* and *TCont* are exhaustive subrelations of *Cont*)

□

This means that *ICont* and *TCont* are exhaustive and disjoint specializations of *Cont*, with *BCont* and $(TCont \wedge \neg BCont)$ further specializing *TCont* into two disjoint and exhaustive types of containment.

9.3.5 Satisfiability of $CODIB_{\downarrow}$

Our main intention of this chapter is to fully capture the class of intended structures $\mathbb{M}_{\text{dense}}$ in a logical theory. The extension of *CODI* to *CODIB* is a step in this direction. But we will also need the closure under intersections and differences as axiomatized in $CODI_{\downarrow}$.

We now introduce one more axiom requiring $BCont(x,y)$ to be defined in terms of boundary-containment of all minimal parts z of x (BC-A6). Of course, this axiom assumes atomicity, i.e., the axiom ME-E1, as well. Without ME-E1, any entity x without any minimal parts would vacuously satisfy the right-hand side and thereby be boundary-contained in all entities—an undesired consequence.

(BC-A6) $BCont(x, y) \leftrightarrow \neg ZEX(x) \wedge \forall z[P(z, x) \wedge Min(z) \rightarrow BCont(z, y)]$
($BCont(x, y)$ defined in terms of boundary-containment of the minimal parts of x)

Axiom Set 9.8: Axiom BC-A6 of the theory $CODIB_{\downarrow}$.

BC-A6 is a quite useful property for many applications. For example, a CAD program may come equipped with a set of predefined basis entities, e.g., polygons, circles, lines, (hyper)cubes, for which the boundaries are well-defined. Then for any more complex entities constructed as compositions of those basic entities, the boundary is equally well-defined.

We define the theory

$$CODIB_{\downarrow} = CODIB \cup CODI_{\downarrow} \cup \{\text{BC-A6, ME-E1, IC-D, TC-D}\}.$$

We can extend the satisfiability result of $CODI_{\downarrow}$ to $CODIB_{\downarrow}$ by showing that the axioms of *BCont* are satisfied by any structure in $\mathbb{M}_{\text{dense}}$. This is only possible once we give an additional intended interpretation of *BCont*. As discussed before and illustrated by Figure 9.7, we have to rely on the correct interpretation of *BCont* to capture boundaries accurately. Any outer boundaries of maximal entities, such as the boundaries of $l3$ and $l4$ in Figure 9.8 cannot be defined without *BCont* as primitive. In the spatial configuration on the right in Figure 9.8 we cannot know whether $\langle q, l3\rangle \in \mathbf{BCont}_{\mathcal{M}}$ and $\langle q, l4\rangle \in \mathbf{BCont}_{\mathcal{M}}$ unless explicitly told.

Theorem 9.2 (Satisfiability of $CODIB_{\downarrow}$)**.** *Let* $\mathfrak{M}$ *be a collection in the class* $\mathbb{M}_{\text{dense}}$ *with domain* $Dom(\mathfrak{M})$ *(as defined in Definition 5.11) and* $\emptyset \in \text{Dom}(\mathfrak{M})$*. Then there exists a corresponding model* $\mathcal{M}$ *of* $CODIB_{\downarrow}$ *with finite domain* $\mathbf{M}$ *such that*

1. $\mu : \text{Dom}(\mathfrak{M}) \rightarrow \mathbf{M}$ *is a bijection;*

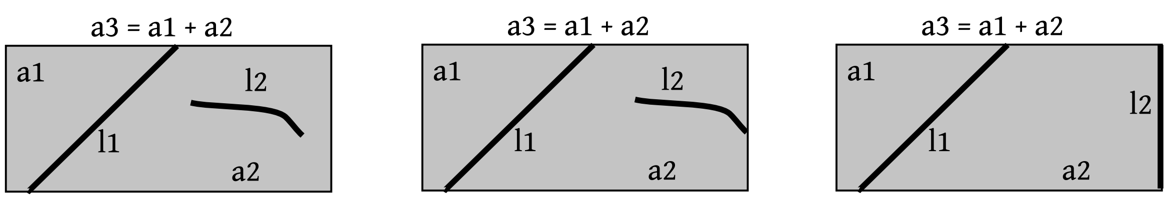


Figure 9.7: Three spatial configurations with different placements of a one-dimensional entity within a two-dimensional entity. In all three spatial configuration it is clear from their extensions in $CODI_{\downarrow}$ that $l1$ must be in the boundary of $a1$ and $a2$, i.e., that $l1$ is an inner boundary of $a3$. However, we only have $\langle l2, a2\rangle \in \mathbf{Cont}_{\mathcal{M}}$ (and, by transitivity, $\langle l2, a3\rangle \in \mathbf{Cont}_{\mathcal{M}}$) but may or may not have $\langle l2, a2\rangle, \langle l2, a3\rangle \in \mathbf{BCont}_{\mathcal{M}}$. In the left and in the center configuration we have $\langle l2, a2\rangle \notin \mathbf{BCont}_{\mathcal{M}}$, while $\langle p, a2\rangle \notin \mathbf{Cont}_{\mathcal{M}}$) is only certain in the left configuration. In the center configuration we cannot tell with certainty whether or not $\langle p, a2\rangle \in \mathbf{BCont}_{\mathcal{M}}$, only if we know that $a2$ is completely displayed we would know. Likewise, in the right configuration, we may have $\langle p, a2\rangle, \langle l, a2\rangle \in \mathbf{BCont}_{\mathcal{M}}$ but it is only certain if we know that $a2$ is completely displayed. The same applies to $\langle p, a3\rangle, \langle l, a3\rangle \in \mathbf{BCont}_{\mathcal{M}}$. In other words, we have no information whether or not $l2$ is located on the outer boundary of the maximal entity $a3$. The justification is that $a2$ (and thus $a3$) may stretch beyond our "map" or "view" of the world so that even though $l2$ is in the boundary of our view, it is not contained in the boundary of $a2$ because $a2$ stretches beyond what we see.

2. *for all* $d \in \mathrm{Dom}(\mathfrak{M})$,

$$\mu(d) \in \mathbf{ZEX}_{\mathcal{M}} \iff \Sigma d = \emptyset;$$

3. *for all* $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$,

$$\langle \mu(d_1), \mu(d_2) \rangle \in (<_{\mathbf{dim}})_{\mathcal{M}} \iff \big(dim(d_1) < dim(d_2)\big) \;\; or \;\; \big(d_1 = \emptyset \;\; and \;\; d_2 \neq \emptyset\big);$$

4. *for all* $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$,

$$\langle \mu(d_1), \mu(d_2) \rangle \in \mathbf{Cont}_{\mathcal{M}} \iff \Sigma d_1 \subseteq \Sigma d_2 \;\; and \;\; d_1 \neq \emptyset;$$

5. *for all* $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$,

$$\langle \mu(d_1), \mu(d_2) \rangle \in \mathbf{BCont}_{\mathcal{M}} \iff \Sigma d_1 \subseteq \Delta d_2 \;\; and \;\; d_1 \neq \emptyset.$$

Proof. Let $\mathfrak{M}$ be a finite collection in the class $\mathbb{M}_{\text{dense}}$ with domain $Dom(\mathfrak{M})$ as defined in Definition 5.11. By Definition 9.1, $\mathfrak{M}$ is also a collection in the class $\mathbb{M}$. Then by Theorem 7.4 a model $\mathcal{M}'$ of $CODI_{\downarrow}$ exists that satisfies Theorem 9.2(1)–(4). If we can extend this model $\mathcal{M}'$ by specifying an interpretation of $\mathbf{BCont}_{\mathcal{M}}$ using Theorem 9.2(5) without changing the extension of any other primitive relation, that is, with

- $\mathbf{Cont}_{\mathcal{M}} = \mathbf{Cont}_{\mathcal{M}}'$,
- $(<_{\mathbf{dim}})_{\mathcal{M}} = (<_{\mathbf{dim}})_{\mathcal{M}'}$, and
- $\mathbf{ZEX}_{\mathcal{M}} = \mathbf{ZEX}_{\mathcal{M}}'$,

to a model of $CODIB_{\downarrow}$, we are done. Because BC-A1 to BC-A6 and ME-E1 are the only additional axioms of $CODIB_{\downarrow}$ and the only ones that mention *BCont*, we only need to prove that those axioms are satisfied by the specification of $\mathbf{BCont}_{\mathcal{M}}$ as specified in Theorem 9.2(5).

Note that ME-E1 is trivially satisfied because the domain $Dom(\mathfrak{M})$ is finite. We now verify that the axioms BC-A1 to BC-A6 are satisfied by the extended model $\mathcal{M}$.

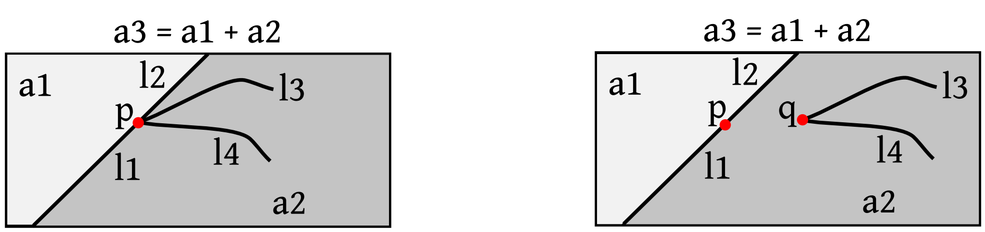


Figure 9.8: Two similar spatial configurations with the right one relying on the correct interpretation of *BCont* that is not definable in $CODI_{\downarrow}$. In the spatial configuration on the left all relevant boundaries, where two entities are superficially connected, are captured properly. In the spatial configuration on the right, the boundary point q of both $l3$ and $l4$ is only captured if the sum $l3 + l4$ exists (as composite manifold). Otherwise, we only know $Cont(q, l3)$ and $Cont(q, l4)$ for sure, but the axioms do not stipulate whether $BCont(q, l3)$ and $BCont(q, l4)$ hold.

(BC-A1): $BCont(x, y) \to Cont(x, y) \wedge Inc(x, y)$.

Assume $\mathbf{BCont}(\mu(d_1), \mu(d_2))$ for some $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$.

We want to prove that $\mathbf{Cont}(\mu(d_1), \mu(d_2))$ and $\mathbf{Inc}(\mu(d_1), \mu(d_2))$.

By $\mathbf{BCont}(\mu(d_1), \mu(d_2))$ we have $d_1 \neq \emptyset$ and $\Sigma d_1 \subseteq \Delta d_2$. Since $\Delta d_1 \subseteq \Sigma d_1$, we immediately obtain $\mathbf{Cont}(\mu(d_1), \mu(d_2))$ by Theorem 7.4(1).

To show $\mathbf{Inc}(\mu(d_1), \mu(d_2))$ we must show that there exists a z such that $z = \mu(d_3)$ for some $d_3 \in \mathrm{Dom}(\mathfrak{M})$ and

$$\mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{Cont}(\mu(d_3), \mu(d_2)) \wedge \mu(d_3) \mathbf{=_{dim}} \mu(d_2) \wedge \mu(d_3) <_{\dim} \mu(d_1) \quad \text{or}$$

$$\mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{Cont}(\mu(d_3), \mu(d_2)) \wedge \mu(d_3) \mathbf{=_{dim}} \mu(d_1) \wedge \mu(d_3) <_{\dim} \mu(d_2)$$

We will show that $d_3 = d_1 \in \mathrm{Dom}(\mathfrak{M})$ satisfies the second condition. We have

- $\mathbf{Cont}(\mu(d_1), \mu(d_1))$ because $d_1 \neq \emptyset$,
- $\mathbf{Cont}(\mu(d_1), \mu(d_2))$ from above,
- $\mu(d_1) \mathbf{<_{dim}} \mu(d_2)$ from Theorem 7.4(3) since $dim(\Sigma d_1) \leq \dim(\Delta d_2) < \dim(\Sigma d_2)$,
- $\mu(d_1) \mathbf{=_{dim}} \mu(d_1)$ trivially;

thus $\mathbf{Inc}(\mu(d_3), \mu(d_2))$.

(BC-A2): $SC(x, y) \wedge Min(x) \wedge P(x, v) \wedge Cont(y, v) \wedge Cont(z, x) \wedge Cont(z, y) \to BCont(z, x)$.

Assume

$$\mathbf{SC}(\mu(d_1), \mu(d_2)) \wedge \mu(d_1) \in \mathbf{Min} \wedge \mathbf{P}(\mu(d_1), \mu(d_4)) \wedge$$
$$\mathbf{Cont}(\mu(d_2), \mu(d_4)) \wedge \mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{Cont}(\mu(d_3), \mu(d_2))$$

for some $d_1, d_2, d_3, d_4 \in \mathrm{Dom}(\mathfrak{M})$. We want to prove that $\mathbf{BCont}(\mu(d_3), \mu(d_1))$.

$\mathbf{P}(\mu(d_1), \mu(d_4))$ implies $\mathbf{Cont}(\mu(d_1), \mu(d_4))$ and $\mu(d_1) \mathbf{=_{dim}} \mu(d_4)$ by EP-D. Additionally, $\mathbf{SC}(\mu(d_1), \mu(d_2))$ implies $\mu(d_1) \cdot \mu(d_2) \mathbf{<_{dim}} \mu(d_2)$ by SC-D, and thus $\mathbf{Cont}(\mu(d_2), \mu(d_4) - \mu(d_1))$ holds by $\mathbf{Cont}(\mu(d_2), \mu(d_4))$ and Dif-A3(a). From the assumption $\mathbf{Cont}(\mu(d_3), \mu(d_2))$ we then obtain $\mathbf{Cont}(\mu(d_3), \mu(d_4) - \mu(d_1))$ by C-A3. As proved in Theorem 7.4, there must exist a manifold $d_5 \in \mathrm{Dom}(\mathfrak{M})$ such that $\mu(d_5) = \mu(d_4) - \mu(d_1)$. Then $\Sigma d_3 \subseteq \Sigma d_5$ by Theorem 9.2(4).

From $\mathbf{Cont}(\mu(d_3), \mu(d_1))$ we also obtain $\Sigma d_3 \subseteq \Sigma d_1$ by Theorem 9.2(4). Because d_4 is a composite manifold, d_1 is an atomic manifold $d_1 \in d_4$ (by $\mu(d_1) \in \mathbf{Min}$ and by $\mathbf{Cont}(\mu(d_1), \mu(d_4))$), and d_5 is a composite submanifold of d_4, we conclude that d_1 and d_5 can only intersect in their boundaries (compare Definition 5.6). With $\Sigma d_3 \subseteq \Sigma d_1$ and $\Sigma d_3 \subseteq \Sigma d_5$ we immediately conclude $d_3 \subseteq \Delta d_1$ and $\mathbf{BCont}(\mu(d_3), \mu(d_1))$ by Theorem 9.2(5).

(BC-A3): $SC(x, y) \wedge P(x, v) \wedge P(y, v) \wedge Cont(z, x) \wedge Cont(z, y) \wedge z\boldsymbol{prec}_{\mathbf{dim}} v \to \neg BCont(z, v)$.

Assume

$$\mathbf{SC}(\mu(d_1), \mu(d_2)) \wedge \mathbf{P}(\mu(d_1), \mu(d_4)) \wedge \mathbf{P}(\mu(d_2), \mu(d_4)) \wedge$$
$$\mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{Cont}(\mu(d_3), \mu(d_2)) \wedge \mu(d_3) \boldsymbol{\prec}_{\mathbf{dim}} \mu(d_4)$$

for arbitrary $d_1, d_2, d_3, d_4 \in \text{Dom}(\mathfrak{M})$.
We want to prove that $\neg\mathbf{BCont}(\mu(d_3), \mu(d_4))$.

Because $\Sigma d_3 \in \Sigma d_1$ and $\Sigma d_3 \in \Sigma d_2$ by $\mathbf{Cont}(\mu(d_3), \mu(d_1))$, $\mathbf{Cont}(\mu(d_3), \mu(d_2))$, where d_1 and d_2 are both subsets of the atomic manifolds in d_4 (by $\mathbf{P}(\mu(d_1), \mu(d_4))$ and by $\mathbf{P}(\mu(d_2), \mu(d_4))$), we can calculate $\Delta_i d_4$ (compare Definition 5.8) using d_1 as base set and successively add manifolds from d_2 until we obtain the interior boundaries of d_4 that are contained in both d_1 and d_2. Every time we come across a part of $\mu(d_3)$, i.e., a manifold d_3' in d_2 and in d_1 that includes a subset of d_3 of the same dimension as d_3, that set d_3' gets added to $\Delta_i d_4$ because for any pair of atomic manifolds in d_1 and d_2 that share such a d_3' we have:

- $\Sigma d_3' \subseteq \partial d_1$,
- $\Sigma d_3' \subseteq \partial d_2$, and
- $\mu(d_3') \prec_{\mathbf{dim}} \mu(d_4)$.

Because every such d_3' will be eventually encountered (recall that $\Sigma d_3 \subseteq \Sigma d_2$), every such part of d_3 will get added to $\Delta_i d_4$, eventually resulting in $\Sigma d_3 \subseteq \Delta_i d_4$. Then by Theorem 9.2(5), we immediately obtain $\mathbf{BCont}(\mu(d_3), \mu(d_4))$, the desired consequence.

(BC-A4): $BCont(x, y) \wedge P(y, z) \wedge \forall v, w[P(v, z) \wedge \neg PO(v, y) \wedge P(w, x) \rightarrow \neg Cont(w, v)] \rightarrow BCont(x, z)$.
Assume that d_1, d_2, d_3 are arbitrary entities in $\text{Dom}(\mathfrak{M})$ with $\mathbf{BCont}(\mu(d_3), \mu(d_1))$, $\mathbf{P}(\mu(d_1), \mu(d_2))$ and such that for all $d_i, d_j \in \text{Dom}(\mathfrak{M})$,

$$\mathbf{P}(\mu(d_i), \mu(d_2)) \wedge \neg\mathbf{PO}(\mu(d_1), \mu(d_i)) \wedge \mathbf{P}(\mu(d_j), \mu(d_3)) \rightarrow \neg\mathbf{Cont}(\mu(d_j), \mu(d_i)).$$

Then by Theorem 9.2(4) and (5) we have $\Sigma d_3 \subseteq \Delta d_1$ and $\Sigma d_1 \subseteq \Sigma d_2$, and thus $\Sigma d_3 \subseteq \Sigma d_2$.

If d_2 is an atomic manifold, we immediately obtain $d_1 = d_2$ and thereby $\Sigma d_3 \subseteq \Delta d_1 = \Delta d_2$ and thus $\mathbf{BCont}(\mu(d_3), \mu(d_2))$ by Theorem 9.2(5). For the remainder of the proof, assume that d_2 is a non-atomic manifold.

Now suppose the consequent is not satisfied, i.e., $\neg\mathbf{BCont}(\mu(d_3), \mu(d_2))$ and therefore $\Sigma d_3 \nsubseteq \Delta d_2$ by Theorem 9.2(5). Because the area Σd_3 is the sum of the areas of d_3's atomic manifolds, some atomic manifold $d_4 \in d_3$ must exist (with $\Sigma d_4 \subseteq \Sigma d_3$) such that $\Sigma d_4 \nsubseteq \Delta d_2$. Recall further that $dim(d_4) < dim(d_2)$ and $\Sigma d_4 \subseteq \Sigma d_3 \subseteq \Delta d_1$ and $\Sigma d_1 \subseteq \Sigma d_2$. Then we still have $\Sigma d_4 \subseteq \Sigma d_2$.

Then by Definitions 5.8 and 5.9 there must exist an atomic manifold $d_5 \neq d_1$ with $d_5 \in d_2$ (with $\Sigma d_5 \subseteq \Sigma d_2$) such that $\Sigma d_4 \subseteq \Sigma d_5$. In other words, d_4 must be contained in atomic manifolds d_1 *and* d_5 that constitute d_2. Thereby $\Sigma d_4 \subseteq \Delta_i d_2$ by Definition 5.8 and thus $\Sigma d_4 \nsubseteq \Delta d_2$ by Definitions 5.9.

Because d_5 and d_1 are distinct atomic manifolds in d_2, they cannot share any manifold of equal dimension, hence $\neg\mathbf{PO}(\mu(d_1), \mu(d_5))$. Since d_4 is an atomic manifold in d_3 they both have the same dimension by Definition 5.6(2). Hence $\mathbf{P}(\mu(d_4), \mu(d_3))$ by Theorem 9.2(4) and EP-D. Moreover, $\Sigma d_4 \subseteq \Sigma d_5$ and thus $\mathbf{Cont}(\mu(d_4), \mu(d_5))$ by Theorem 9.2(4). Altogether, we have

$$\mathbf{P}(\mu(d_5), \mu(d_2)) \wedge \neg\mathbf{PO}(\mu(d_1), \mu(d_5)) \wedge \mathbf{P}(\mu(d_4), \mu(d_3)) \wedge \mathbf{Cont}(\mu(d_4), \mu(d_5))$$

which contradicts our assumption

$$[\mathbf{P}(\mu(d_i), \mu(d_2)) \wedge \neg\mathbf{PO}(\mu(d_1), \mu(d_i)) \wedge \mathbf{P}(\mu(d_j), \mu(d_3)) \rightarrow \neg\mathbf{Cont}(\mu(d_j), \mu(d_i))]$$

for $d_i = d_5$ and $d_j = d_4$. Hence our supposition $\neg\mathbf{BCont}(\mu(d_3), \mu(d_2))$ was false and BC-A4 is satisfied.

(BC-A5): $BCont(x, y) \wedge Cont(z, x) \rightarrow BCont(z, y)$.
Assume $\mathbf{BCont}(\mu(d_1), \mu(d_2))$ and $\mathbf{Cont}(\mu(d_3), \mu(d_1))$ for arbitrary $d_1, d_2, d_3 \in \mathrm{Dom}(\mathfrak{M})$.
Then $\Sigma d_3 \subseteq \Sigma d_1 \subseteq \Delta d_2$ and by transitivity of the subset relation we immediately obtain $\Sigma d_3 \subseteq \Delta d_2$ which amounts to $\mathbf{BCont}(\mu(d_3), \mu(d_2))$ by Theorem 9.2(5).

(BC-A6): $BCont(x, y) \leftrightarrow \neg ZEX(x) \wedge \forall z[P(z, x) \wedge Min(z) \rightarrow BCont(z, y)]$.
We prove the two directions of the biconditional individually.

Direction (a): $BCont(x, y) \rightarrow \neg ZEX(x) \wedge \forall z[P(z, x) \wedge Min(z) \rightarrow BCont(z, y)]$.
Assume $\mathbf{BCont}(\mu(d_1), \mu(d_2))$, $\mathbf{P}(\mu(d_3), \mu(d_1))$, and $\mu(d_3) \in \mathbf{Min}$ for arbitrary $d_1, d_2, d_3 \in \mathrm{Dom}(\mathfrak{M})$.
From Theorem 9.2(5) and (1) $\mu(d_1) \notin \mathbf{ZEX}$ follows immediately.
We have $\Sigma d_3 \subseteq \Sigma d_1$ and $\Sigma d_1 \subseteq \Delta d_2$ by Theorem 9.2(4) and (5). Thus $\Sigma d_3 \subseteq \Sigma d_1 \subseteq \Delta d_2$ and we immediately obtain $\mathbf{BCont}(\mu(d_3), \mu(d_2))$ by Theorem 9.2(5).

Direction (b): $BCont(x, y) \leftarrow \neg ZEX(x) \wedge \forall z[P(z, x) \wedge Min(z) \rightarrow BCont(z, y)]$.
Assume d_1, d_2 are arbitrary entities in $\mathrm{Dom}(\mathfrak{M})$ such that $\mu(d_1) \notin \mathbf{ZEX}$ and for all $d_i \in \mathrm{Dom}(\mathfrak{M})$, $[\mathbf{P}(\mu(d_i), \mu(d_1)) \wedge \mathbf{Min}(z) \rightarrow \mathbf{BCont}(\mu(d_i), \mu(d_2))]$.
Then any atomic manifold e_i with $\Sigma e_1 \subseteq \Sigma d_1$ is contained in the boundary of d_2, i.e., $\Sigma e_i \subseteq \Delta d_2$. Because $\Sigma d_1 = \bigcup_i \Sigma e_i$ by Definition 5.7, we immediately obtain $\Sigma d_1 \subseteq \Delta d_2$ and thus $\mathbf{BCont}(\mu(d_1), \mu(d_2))$ (with $\mu(d_1) \notin \mathbf{ZEX}$ implying $d_1 \neq \emptyset$).

The two directions together prove BC-A6.

By the theorem's assumption the extended model satisfies all axioms of $CODI_{\downarrow}$ and we only added an interpretation of the new primitive relation *BCont* without altering the interpretations of the other relations and functions. For *BCont* we proved that the axioms BC-A1 to BC-A6 are satisfied. Hence, it is guaranteed that the extended structure satisfies all axioms of $CODI_{\downarrow} \cup \{$BC-A1 – BC-A6, ME-E1$\}$ and thus of $CODIB_{\downarrow}$. ☐

While Theorem 9.2 proves that every structure $\mathfrak{M}_{\text{dense}}$ in the class of intended structures $\mathbb{M}_{\text{dense}}$ corresponds to a model of $CODIB_{\downarrow}$ (satisfiability), the reverse (axiomatizability) is not true. We would need to restrict the models to those with finite domain, which we cannot do in first-order logic by the Löwenheim-Skolem Theorem: every theory with an arbitrarily large finite model must have an infinite model. But whether there are finite models of $CODIB_{\downarrow}$ that do not correspond to structures in the class $\mathbb{M}_{\text{dense}}$ is an open question. But at least the condition that two atomic manifolds that are part of a composite manifold can only meet in their boundaries is enforced in $CODIB_{\downarrow}$, which we could not enforce previously in $CODI_{\downarrow}$. More precisely, for any $a, b \in \mathbf{Min}_{\mathcal{M}}$ for some model $\mathcal{M}$ of $CODIB_{\downarrow}$, we have $\varrho(a)^{-} \cap \Delta\varrho(b) = \emptyset$. This captures an important condition imposed upon the class of intended structures in Definition 5.11.

9.3.6 Thin boundary

For any non-closed, i.e., bounded entity x, we may require an entity to exist that denotes the complete 'thin' boundary as a function boundary(x) that satisfies BC-A1 but contains all entities contained in the boundary of x (BC-E1). BC-E1 nonconservatively extends $CODIB_{\downarrow}$ because it forces some special sums to exist. For closed entities, the boundary will be the zero entity that must exist in $CODI_{\downarrow}$, a subtheory of $CODIB_{\downarrow}$.

(BC-E1) $Cont(x, \text{boundary}(y)) \leftrightarrow BCont(x, y)$
(boundary(y) is the maximal 'thin' boundary defined in terms of $BCont(x, y)$)

Axiom Set 9.9: Extension axiom BC-E1 of the theory *CODIB*.

The following three properties verify that the boundary function behaves as expected for closed and bounded, that is, non-closed, entities and that entities of minimal dimension are closed.

(BC-T8) $\neg Closed(x) \rightarrow BCont(\text{boundary}(x), x)$ (the boundary of x is boundary-contained in x)

(BC-T9) $Closed(x) \rightarrow ZEX(\text{boundary}(x))$ (zero region as boundary of closed entities)

Lemma 9.7. *$CODIB_{\downarrow} \cup$ BC-E1 $\vDash$ {BC-T8, BC-T9}*

Proof. **(BC-T8):** $\neg Closed(x) \rightarrow BCont(\text{boundary}(x), x)$.
Assume $x \notin \mathbf{Closed}_{\mathcal{M}}$.
We distinguish two cases.

Case (I): Suppose $\mathbf{boundary}(x) \in ZEX_{\mathcal{M}}$.
Then by BC-E1, for all $y \in \mathbf{M}$ we have $\neg\mathbf{BCont}(y, x)$ and hence $x \in \mathbf{Closed}_{\mathcal{M}}$—a contradiction to our assumption.

Case (II): Suppose $\mathbf{boundary}(x) \notin ZEX_{\mathcal{M}}$.
Then $\mathbf{Cont}(\mathbf{boundary}(x), \mathbf{boundary}(x))$, leading to $\mathbf{BCont}(\mathbf{boundary}(x), x)$ by B-D1.

Clearly, the two cases are exhaustive.

(BC-T9): $Closed(x) \rightarrow ZEX(\text{boundary}(x))$.
Assume $x \in \mathbf{Closed}_{\mathcal{M}}$ and suppose $\mathbf{boundary}(x) \notin ZEX_{\mathcal{M}}$.
Then $\mathbf{Cont}(\mathbf{boundary}(x), \mathbf{boundary}(x))$ by C-A1 and $\mathbf{BCont}(\mathbf{boundary}(x), x)$ by BC-E1, the latter contradicting CL-D, which requires $\forall y[\neg BCont(y, x)]$.

□

9.3.7 Closure under sums representing composite manifolds

One of our motivations of introducing boundary-containment was to disallow sums of entities that intersect in their interior (compare Definition 5.6(1),(2)) in our axiomatic theory because those sums do not correspond to m-manifolds in the class of intended structures. The axiom BC-A2 achieves this.

Now, we can close the models of the theory $CODIB_{\downarrow}$ by adding axioms that require sums of entities x and y to exist if x and y do not meet in interiors of any of their minimal parts. Obviously, the axioms Sum-A1 – Sum-A4 of $CODI_{\updownarrow}$ are too strong. Instead of requiring sums to exist for arbitrary entities as in $CODI_{\updownarrow}$, we only want sums that themselves represent composite manifolds again. To achieve this, we will use the axioms Sum′-A0 – Sum′-A5 introduced at the end of Chapter 7 in the theory $CODI'_{\downarrow}$. Those axiomatize the properties of sums through a ternary relation $Sum(x, y, z)$ read as $x + y = z$, but $CODI'_{\downarrow}$ does not force any sums to exist. It only forces sums that already exist to be called so (compare Sum′-A2, Sum′-A5). We now introduce Sum'-A6, which requires sums of entities that do not meet in their interior to exist as well. This is a weaker form of the axioms requiring arbitrary sums to exist.

(Sum′-A6) $x =_{\dim} y \wedge \forall z\big[Cont(z,x) \wedge Cont(z,y) \wedge Min(z) \rightarrow \exists u,v[P(u,x) \wedge P(v,y) \wedge BCont(z,u) \wedge BCont(z,v)]\big] \rightarrow \exists z[Sum(x,y,z)]$ (the sum of equidimensional entities x, y must exist if x and y only touch in boundaries of minimal parts)

Axiom Set 9.10: Axiom Sum′-A6 of the theory $CODIB_{\updownarrow}$.

We define the theory

$$CODIB_{\updownarrow} = CODIB_{\downarrow} \cup \{\text{Sum}'\text{-A0} - \text{Sum}'\text{-A6, U-A1}\},$$

which is intuitively the theory $CODI_{\downarrow}$ extended by the axiomatization of *BCont* and closed under universals and under sums where the sum is guaranteed to represent a composite manifold. As in $CODI'_{\downarrow}$, the operation + defined as $x + y = z \leftrightarrow Sum(x, y, z)$ is a nontotal function in $CODIB_{\updownarrow}$.

$CODIB_{\updownarrow}$ excludes certain sums (Sum‘-T1), namely the sums of entities that intersect in the interior of constituent minimal parts.

(Sum′-T1) $x =_{\dim} y \wedge P(v,x) \wedge Min(v) \wedge Cont(z,v) \wedge Cont(z,y) \wedge SC(x,y) \wedge \neg BCont(z,v) \rightarrow \neg\exists w[Sum(x,y,w)]$ (no sum of equidimensional entities x, y in superficial contact exists if some minimal part v of x shares a z with y that is not contained in the boundary of v)

Lemma 9.8. $CODIB_{\updownarrow} \models$ *Sum′-T1*

Proof. Assume x, y, z, v are arbitrary entities in $\mathbf{M}$ such that $x {=_{\mathbf{dim}}} y$, $\mathbf{P}(v, x)$, $v \in \mathbf{Min}_{\mathcal{M}}$, $\mathbf{Cont}(z, v)$, $\mathbf{Cont}(z, y)$, $\mathbf{SC}(x, y)$, and $\neg\mathbf{BCont}(z, v)$.
Suppose some $w \in \mathbf{M}$ exists such that $\mathbf{Sum}(x, y, w)$ contrary to the consequent of Sum′-T1. From $\mathbf{Sum}(x, y, w)$ and $x {=_{\mathbf{dim}}} y$ we obtain $\mathbf{P}(x, w)$ and $\mathbf{P}(y, w)$ by Sum′-A3. By transitivity of containment we then get, in contradiction to BC-A2,

$$\mathbf{SC}(v, y) \wedge \mathbf{P}(v, w) \wedge v \in \mathbf{Min}_{\mathcal{M}} \wedge \mathbf{P}(y, w) \wedge \mathbf{Cont}(z, w) \wedge \mathbf{Cont}(z, y) \wedge \neg\mathbf{BCont}(z, v).$$

□

Note that for a given intended structure, the resulting model of $CODIB_{\updownarrow}$ may not be an extension of the resulting model of $CODI_{\updownarrow}$. Consider a spatial configuration in which a manifold $l1$ with empty boundary (such as a circle) which intersects another manifold $l2$ of equal dimension. In the corresponding model of $CODI_{\updownarrow}$, the sum $l1 + l2$ would be forced to exist, but in the corresponding model of $CODIB_{\updownarrow}$ no sum $l1 + l2$ can exist. However, the corresponding model of $CODI_{\updownarrow}$ can also be interpreted in a intended structure where $l1$ has a nonempty boundary at which it intersects $l2$, compare Figure 9.9.

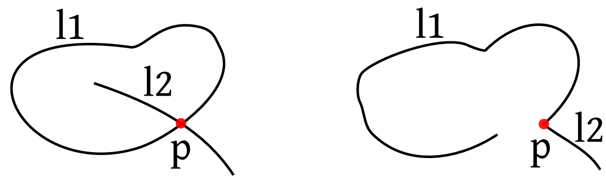


Figure 9.9: Two spatial configurations that have elementarily equivalent models of $CODI_{\updownarrow}$ but distinct models of $CODIB_{\updownarrow}$. Both spatial configurations are elementarily equivalent models of $CODI_{\updownarrow}$ with $\mathbf{Cont}_{\mathcal{M}} = \{\langle l1, l1\rangle, \langle l1, l1+l2\rangle, \langle l2, l2\rangle, \langle l2, l1+l2\rangle, \langle p, p\rangle, \langle p, l1\rangle, \langle p, l2\rangle, \langle p, l1+l2\rangle\}$ and $(<_{\mathbf{dim}})_{\mathcal{M}} = \{\langle p, l1\rangle, \langle p, l2\rangle, \langle p, l1+l2\rangle, \langle \mathrm{ze}, p\rangle, \langle \mathrm{ze}, l1\rangle, \langle \mathrm{ze}, l2\rangle, \langle \mathrm{ze}, l1+l2\rangle\}$. However, their models of $CODIB_{\updownarrow}$ differ: the left configuration results in model $\mathcal{M}'$ with $\mathbf{Cont}_{\mathcal{M}'} = \{\langle l1, l1\rangle, \langle l2, l2\rangle, \langle p, p\rangle, \langle p, l1\rangle, \langle p, l2\rangle\}$, $(<_{\mathbf{dim}})_{\mathcal{M}'} = \{\langle p, l1\rangle, \langle p, l2\rangle, \langle \mathrm{ze}, p\rangle, \langle \mathrm{ze}, l1\rangle, \langle \mathrm{ze}, l2\rangle\}$, and $\mathbf{BCont}_{\mathcal{M}'} = \{\}$ whereas the right configuration results in model $\mathcal{M}''$ with $\mathbf{Cont}_{\mathcal{M}''} = \mathbf{Cont}_{\mathcal{M}}$, $(<_{\mathbf{dim}})_{\mathcal{M}''} = (<_{\mathbf{dim}})_{\mathcal{M}}$, and $\mathbf{BCont}_{\mathcal{M}''} = \{\langle p, l1\rangle, \langle p, l2\rangle\}$. While $\mathcal{M}$ is clearly a substructure of $\mathcal{M}''$, it is not a substructure of $\mathcal{M}'$. That is, only $\mathcal{M}''$ is an extension of the model $\mathcal{M}$.

It is not difficult to see that extending any model of $\mathcal{M}$ of $CODIB_{\updownarrow}$ that corresponds to a structure $\mathfrak{M}_{\text{dense}}$ in the class of intended structures $\mathbb{M}_{\text{dense}}$ to a minimal model that satisfies the axioms Sum′-A0 – Sum′-A6 and U-A1 will again correspond to a structure in the class $\mathbb{M}_{\text{dense}}$. However, this result is only of interest once we prove that *every* model of $CODIB_{\downarrow}$ corresponds to some structures in the class $\mathfrak{M}_{\text{dense}}$, i.e., once we have an axiomatizability proof for $CODIB_{\downarrow}$.

The restriction of the intended models of $CODIB_{\updownarrow}$ to structures in $\mathbb{M}_{\text{dense}}$ allows us to place more stringent requirements on the universal entity, such as requiring it to be internally self-connected (U-E3) instead of only being self-connected as proposed as axiom U-E2 in Chapter 7. Such axiom can be added as seems fit for particular applications.

(U-E3) $ICon(U)$ (universal entity is internally self-connected)

Axiom Set 9.11: Extension axiom U-E3 of the theory $CODIB_{\updownarrow}$.

For the remainder of this chapter we will work with the theory $CODIB_{\downarrow}$ and not with $CODIB_{\updownarrow}$. But because $CODIB_{\updownarrow}$ is an extension of $CODIB_{\downarrow}$, all results equally apply to $CODIB_{\updownarrow}$. All definitions we will introduce—except for that of external overlap EO—can also be used in the theory $CODIB$.

9.4 Equidimensional boundaries

Alternatively to lower-dimensional, i.e., ‘thin’ boundaries, so-called ‘bulky’ or ‘thick’ boundaries are also of interest. If a region of space represents a physical object, the thick boundary of the region could be used to represent the space associated with the surface (or ‘physical boundary’) of the object, compare Section 11.3. Contrary to a thin boundary, which is a pure abstraction with a lower dimension than the bounded region, a thick boundary is indeed part of the bounded region in that it is of the same dimension as the bounded region. Taking the United States as example, we can determine its land border states, i.e., its states that border some other country (Canada or Mexico) as in Figure 9.10, or its coastal border, i.e., its states that have a ocean coastline. Equally, we can determine the states in the United States and the provinces in Canada that form the land border of the Great Lakes, compare Figure 9.11. As another example, we may want to capture all the walls in a building, i.e., the space that separates rooms from one another or from the outside—assuming we rely on a construction drawing where the walls have non-negligible thickness. While for many purposes the thickness of walls may be negligible, for other purposes we may treat walls as 'parts' of the building that occupy a three-dimensional chunk of space to, e.g., account for the reduced effectively usable space.

9.4.1 Tangential and interior parts

First, we define tangential and interior parthood as the equidimensional equivalents of tangential and interior containment (TP-D, IP-D).

(TP-D) $TP(x,y) \leftrightarrow P(x,y) \wedge TCont(x,y)$ (tangential part)

(IP-D) $IP(x,y) \leftrightarrow P(x,y) \wedge ICont(x,y)$ (interior part)

Axiom Set 9.12: Definitions TP-D and IP-D of the *CODIB* hierarchy.

We can prove the following properties for tangential and interior parthood, which are analogue to those for tangential and interior containment, compare Sections 9.3.2 and 9.3.3.

(TP-T1) $Closed(x) \rightarrow \forall y[\neg TP(y,x)]$ (closed entity has no tangential parts)

(TP-T2) $\neg Closed(x) \rightarrow TP(x,x)$ (TP reflexive for non-closed entity)

(TP-T3) $Closed(x) \wedge \neg ZEX(x) \leftrightarrow IP(x,x)$ (IP reflexive iff nonzero entity is closed)

Lemma 9.9. $CODIB_{\downarrow} \cup \{TP\text{-}D, IP\text{-}D\} \models \{TP\text{-}T1 – TP\text{-}T3\}$

Considering the definitions TP-D and IP-D and the fact that P specializes *Cont*, an immediate consequence of Theorem 9.1 is that tangential and interior parthood are jointly exhaustive and disjoint subrelations of parthood in $CODIB_{\downarrow} \cup \{\text{TP-D, IP-D}\}$.

Theorem 9.3. *In a model $\mathcal{M}$ of $CODIB_{\downarrow} \cup \{TP\text{-}D, IP\text{-}D\}$, $\mathbf{IP}_{\mathcal{M}}$ and $\mathbf{TP}_{\mathcal{M}}$ partition $\mathbf{P}_{\mathcal{M}}$.*

Proof. Because by the definitions TP-D and IP-D we already know that TP and IP are specializations of P, it suffices to prove the following theorems:

(TP-T4) $\neg TP(x,y) \vee \neg IP(x,y)$ (TP and IP are disjoint)

(TP-T5) $P(x,y) \rightarrow TP(x,y) \vee IP(x,y)$ (TP and IP are exhaustive subrelations of P)

These follow from Theorem 9.1. □

Note that if in a model all entities—apart from the zero entity—are of equal dimension, then no entity can have a tangential part. This is because tangential containment is defined in terms of boundary-containment, which in turn can only hold between entities of differing dimension. This is a limitation of our axiomatization based on boundary-containment as primitive relation. In the case of all entities being equidimensional, it would be more appropriate to use boundary parthood (either in its strong or weak form, compare BP-D and SBP-D further down) as primitive relation instead of boundary-containment. We do not explore this alternative approach here, it has already been studied in the various equidimensional mereotopologies, such as [AV95; RCC92] that define relations of tangential and nontangential parthood.

Figure 9.10: Thick boundaries of the United States. If states are the smallest spatial units (minimal entities), then the olive shaded states represent the thick land border of the United States. Equally, the thick border of Mexico (with the US only) is shaded turquoise, treating again states as minimal entities. In this example the thick border of Mexico is incomplete: it is only with respect to the displayed countries.

There are actually two slightly different kinds of tangential parthood, namely weak and strong tangential parthood, based on the dimension of the boundary that the tangential part contains. While TP-D captures the notion of weak tangential parthood (simply referred to as tangential parthood in the sequel), strong tangential parthood additionally requires in addition that the tangential part contains a part of the original entity's boundary, not just an entity of a lower dimension than the boundary but contained in the boundary. For example, in a 2D area, a part is a strong tangential part if it contains a 1D part of the boundary of the 2D area. A part that only contains a single point (or a set of points) of the boundary of the 2D area is not strongly tangential. The definition STP-D captures this notion of strong tangential parthood.

(STP-D) $STP(x,y) \leftrightarrow TP(x,y) \wedge \exists z[Cont(z,x) \wedge BCont(z,y) \wedge z \prec_{\text{dim}} y]$ (strong tangential parthood)

Axiom Set 9.13: Definition STP-D of the *CODIB* hierarchy.

Figure 9.11: The thick border of North America with respect to the Great Lakes with states as smallest spatial units (minimal entities).

Clearly, STP specializes TP thus all properties of TP also apply to STP.

9.4.2 Boundary parts

From tangential and strong tangential parthood two kinds of boundary parthood arise. First, consider the weak case: we want every minimal tangential part and sums of minimal tangential parts to be boundary parts. We can express this axiomatically as BP-D: x is a boundary part of y if every part z of x is a tangential part of y. Likewise, x is a strong boundary part of y if every part z of x is a strong tangential part of y (SBP-D).

(BP-D) $BP(x,y) \leftrightarrow P(x,y) \land \forall z[P(z,x) \rightarrow TP(z,y)]$ ((weak) boundary part)

(SBP-D) $SBP(x,y) \leftrightarrow P(x,y) \land \forall z[P(z,x) \rightarrow STP(z,y)]$ (strong boundary part)

Axiom Set 9.14: Definitions BP-D and SBP-D of the *CODIB* hierarchy.

The following properties become provable in the definitional extension of $CODIB_{\downarrow}$.

(BP-T1) $BP(x,y) \rightarrow TP(x,y)$ (BP specializes TP)

(BP-T2) $SBP(x,y) \rightarrow BP(x,y)$ (SBP specializes BP)

(BP-T3) $TP(x,y) \land Min(x) \rightarrow BP(x,y)$ (minimal tangential parts are boundary parts)

(BP-T4) $STP(x,y) \land Min(x) \rightarrow SBP(x,y)$

(minimal strong tangential parts are strong boundary parts)

(BP-T5) $Closed(x) \rightarrow \forall y[\neg BP(y,x)]$ (closed entities have no boundary parts)

(BP-T6) $Closed(x) \rightarrow \forall y[\neg SBP(y,x)]$ (closed entities have no strong boundary parts)

Lemma 9.10. *$CODIB_{\downarrow} \cup$ {TP-D, IP-D, STP-D, BP-D, SBP-D} $\models$ {BP-T1 – BP-T6}*

Proof. **(BP-T1):** $BP(x,y) \rightarrow TP(x,y)$.

Assume $x, y \in \mathbf{M}$ with $\mathbf{BP}(x,y)$.

We can choose $z := x$ in BP-D to obtain $\mathbf{TP}(x,y)$.

(BP-T2): $SBP(x,y) \rightarrow BP(x,y)$.

Assume $x, y \in \mathbf{M}$ with $\mathbf{SBP}(x,y)$.

Then for all $z \in \mathbf{M}$ $\forall z[P(z,x) \rightarrow STP(z,y)]$ by SBP-D and with STP-D we get for all $z \in \mathbf{M}$ $\mathbf{P}(z,x)$ implies $\mathbf{TP}(z,y)]$ which requires $\mathbf{BP}(x,y)$ by BP-D.

(BP-T3): $TP(x,y) \land Min(x) \rightarrow BP(x,y)$.

Assume $x, y \in \mathbf{M}$ with $\mathbf{TP}(x,y)$ and $x \in \mathbf{Min}_{\mathcal{M}}$.

Then by definition of $x \notin \mathbf{Min}_{\mathcal{M}}$ no proper part of x can exist, i.e., $\mathbf{P}(z,x) \rightarrow z = x$. Then for all $z \in \mathbf{M}$, $\mathbf{P}(z,x) \rightarrow \mathbf{TP}(z,y)$ is equivalent to $\mathbf{TP}(x,y)$ which holds by assumption; hence $\mathbf{BP}(x,y)$ by definition BP-D.

(BP-T4): $STP(x,y) \land Min(x) \rightarrow SBP(x,y)$.

Analogue to BP-T3.

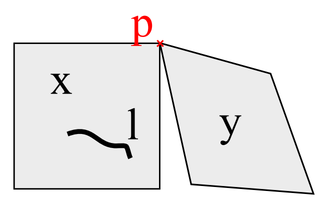


Figure 9.12: A spatial configuration in which a non-closed minimal entity is not tangentially contained in itself. This configuration can be directly captured as a model of $CODIB_{\downarrow}$. We do not specify the complete model here, let it suffice that $\langle p, l\rangle, \langle l, x\rangle, \langle l, y\rangle, \langle l, x+y\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$, $x, y \in \mathbf{Min}_{\mathcal{M}}$ $\langle p, x\rangle, \langle p, y\rangle, \langle l, x\rangle, \langle x, x+y\rangle, \langle y, x+y\rangle \in \mathbf{Cont}_{\mathcal{M}}$, and $\langle p, x\rangle, \langle p, y\rangle \in \mathbf{BCont}_{\mathcal{M}}$. Then $\mathbf{boundary}_{\mathcal{M}}(x) = \mathbf{boundary}_{\mathcal{M}}(y) = p$ and thus $x \notin \mathbf{Closed}_{\mathcal{M}}$. But $\langle p, x\rangle \in (\prec_{\mathbf{dim}})_{\mathcal{M}}$ and $\langle l, x\rangle \notin \mathbf{BCont}_{\mathcal{M}}$, thus p and l are the only lower-dimensional entities contained in x, hence $\langle x, x\rangle \notin \mathbf{SBP}_{\mathcal{M}}$.

(BP-T5): $\mathit{Closed}(x) \rightarrow \forall y[\neg BP(y, x)]$.

Assume $x \in \mathbf{M}$ with $x \in \mathbf{Closed}_{\mathcal{M}}$.

Then by TP-T1, we have for all $z \in \mathbf{M}$, $\neg\mathbf{TP}(y, x)$ and by the contrapositive of BP-T1 further for all $z \in \mathbf{M}$, $\neg\mathbf{BP}(y, x)$.

(BP-T6): $\mathit{Closed}(x) \rightarrow \forall y[\neg SBP(y, x)]$.

Assume $x \in \mathbf{M}$ with $x \in \mathbf{Closed}_{\mathcal{M}}$.

Then by TP-T1, we have for all $z \in \mathbf{M}$ $\neg\mathbf{TP}(y, x)$ and thus $\neg\mathbf{STP}(y, x)$ for all $z \in \mathbf{M}$ by STP-D and by the contrapositive of BP-T2 with BP-D1 we also have $\neg\mathbf{SBP}(y, x)$ for all $z \in \mathbf{M}$.

□

Notice that some non-closed minimal entity are not their own strong tangential part, that is

$$\forall x[\neg \mathit{Closed}(x) \wedge \mathit{Min}(x) \rightarrow \mathit{SBP}(x, x)]$$

is not entailed by $CODIB_{\downarrow}$. Figure 9.12 gives an example in which the property fails. This example should suffice to convince the reader that the property is not valid in $CODIB_{\downarrow}$.

9.4.3 Thick boundary

Just as BC-E1 requires a 'thin' boundary of non-closed entities to exist, we can require a 'think' boundary (BC-E2) or a strong 'thick' boundary (BC-E3) to exist.

(BC-E2)	$\mathit{Cont}(x, \text{thickboundary}(y)) \leftrightarrow \mathit{BP}(x, y)$	('thick' boundary)
(BC-E3)	$\mathit{Cont}(x, \text{strongthickboundary}(y)) \leftrightarrow \mathit{SBP}(x, y)$	(strong 'thick' boundary)

Axiom Set 9.15: Extension axioms BC-E2 and BC-E3 of the *CODIB* hierarchy.

By defining a notion of (strong) thick boundary we have achieved our goal for this section. It is not clear which of the two definitions—thick boundary or strong thick boundary—is better suited for practical matters. It may even turn out that their minuscule difference does not matter for most practical purposes.

Next, we will use the definitions of boundaries to distinguish types of contacts that manifest themselves in whether interiors or boundaries are in contact. For this purpose, we will exclusively use the lower-dimensional, 'thin' boundaries. However, we could define the same set of contact relations using 'thick' boundaries instead.

9.5 Boundary and interior contact

The main purpose of this section is to show that we can define Egenhofer's nine topological relations between two entities [Ege89; Ege91; EF91; EH91] that distinguish interior, boundary, and exterior contact in multidimensional configurations of space independent of the dimensions and codimensions of either entity. More precisely, we give six definitions (three symmetric ones and three nonsymmetric ones) that suffice to define Egenhofer's nine topological relations in models of $CODIB_{\downarrow}$.

(IO-D) $IO(x, y) \leftrightarrow \exists z[Min(z) \wedge Cont(z, x) \wedge Cont(z, y) \wedge \neg BCont(z, x) \wedge \neg BCont(z, y)]$
(interior overlap: the interiors of x and y are in contact)

(IBC-D) $IBC(x, y) \leftrightarrow \exists z[Cont(z, x) \wedge \neg BCont(z, x) \wedge BCont(z, y)]$
(interior-boundary contact: the interior of x is in contact to the boundary of y)

(BO-D) $BO(x, y) \leftrightarrow \exists z[BCont(z, x) \wedge BCont(z, y)]$
(boundary overlap: the boundaries of x and y are in contact)

(IEC-D) $IEC(x, y) \leftrightarrow \neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg Cont(x, y)$
(interior-exterior contact: the interior of x is in contact to the exterior of y)

(BEC-D) $BEC(x, y) \leftrightarrow \exists z[BCont(z, x) \wedge \neg Cont(z, y)]$
(boundary-exterior contact: the boundary of x is in contact to the exterior of y)

(EO-D) $EO(x, y) \leftrightarrow \exists z[\neg ZEX((z - x) - y)]$ (exterior overlap: the exteriors of x and y are in contact)

Axiom Set 9.16: Definitions IO-D, IBC-D, BO-D, IEC-D, BEC-D, and EO-D of $CODIB_{\downarrow}$.

From the definitions of IO, BO and EO it is easy to see that those relations are symmetric. We can further prove that the relations IO, BO, IBC and the inverse IBC^{-1} are exhaustive subrelations of contact C in $CODIB_{\downarrow}$. Note that they are not disjoint subrelations of contact, i.e., more than one of them can hold simultaneously between two entities.

Theorem 9.4. *In a model $\mathcal{M}$ of $CODIB_{\downarrow} \cup$ {IO-D, IBC-D, BO-D} for all $x, y \in \mathbf{M}$,*

$$\langle x, y\rangle \in \mathbf{C}_{\mathcal{M}} \iff \langle x, y\rangle \in \mathbf{IO}_{\mathcal{M}} \text{ or } \langle x, y\rangle \in \mathbf{IBC}_{\mathcal{M}} \text{ or } \langle y, x\rangle \in \mathbf{IBC}_{\mathcal{M}}, \text{ or } \langle x, y\rangle \in \mathbf{BO}_{\mathcal{M}}.$$

Proof. We need to prove that the relations IO, BO, and IBC as well as the inverse relation IBC^{-1} are subrelations of C and that those are exhaustive subrelations of C.

(BC-T10) $IO(x, y) \rightarrow C(x, y)$ (IO is a subrelation of C)

(BC-T11) $BO(x, y) \rightarrow C(x, y)$ (BO is a subrelation of C)

(BC-T12) $IBC(x, y) \rightarrow C(x, y)$ (IBC is a subrelation of C)

(BC-T13) $IBC(y, x) \rightarrow C(x, y)$ (IBC^{-1} is a subrelation of C)

(BC-T14) $C(x, y) \rightarrow IO(x, y) \vee BO(x, y) \vee IBC(x, y) \vee IBC(y, x)$ (exhaustive subrelations of C)

BC-T10 to BC-T13 are automatically provable.

To prove BC-T14, assume x, y to be arbitrary entities in $\mathbf{M}$ such that $\mathbf{C}(x, y)$. Then by C-D some $z \in \mathbf{M}$ exists such that $\mathbf{Cont}(z, x)$ and $\mathbf{Cont}(z, y)$. Let us consider the set of all entities z that are contained in x and y. We distinguish the following cases.

Case (I): some $z \in \mathbf{M}$ exists such that $\mathbf{BCont}(z, x)$ and $\mathbf{TCont}(z, y)$.
Then by TC-D, some v is contained in z and is boundary-contained in y. Hence $\mathbf{BCont}(v, x)$ and $\mathbf{BCont}(v, y)$, thus $\langle x, y\rangle \in \mathbf{BO}_{\mathcal{M}}$ by BO-D.

Case (II): some $z \in \mathbf{M}$ exists such that $\mathbf{TCont}(z, x)$ and $\mathbf{BCont}(z, y)$.
Analogous to Case (I) we obtain $\langle x, y\rangle \in \mathbf{BO}_{\mathcal{M}}$.

Case (III): not Case (I) or (II) but some $z \in \mathbf{M}$ exists such that $\mathbf{BCont}(z, x)$ and $\mathbf{BCont}(z, y)$.
Then there must exist an entity v such that $\mathbf{BCont}(v, z)$ and $\mathbf{BCont}(v, x)$. Because the conditions of the Cases (I) and (II) are not satisfied, we must have $\neg\mathbf{BCont}(v, y)$ and thus $\langle y, x\rangle \in \mathbf{IBC}_{\mathcal{M}}$ by IBC-D.

Case (IV): not Case (I)–(III) but some $z \in \mathbf{M}$ exists such that $\mathbf{ICont}(z, x)$ and $\mathbf{TCont}(z, y)$.
Then there must exist an entity v such that $\mathbf{Cont}(v, z)$ and $\mathbf{BCont}(v, y)$. By $ICont(z, x)$ we further have $\neg\mathbf{BCont}(z, x)$ and thus $\neg\mathbf{BCont}(v, x)$. Then $\langle x, y\rangle \in \mathbf{IBC}_{\mathcal{M}}$ by IBC-D.

Case (V): not Case (I)–(IV) but some $z \in \mathbf{M}$ exists such that $\mathbf{TCont}(z, x)$ and $\mathbf{ICont}(z, y)$.
Analogous to Case (IV) we obtain $IBC(y, x)$.

Case (VI): not Case (I)–(V).
Because no z satisfies either $\mathbf{TCont}(z, x)$ or $\mathbf{TCont}(z, y)$ (because the Cases (III)–(V) do not apply), no $z \in \mathbf{M}$ satisfies $\mathbf{TCont}(z, x)$ and $\mathbf{BCont}(z, y)$ (TC-T2). Because some $z \in \mathbf{M}$ exists with $\mathbf{BCont}(z, x)$ and $\mathbf{BCont}(z, y)$], a $v \in \mathbf{M}$ that is a minimal part of z is guaranteed to exist by ME-E1 and satisfies

$$v \in \mathbf{Min}_{\mathcal{M}} \wedge \mathbf{Cont}(v, x) \wedge \mathbf{Cont}(v, y) \wedge \neg\mathbf{BCont}(v, x) \wedge \neg\mathbf{BCont}(v, y)]$$

by transitivity of *Cont* (C-A3) and of *BCont* (BC-A5). We immediately conclude $\langle x, y\rangle \in \mathbf{IO}_{\mathcal{M}}$ by IO-D.

Recall that by BC-T7 the sentence

$$Cont(x, y) \rightarrow ICont(x, y) \vee TCont(x, y)$$

holds. Therefore, by the construction of the Cases (I)–(VI), those six cases are exhaustive. In either case we have at least one of $\langle x, y\rangle \in \mathbf{BO}_{\mathcal{M}}$, $\langle x, y\rangle \in \mathbf{IBC}_{\mathcal{M}}$, $\langle y, x\rangle \in \mathbf{IBC}_{\mathcal{M}}$, or $\langle x, y\rangle \in \mathbf{IO}_{\mathcal{M}}$, i.e., BC-T4 is always satisfied. □

Note that the relations *IEC*, *BEC*, their inverses IEC^{-1} and BEC^{-1}, and *EO*, which all involve the exterior of one of the two entities in relation, do not make any claim about the entities being in contact; instead they only say that one of the entities in question is not contained in the other. In that sense, the four relations *IO*, *IBC*, IBC^{-1}, and *BO* are sufficient to describe the *4-intersection* relations [EF91] between two composite manifolds of possibly different dimension in any space of the same or higher dimension. That is, all 16 relations identified between two spatial objects can be realized by manifolds, not just the eight relations that apply to two spatial entities of equal dimension in a space of equal dimension, such as between two-dimensional areas in $\mathbb{R}^n$ [EF91].

In our final step for this chapter, we show that the nine relations *IO*, *IBC*, IBC^{-1}, *IEC*, IEC^{-1}, *BO*, *BEC*, BEC^{-1}, and *EO* capture the intended topological meaning of whether one entity's interior, boundary, or exterior shares a point with a second entity's interior, boundary, or exterior, where the entities can be arbitrary composite manifolds of different dimensions located in a space that is of equal or higher dimension. To achieve this, we map the six relations *IO*, *IBC*, *IEC*, *BO*, *BEC*, and *EO*—we do not need to consider the inverse relations separately—to their manifold counterparts expressed using the manifold operations Θ, Δ, and $^-$. Θ, Δ, and $^-$ denote the interior, boundary, and exterior of a manifold, respectively, analogue to the topological operations A°, ∂A, and A^-. Through this work we generalize the *9-intersection* relations of Egenhofer and associates [EF91; EH91] that was originally restricted to regions of equal dimension in a space of codimension zero. Specifically, we can define the nine intersection relations, often captured as a 3×3 matrix called the *9-intersection matrix*, as logical relations. Each value in the matrix indicates whether the intersection of A's interior (top row), boundary (center row), and exterior (bottom row) with the interior (left column), boundary (center column), and exterior (right column) of B is nonempty. Expressed using the logical relations, we want to prove that the 9-intersection matrix looks as follows:

$$R(A,B) = \begin{pmatrix} IO(A,B) & IBC(A,B) & IEC(A,B) \\ IBC(B,A) & BO(A,B) & BEC(A,B) \\ IEC(B,A) & BEC(B,A) & EO(A,B) \end{pmatrix}$$

We formalize this relationship by using the definitions of interior Θ, boundary Δ, and exterior $^-$ of composite manifolds in a complex manifolds (compare Sections 5.3 and 5.4).

Theorem 9.5. *Let $\mathfrak{M}$ be a structure in $\mathbb{M}_{\text{dense}}$ with domain $Dom(\mathfrak{M})$ of composite manifolds and let $\mathcal{M}$ be the corresponding model of $CODIB_{\downarrow}$ as constructed in Theorem 9.2.*
Then for all $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$,

1. $\langle d_1, d_2\rangle \in \mathbf{IO}_{\mathcal{M}} \iff \Theta d_1 \cap \Theta d_2 \neq \emptyset$;
2. $\langle d_1, d_2\rangle \in \mathbf{IBC}_{\mathcal{M}} \iff \Theta d_1 \cap \Delta d_2 \neq \emptyset$;
3. $\langle d_1, d_2\rangle \in \mathbf{IEC}_{\mathcal{M}} \iff \Theta d_1 \cap d_2^- \neq \emptyset$ *and* $d_2 \neq \emptyset$;
4. $\langle d_1, d_2\rangle \in \mathbf{BO}_{\mathcal{M}} \iff \Delta d_1 \cap \Delta d_2 \neq \emptyset$;
5. $\langle d_1, d_2\rangle \in \mathbf{BEC}_{\mathcal{M}} \iff$ *there exists a* $d_3 \in \mathrm{Dom}(\mathfrak{M})$ *with* $d_3 \neq d_1$ *and* $\Delta d_1 \cap d_2^- \cap \Sigma d_3 \neq \emptyset$;
6. $\langle d_1, d_2\rangle \in \mathbf{EO}_{\mathcal{M}} \iff d_1^- \cap d_2^- \neq \emptyset$.

Proof. We prove each of the properties (1)–(6) individually in each direction.

1. $\langle d_1, d_2\rangle \in \mathbf{IO}_{\mathcal{M}} \iff \Theta d_1 \cap \Theta d_2 \neq \emptyset$.

   Direction (a): $\langle d_1, d_2\rangle \in \mathbf{IO}_{\mathcal{M}} \Rightarrow \Theta d_1 \cap \Theta d_2 \neq \emptyset$.
   Assume $\langle d_1, d_2\rangle \in \mathbf{IO}_{\mathcal{M}}$.

Then there exists a $d_i \in \mathrm{Dom}(\mathfrak{M})$ such that

$$\mu(d_i) \in \mathbf{Min}_{\mathcal{M}} \wedge \mathbf{Cont}(\mu(d_i), \mu(d_1)) \wedge \mathbf{Cont}(\mu(d_i), \mu(d_2)) \wedge \neg\mathbf{BCont}(\mu(d_i), \mu(d_1)) \wedge \neg\mathbf{BCont}(\mu(d_i), \mu(d_2))$$

by IO-D. Then there exists an atomic nonzero manifold $d_3 \in \mathrm{Dom}(\mathfrak{M})$ such that

$$\mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{Cont}(\mu(d_3), \mu(d_2)) \wedge \neg\mathbf{BCont}(\mu(d_3), \mu(d_1)) \wedge \neg\mathbf{BCont}(\mu(d_3), \mu(d_2)).$$

Assume such d_3 exists. Then we can distinguish three exhaustive cases.

Case (a.i): Assume $\mathbf{ICont}(\mu(d_3), \mu(d_1))$.

Then $\Sigma d_3 \subseteq \Sigma d_1$ by Theorem 9.2(4) from $\mathbf{Cont}(\mu(d_3), \mu(d_1))$.

Now suppose $\Sigma d_3 \cap \Delta d_1 \neq \emptyset$. Then $\Sigma d_3 \cap \Sigma d_1 \neq \emptyset$ However, because d_3 is an atomic manifold, it cannot share a manifold of its own dimension with d_1. Hence $dim(\Sigma d_3 \cap \Sigma d_1) < dim(d_3)$. Then we also have $dim(\Sigma d_3 \cap \Delta d_1) < dim(d_3)$, so that we can apply Definition 5.11(3): there exists a collection $\mathfrak{M}' \subseteq \mathfrak{M}$ such that $\left(\bigcup_{d \in \mathfrak{M}'} \Sigma d\right) = \Sigma d_3 \cap \Delta d_1$. Any $d_4 \in \mathfrak{M}'$ satisfies $\Sigma d_4 \subseteq \Sigma d_3$ and $\Sigma d_4 \subseteq \Delta d_1$. But then by Theorem 9.2(4),(5):

$$\mathbf{Cont}(\mu(d_4), \mu(d_3)) \ \text{ and } \mathbf{BCont}(\mu(d_4), \mu(d_1))$$

and thus $\neg\mathbf{ICont}(\mu(d_3), \mu(d_1))$ by IC-D. This contradictions our assumption; hence we cannot have $\Sigma d_3 \cap \Delta d_1 \neq \emptyset$ and therefore $\Sigma d_3 \cap \Delta d_1 = \emptyset$, which together with $\Sigma d_3 \subseteq \Sigma d_1$ let us conclude $\Sigma d_3 \subseteq \Theta d_1$ as expected.

Then $\Sigma d_3 \subseteq \Sigma d_2$ follows from our assumption $\mathbf{Cont}(\mu(d_3), \mu(d_2))$, and $\Sigma d_3 \not\subseteq \Delta d_2$ follows from $\neg\mathbf{BCont}(\mu(d_3), \mu(d_2))$. Hence $\Sigma d_3 \cap \Theta d_2 \neq \emptyset$ so that $\Sigma d_3 \cap \Theta d_2 \cap \Theta d_1 \neq \emptyset$ and thereby $\Theta d_1 \cap \Theta d_2 \neq \emptyset$, our desired consequence.

Case (a.ii): Assume $\mathbf{ICont}(\mu(d_3), \mu(d_2))$.

Analogously to Case (a.i).

Case (a.iii): Assume $\neg\mathbf{ICont}(\mu(d_3), \mu(d_1))$ and $\neg\mathbf{ICont}(\mu(d_3), \mu(d_2))$.

We must have $\mathbf{TCont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{TCont}(\mu(d_3), \mu(d_2))$ by BC-T7. But because d_3 is an atomic manifold, d_3 cannot contain other manifolds of equal dimension. That means only lower-dimensional entities contained in d_3 can be contained in the boundary of d_1 or d_2. But since the area Σd_3 can never be covered by a finite collection of lower-dimensional manifolds; there will always exist a point in Σd_3 that is in the interior of d_1 and d_2. Hence $\Theta d_1 \cap \Theta d_2 \neq \emptyset$.

The three cases (a.i)–(a.iii) are trivially exhaustive. Hence we can always conclude $\Theta d_1 \cap \Theta d_2 \neq \emptyset$ when $\langle d_1, d_2 \rangle \in \mathbf{IO}_{\mathcal{M}}$.

Direction (b): $\langle d_1, d_2 \rangle \in \mathbf{IO}_{\mathcal{M}} \Leftarrow \Theta d_1 \cap \Theta d_2 \neq \emptyset$.

Assume $\Theta d_1 \cap \Theta d_2 \neq \emptyset$.

Then $\Sigma d_1 \cap \Sigma d_2 \neq \emptyset$ and hence there exists a collection of manifolds $\mathfrak{M}'$ such that

$$\bigcup_{d \in \mathfrak{M}'} \Sigma d = \Sigma d_1 \cap \Sigma d_2$$

by Definition 5.11(2) with some atomic manifold $d_3 \in \mathfrak{M}'$ such that $\Sigma d_3 \subseteq \Sigma d_1$, $\Sigma d_3 \subseteq \Sigma d_2$, and $\Sigma d_3 \cap \Theta d_1 \cap \Theta d_2 \neq \emptyset$ (by $\Theta d_1 \cap \Theta d_2 \neq \emptyset$. By the latter we can have neither $\Sigma d_3 \subseteq \Delta d_1$ nor $\Sigma d_3 \subseteq \Delta d_2$. Hence,

$$\neg\mathbf{BCont}(\mu(d_3), \mu(d_1)) \wedge \neg\mathbf{BCont}(\mu(d_3), \mu(d_2))$$

by Theorem 9.2(5). From $\Sigma d_2 \subseteq \Sigma d_1$ and $\Sigma d_3 \subseteq \Sigma d_2$ we also obtain

$$\mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{Cont}(\mu(d_3), \mu(d_2))$$

by Theorem 9.2(4). Those two conditions together and the fact that d_3 is atomic and thus $\mu(d_3) \in \mathbf{Min}_{\mathcal{M}}$ let us conclude $\langle d_1, d_2\rangle \in \mathbf{IO}_{\mathcal{M}}$ according to IO-D.

2. $\langle d_1, d_2\rangle \in \mathbf{IBC}_{\mathcal{M}} \iff \Theta d_1 \cap \Delta d_2 \neq \emptyset$.

Direction (a): $\langle d_1, d_2\rangle \in \mathbf{IBC}_{\mathcal{M}} \Rightarrow \Theta d_1 \cap \Delta d_2 \neq \emptyset$.
Assume $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$ with $\langle d_1, d_2\rangle \in \mathbf{IBC}_{\mathcal{M}}$.
Then by IBC-D there exists a z with $z = \mu(d_3)$ where $d_3 \in \mathrm{Dom}(\mathfrak{M})$ such that

$$\mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \neg\mathbf{BCont}(\mu(d_3), \mu(d_1) \wedge \mathbf{BCont}(\mu(d_3), \mu(d_2))$$

Then $\Sigma d_3 \neq \emptyset$, $\Sigma d_3 \subseteq \Sigma d_1$, $\Sigma d_3 \nsubseteq \Delta d_1$, and $\Sigma d_3 \subseteq \Delta d_2$ by Theorem 9.2(4) and (5). Hence, $\Sigma d_3 \cap \Theta d_1 \neq \emptyset$ and thus $\Theta d_1 \cap \Delta d_2 \neq \emptyset$, the desired conclusion.

Direction (b): $\langle d_1, d_2\rangle \in \mathbf{IBC}_{\mathcal{M}} \Leftarrow \Theta d_1 \cap \Delta d_2 \neq \emptyset$.
Assume $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$ with $\Theta d_1 \cap \Delta d_2 \neq \emptyset$.
Then by Definition 5.11(2) there exists a collection of atomic manifolds $\mathfrak{M}' \subseteq \mathfrak{M}$ such that $\left(\bigcup_{d\in\mathfrak{M}'} \Sigma d\right) = \Sigma d_1 \cap \Sigma d_2$. Because $\Theta d_1 \cap \Delta d_2 \neq \emptyset$, some $d_3 \in \mathfrak{M}'$ exists with $d_3 \cap \Theta d_1 \cap \Delta d_2 \neq \emptyset$. Then there exists a collection of manifolds $\mathfrak{M}'' \subseteq \mathfrak{M}$ such that $\left(\bigcup_{d\in\mathfrak{M}''} \Sigma d\right) = \Sigma d_3 \cap \Delta d_2$, again by Definition 5.11(2). In particular, some $d_4 \in \mathfrak{M}''$ exists with $d_4 \cap \Theta d_1 \neq \emptyset$ and thus $\Sigma d_4 \nsubseteq \Delta d_1$. We also have $\Sigma d_4 \subseteq \Sigma d_3 \subseteq \Sigma d_1$ and $\Sigma d_4 \subseteq \Delta d_2$. By Theorem 9.2(4) and (5) we obtain:

$$\mathbf{Cont}(\mu(d_4), \mu(d_1)) \wedge \neg\mathbf{BCont}(\mu(d_4), \mu(d_1) \wedge \mathbf{BCont}(\mu(d_4), \mu(d_2))$$

which proves $\langle d_1, d_2\rangle \in \mathbf{IBC}_{\mathcal{M}}$ by IBC-D.

3. $\langle d_1, d_2\rangle \in \mathbf{IEC}_{\mathcal{M}} \iff \Theta d_1 \cap d_2^- \neq \emptyset$ and $d_2 \neq \emptyset$.

Direction (a): $\langle d_1, d_2\rangle \in \mathbf{IEC}_{\mathcal{M}} \Rightarrow \Theta d_1 \cap d_2^- \neq \emptyset$.
Assume $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$ with $\langle d_1, d_2\rangle \in \mathbf{IEC}_{\mathcal{M}}$.
Then $\neg\mathbf{Cont}(\mu(d_1), \mu(d_2))$, $\mu(d_1) \notin \mathbf{ZEX}_{\mathcal{M}}$, and $\mu(d_2) \notin \mathbf{ZEX}_{\mathcal{M}}$ by IEC-D and there exists a $z = \mu(d_3)$ with $d_3 \in \mathrm{Dom}(\mathfrak{M})$ such that

$$\mu(d_3) \in \mathbf{Min}_{\mathcal{M}} \wedge \mathbf{P}(\mu(d_3), \mu(d_1)) \wedge \neg\mathbf{Cont}(\mu(d_3), \mu(d_2))$$

by EP-E3 and ME-E1. By Theorem 9.2(1),(3),(4) $d_3 \in \mathfrak{M}$ must be an atomic manifold such $\Sigma d_3 \subseteq \Sigma d_1$ and $\Sigma d_3 \nsubseteq \Sigma d_2$. Because d_3 is atomic, either d_2 is of lower dimension than d_3 or d_3 and d_2 can only meet in the boundary of d_3. In the latter case, any interior point

$p \in \Theta d_3$ satisfies $p \notin \Sigma d_2$. In the former case d_2 cannot completely cover the interior of d_3 and thus some point $p \in \Theta d_3$ satisfies $p \notin \Sigma d_2$. In either case $p \in \Theta d_1$ and $p \in d_2^-$ and thus $\Theta d_1 \cap d_2^- \neq \emptyset$ as well as $d_2 \neq \emptyset$.

Direction (b): $\langle d_1, d_2 \rangle \in \mathbf{IEC}_{\mathcal{M}} \Leftarrow \Theta d_1 \cap d_2^- \neq \emptyset$.

Assume $\Theta d_1 \cap d_2^- \neq \emptyset$ and $d_2 \neq \emptyset$.

Then some point p exists such that $p \in \Theta d_1 \subseteq \Sigma d_1$ and $p \notin \Sigma d_2$. Hence $\Sigma d_1 \nsubseteq \Sigma d_2$ and thus $\neg\mathbf{Cont}(\mu(d_1), \mu(d_2))$ by Theorem 9.2(4). By IEC-D, we get $\langle d_1, d_2 \rangle \in \mathbf{IEC}_{\mathcal{M}}$ because $\Sigma d_1 \neq \emptyset$ by $\Theta d_1 \cap \Theta d_2^- \neq \emptyset$ and $d_2 \neq \emptyset$.

4. $\langle d_1, d_2 \rangle \in \mathbf{BO}_{\mathcal{M}} \quad \Longleftrightarrow \quad \Delta d_1 \cap \Delta d_2 \neq \emptyset$.

Direction (a): $\langle d_1, d_2 \rangle \in \mathbf{BO}_{\mathcal{M}} \Rightarrow \Delta d_1 \cap \Delta d_2 \neq \emptyset$.

Assume $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$ with $\langle d_1, d_2 \rangle \in \mathbf{BO}_{\mathcal{M}}$.

Then by BO-D there exists a $d_3 \in \mathrm{Dom}(\mathfrak{M})$ such that

$$\mathbf{BCont}(\mu(d_3), \mu(d_1)) \wedge \mathbf{BCont}(\mu(d_3), \mu(d_2))$$

Then by Theorem 9.2(5) we have $\Sigma d_3 \subseteq \Delta d_1$, $\Sigma d_3 \subseteq \Delta d_2$, and $\Sigma d_3 \neq \emptyset$. We conclude $\emptyset \neq \Sigma d_3 \subseteq \Delta d_1 \cap \Delta d_2$.

Direction (b): $BO(\mu(d_1), \mu(d_2)) \Leftarrow \Delta d_1 \cap \Delta d_2 \neq \emptyset$.

Assume $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$ with $\Delta d_1 \cap \Delta d_2 \neq \emptyset$.

We distinguish three cases.

Case (b.i): Assume $dim(\Sigma d_1 \cap \Delta d_2) < dim(d_1)$.

Then by Definition 5.11(3) some collection $\mathfrak{M}' \subseteq \mathfrak{M}$ exists such that $\left(\bigcup_{d \in \mathfrak{M}'} \Sigma d\right) = \Sigma d_1 \cap \Delta d_2$. Some $d_3 \in \mathrm{Dom}(\mathfrak{M}')$ additionally satisfies $\Sigma d_3 \cap \Delta d_1 \neq \emptyset$ and since Σd_3 is the sum of the areas of atomic manifolds, there exists an atomic manifold $d_4 \in d_3$ such that $\Sigma d_4 \cap \Delta d_1 \neq \emptyset$. This atomic manifold cannot share a manifold of equal dimension with Δd_1. Hence we have $dim(\Sigma d_4 \cap \Delta d_1) < dim(d_4)$ and we can apply Definition 5.11(3) again: some collection $\mathfrak{M}'' \subseteq \mathfrak{M}$ exists such that $\left(\bigcup_{d \in \mathfrak{M}''} \Sigma d\right) = \Sigma d_4 \cap \Delta d_1$. Any $d_5 \in \mathrm{Dom}(\mathfrak{M}'')$ satisfies

$$\Sigma d_5 \subseteq \Delta d_1 \text{ and } \Sigma d_6 \subseteq \Sigma d_5 \subseteq \Sigma d_4 \subseteq \Delta d_2$$

Hence we obtain $\mathbf{BCont}(\mu(d_5), \mu(d_1)) \wedge \mathbf{BCont}(\mu(d_5), \mu(d_2))$ by Theorem 9.2(5) and thus $BO(\mu(d_1), \mu(d_2))$.

Case (b.ii): Assume $dim(\Sigma d_2 \cap \Delta d_1) < dim(d_2)$.

Analogously to Case (b.ii).

Case (b.iii): Assume $dim(\Sigma d_1 \cap \Delta d_2) \not< \dim(d_1)$ and $dim(\Sigma d_2 \cap \Delta d_1) \not< \dim(d_2)$. We also have $dim(d_2) \gneq \dim(\Delta d_2)$.

Then

$$dim(\Delta d_2) \geq \dim(d_1) \gneq \dim(\Delta d_1) \geq \dim(d_2) \gneq \dim(\Delta d_2),$$

which is self-contradictory. Hence this case is ruled out.

These three cases are trivially exhaustive, hence we obtain $BO(\mu(d_1), \mu(d_2))$ in any case.

5. $\langle d_1, d_2\rangle \in \mathbf{BEC}_\mathcal{M} \iff$ there exists a $d_3 \in Dom(\mathfrak{M})$ such that $\left[d_3 \neq d_1 \wedge \Delta d_1 \cap d_2^- \cap \Sigma d_3 \neq \emptyset\right]$.

Direction (a): $\langle d_1, d_2\rangle \in \mathbf{BEC}_\mathcal{M} \Rightarrow$ there exists a $d_3 \in Dom(\mathfrak{M})$ such that $\left[d_3 \neq d_1 \wedge \Delta d_1 \cap d_2^- \cap \Sigma d_3 \neq \emptyset\right]$.

Assume $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$ with $\langle d_1, d_2\rangle \in \mathbf{BEC}_\mathcal{M}$.

Then there exists a $z = \mu(d_3)$ with $d_3 \in \mathrm{Dom}(\mathfrak{M})$ such that

$$\mathbf{BCont}(\mu(d_3), \mu(d_1)) \wedge \neg\mathbf{Cont}(\mu(d_3), \mu(d_2))$$

By Theorem 9.2(4),(5) we obtain $\Sigma d_3 \subseteq \Delta d_1$ and $\Sigma d_3 \nsubseteq \Sigma d_2$. Thus $\Sigma d_3 \cap d_2^- \neq \emptyset$ and thus $\Delta d_1 \cap d_2^- \cap \Sigma d_3 \neq \emptyset$. From $\Sigma d_3 \subseteq \Delta d_1$ we also obtain $dim(d_3) \lneq \mathrm{dim}(d_1)$ and hence $d_3 \neq d_1$.

Direction (b): $\langle d_1, d_2\rangle \in \mathbf{BEC}_\mathcal{M} \Leftarrow$ there exists a $d_3 \in Dom(\mathfrak{M})$ such that $\left[d_3 \neq d_1 \wedge \Delta d_1 \cap d_2^- \cap \Sigma d_3 \neq \emptyset\right]$.

Assume $d_1, d_2, d_3 \in \mathrm{Dom}(\mathfrak{M})$ with $d_1 \neq d_3$ and $\Delta d_1 \cap d_2^- \cap \Sigma d_3 \neq \emptyset$.

Then some atomic manifold $d_4 \in d_3$ exist such that $\Sigma d_4 \cap \Delta d_1 \cap d_2^- \neq \emptyset$. Consider the nonempty intersection $\Sigma d_4 \cap \Delta d_1$. We distinguish two cases.

Case (b.i): Assume $dim(\Sigma d_4 \cap \Delta d_1) = dim(d_4)$.

Then $\Sigma d_4 \subseteq \Delta d_1$ because d_4 does not contain any other atomic manifold than itself. Then we have $\mathbf{BCont}(\mu(d_4), \mu(d_1))$ by Theorem 9.2(5). From $\Sigma d_4 \cap \Delta d_1 \cap d_2^- \neq \emptyset$ we further obtain $\Sigma d_4 \cap d_2^- \neq \emptyset$ and thus $\Sigma d_4 \nsubseteq \Sigma d_2$ by Definition 5.14. Thus $\neg\mathbf{Cont}(\mu(d_4), \mu(d_2))$ and thus with $\mathbf{BCont}(\mu(d_4), \mu(d_1))$ we conclude $\langle d_1, d_2\rangle \in \mathbf{BEC}_\mathcal{M}$ by BEC-D.

Case (b.i): Assume $dim(\Sigma d_4 \cap \Delta d_1) < dim(d_4)$.

Then there exists some collection $\mathfrak{M}' \subseteq \mathfrak{M}$ such that $\left(\bigcup_{d \in \mathfrak{M}'} \Sigma d\right) = \Sigma d_4 \cap \Delta d_1$. In particular, some $d_5 \in \mathfrak{M}'$ satisfies $\Sigma d_5 \cap d_2^- \neq \emptyset$ and thus $\Sigma d_5 \nsubseteq \Sigma d_2$ by Definition 5.14, so that $\neg\mathbf{Cont}(\mu(d_5), \mu(d_2))$. Moreover, because $\Sigma d_5 \subseteq \Delta d_1$ we obtain $\mathbf{BCont}(\mu(d_5), \mu(d_1))$, so that we conclude $\langle d_1, d_2\rangle \in \mathbf{BEC}_\mathcal{M}$ by BEC-D.

Clearly, the two cases (b.i) and (b.ii) are exhaustive because the intersection $\Sigma d_4 \cap \Delta d_1$ can never be of greater dimension than d_4 itself. Hence in any case $\langle d_1, d_2\rangle \in \mathbf{BEC}_\mathcal{M}$.

6. $\langle d_1, d_2\rangle \in \mathbf{EO}_\mathcal{M} \iff d_1^- \cap d_2^- \neq \emptyset$.

Direction (a): $\langle d_1, d_2\rangle \in \mathbf{EO}_\mathcal{M} \Rightarrow d_1^- \cap d_2^- \neq \emptyset$.

Assume $d_1, d_2 \in \mathrm{Dom}(\mathfrak{M})$ with $\langle d_1, d_2\rangle \in \mathbf{EO}_\mathcal{M}$.

Then there exists a $d_3 \in \mathrm{Dom}(\mathfrak{M})$ such that $((\mu(d_3) - \mu(d_1)) - \mu(d_2)) \notin \mathbf{ZEX}_\mathcal{M}$ by EO-D. Because $\neg\mathbf{PO}(((\mu(d_3) - \mu(d_1)) - \mu(d_2), \mu(d_1))$ and $\neg\mathbf{PO}((\mu(d_3) - \mu(d_1)) - \mu(d_2), \mu(d_2))$ (both by Dif-T3), there must exist a d_4 such that

$$\mu(d_4) \in \mathbf{Min}_\mathcal{M} \wedge \mathbf{P}(\mu(d_4), \mu(d_3)) \wedge \neg\mathbf{Cont}(\mu(d_4), \mu(d_1)) \wedge \neg\mathbf{Cont}(\mu(d_4), \mu(d_2)).$$

By Theorem 9.2(1),(4) d_4 must be an atomic manifold such that $\Sigma d_4 \nsubseteq \Sigma d_1$ and $\Sigma d_4 \nsubseteq \Sigma d_2$. Because d_4 cannot intersect d_1 or d_2 in a manifold of the dimension of d_4, d_4 can only share lower-dimensional manifolds with d_1 and d_2. No finite set of such lower-dimensional manifolds is able to cover all interior points of d_4, so that some interior point $p \in \Sigma d_4$ exists such that $p \notin \Sigma d_1$ and $p \notin \Sigma d_2$. Then $d_1^- \cap d_2^- \neq \emptyset$.

Direction (b): $\langle d_1, d_2\rangle \in \mathbf{EO}_{\mathcal{M}} \Leftarrow d_1^- \cap d_2^- \neq \emptyset$.
Assume $d_1, d_2 \in \text{Dom}(\mathfrak{M})$ with $d_1^- \cap d_2^- \neq \emptyset$.
Then there exists a point $p \in \bigcup_{d\in\mathfrak{M}} \Sigma d$ such that $p \notin \Sigma d_1$ and $p \notin \Sigma d_2$. Then some atomic manifold $d_3 \in \mathfrak{M}$ exists such that $\Sigma d_3 \nsubseteq \Sigma d_1$ and $\Sigma d_3 \nsubseteq \Sigma d_2$. Then

$$\mu(d_3)\in\mathbf{Min}_{\mathcal{M}} \wedge \neg\mathbf{Cont}(\mu(d_3), \mu(d_1)) \wedge \neg\mathbf{Cont}(\mu(d_3), \mu(d_2))$$

by Theorem 9.2(1),(4). From $\mu(d_3)\in\mathbf{Min}_{\mathcal{M}}$ we obtain $\mu(d_3)\cdot\mu(d_1) <_{\text{dim}} \mu(d_3)$ and $\mu(d_3)\cdot\mu(d_2) <_{\text{dim}} \mu(d_3)$. Hence $\mathbf{Cont}(\mu(d_3), \mu(d_3) - \mu(d_1))$ by Dif-A3a and $\mu(d_3) - \mu(d_1) =_{\text{dim}} \mu(d_2) >_{\mathbf{dim}} (\mu(d3) - \mu(d_1)) \cdot \mu(d_2)$ by Dif-A1. Hence $\mathbf{Cont}(\mu(d_3), (\mu(d_3) - \mu(d_1)) - \mu(d_2))$ by Dif-A3a and thus $((\mu(d_3) - \mu(d_1)) - \mu(d_2)) \notin \mathbf{ZEX}_{\mathcal{M}}$ by C-A4.

This completes the proof of all six mappings. □

Except for the mapping (5) of *BEC*, all mappings confirm our intuitions. We have thereby proved that the topological characterization of the relationship between two entities can be achieved in the general multidimensional setting in $\mathit{CODIB}_{\downarrow}$ without restricting the dimension or codimension of the two entities. That means we can describe how two complex manifolds are spatially located to one another using conjunctions of those nine logical relations and their negations. This allows us to give explicit definitions for all 16 relations arising from the 4-intersection relations and define all of the 512 relations arising from the 9-intersection relations that are physically possible in a general multidimensional setting[3]. Thereby, $\mathit{CODIB}_{\downarrow}$ also generalizes the mereotopological relations between areas, lines, and points in two-dimensional space from [CDFO93; Ege91; EH91; McK+05]. Our approach has two advantages over the earlier work: (1) it works independent of the dimension of the space and independent of the dimensions of the two spatial entities whose relation we want to describe. We no longer have to treat the relations between lines and areas different that the relations between areas and areas; all those relations can be described in our logical theory. Recall that we can describe the dimension of the contact—the shared region—as well, so that the dimension-refinement proposed in [McK+05] can be easily defined in $\mathit{CODIB}_{\downarrow}$.

One question remains: why does the mapping (5), of *BEC*, differ from the other mappings? This is due to the fact that some points $p \in \Delta d_1$ may not be contained in any manifold d_3 such that $\langle d_3, d_1\rangle \in \mathbf{BCont}_{\mathcal{M}}$. That is, the direction

$$\langle d_1, d_2\rangle \in \mathbf{BEC}_{\mathcal{M}} \Leftarrow \Delta d_1 \cap d_2^- \neq \emptyset$$

may fail if no $d_3 \in \text{Dom}(\mathfrak{M})$ exists with $\Delta d_1 \cap d_2^- \cap \Sigma d_3 \neq \emptyset$, while the direction

$$\langle d_1, d_2\rangle \in \mathbf{BEC}_{\mathcal{M}} \Rightarrow \Delta d_1 \cap d_2^- \neq \emptyset$$

still holds. This is a limitation of our axiomatization of boundaries, in which boundaries may not be completely captured unless it is unambiguous from the spatial configuration in that two entities meet in the boundary of one. Any boundary point of a manifold that is not shared with another nonoverlapping

[3] It is left to investigate how many of the theoretically possible 512 relations are physically possible. These are at least all the relations between two two-dimensional areas (8 relations), two one-dimensional lines (57 relations), and between a two-dimensional area and a one-dimensional line (20 relations) in $\mathbb{R}^2$, for which [EH91] provide examples. Note though that a relation between two lines may also apply to two areas or to a line and an area. Therefore, without closer investigation only 57, and not 85, different relations must definitely exist.

manifold is not treated as such, i.e., such boundary point may end up in the interior in the corresponding logical model.

9.6 Summary

In this chapter, we extended our multidimensional mereotopologies from the previous chapters by a primitive notion of boundary-containment in order to formalize the distinction between the interior and the boundary of a composite m-manifold. A key finding of this chapter is that the bodiless notion of boundary (captured by the boundary function) as well as the bulky notion of boundary (captured by either of the functions thickboundary or strongthickboundary) are definable as mutually consistent extensions of the same theory, *CODIB*. We can thus say that both conceptions of boundaries coexist in the same theory, they are in no way incompatible. Moreover, once we ground physical space, i.e., the space that talks about arrangements of physical objects, we are free to choose a material or an abstract notion of boundary; the former would correspond to bulky boundaries, whereas the latter corresponds to bodiless boundaries. We will discuss the grounding of physical surfaces and boundaries in more detail in Section 11.3, after we formally relate physical objects to the regions of abstract space they occupy.

The extension of the language of *CODI* by the primitive relation *BCont* led to the new theory *CODIB*. We further extended *CODIB* to $CODIB_{\downarrow}$ analogue to the extension of *CODI* to $CODI_{\downarrow}$. If we want entire boundaries to exist, we can further extend $CODIB_{\downarrow}$ by one or more of BC-E1 – BC-E3. The hierarchy of *CODIB* theories developed in this chapter and its relationships to the theories of the *CODI* hierarchy are illustrated in Figure 9.13.

As a central theoretical result, we showed in Theorem 9.2 that our proposed extension of $CODI_{\downarrow}$ to $CODIB_{\downarrow}$ preserves satisfiability with respect to the class $\mathbb{M}_{\text{dense}}$, a subset of the class $\mathbb{M}$ of intended structures presented in Chapter 5. Compared to $CODI_{\downarrow}$, $CODIB_{\downarrow}$ ensures that more of the properties of the intended structures —in particular condition (3) of the definition of complex manifolds (Definition 5.11)—are satisfied in the logical theories. Still, axiomatizability as guarantee that every model of the theory $CODIB_{\downarrow}$ corresponds to some structure in the class $\mathbb{M}_{\text{dense}}$ suffers from the same problems that already prevented us from proving axiomatizability for the theory $CODI_{\downarrow}$ in Section 7.2.5: we would need to show that all domain elements in a model of $CODIB_{\downarrow}$ are representable as manifolds with boundaries.

We have also proved that the extensions of the defined relations *ICont* (interior containment) and *TCont* (tangential containment) partition the extension of containment (Theorem 9.1) in any model of *CODIB*. Likewise, the extensions of *IP* (interior parthood) and *TP* (tangential parthood) partition the extension of parthood (Theorem 9.3). boundary-containment and boundary parthood specialize tangential containment and parthood, respectively. This results in a generalized, dimension-independent version of the distinctions between tangential and non-tangential parthood known from the RCC and from other equidimensional mereotopologies. Specifically, we can classify containment (and its inverse) as illustrated in Figure 9.14, with Figure 9.15 restricting the lattice of relations to the equidimensional case, resulting in a refinement of the partial overlap relation.

We have provided an even finer classifications of mereotopological relations in multidimensional space by distinguishing whether an entity's interior, boundary, or exterior meets another entity's interior, boundary, or exterior. We defined three symmetric (*IO*, *BO*, *EO*) and three non-symmetric relations (*IBC*, *IEC*, *BEC*) that generalize the nine topological intersection relations from [Ege91; EH91], which

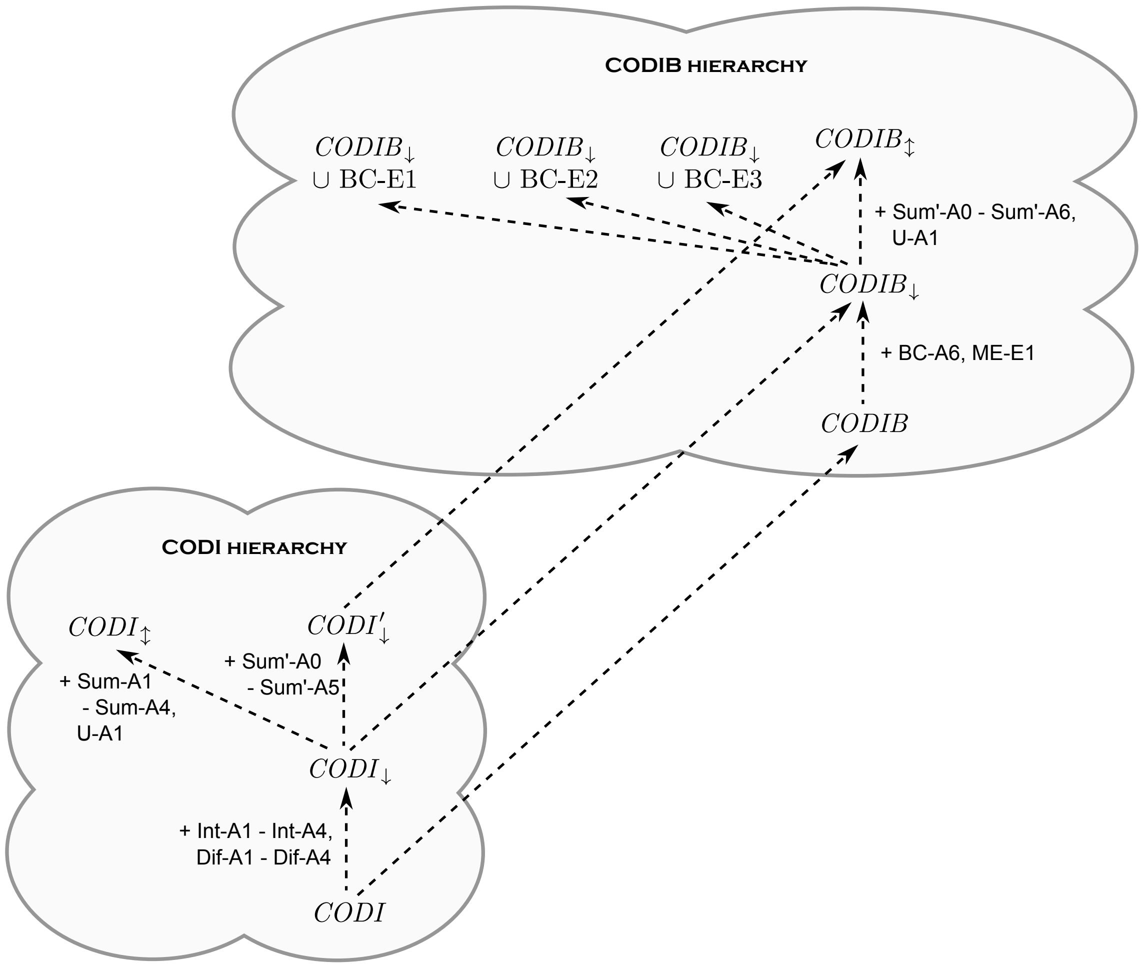


Figure 9.13: The theories of the *CODIB* hierarchy and their relationships to the theories of the *CODI* hierarchy.

have been widely studied and used in Geographic Information Science, to the most general multidimensional setting. Theorem 9.5 proves that our so-defined relations correctly capture the nine topological intersections as expressed for manifolds. Our relations do not rely on references to absolute dimensions or codimensions of the two involved entities or of their shared entities. This generalizes the earlier work of [CDF98; CDFO93; Ege91; EH91; EM95; ME94; McK+05], which defined subsets of the same relations in more restricted settings by explicitly distinguishing relations between two spatial entities for any combination of the absolute dimensions 0, 1, and 2. Since we can already express the relative dimension of the intersection of two entities in contact—using *PO*, *Inc*, and *SC* from Section 6.3 together with Int-T7 – Int-T9 from Section 7.1—we can define the even finer "dimension-refined" distinctions studied by McKenney et al. [McK+05] without reference to absolute dimensions or codimensions as well. This makes our logical theory applicable to any finite-dimensional setting, the three-dimensional setting being probably most relevant. The different spatial configurations in Figures 9.16 and 9.17, which have been used as examples for the 9-intersection relations, are also distinguishable in $CODIB_{\downarrow}$. A three-dimensional example that cannot be modelled in any of the previously studied theories is presented in Figure 9.18. In this example, we demonstrate some spatial aspects of a three-dimensional building may

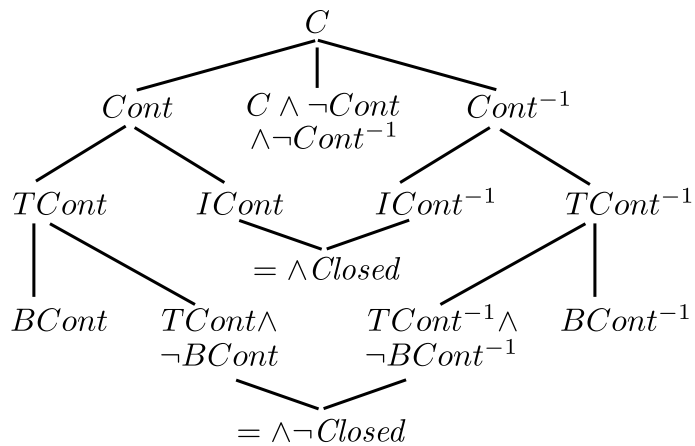


Figure 9.14: The dimension-independent lattice of jointly exhaustive, pairwise disjoint binary relations that refine contact in terms of different kinds of containment and its inverse in *CODIB*. It generalizes the five refinements of overlap in the spatial calculus RCC-8, namely PO, TPP, $NTPP$, TPP^{-1}, $NTPP^{-}1$, to the multidimensional case, where they correspond (in the same order) to $C \wedge \neg Cont \wedge \neg Cont^{-1}$, and to $TCont \wedge \neg BCont$, $ICont$, $TCont^{-1} \wedge \neg BCont^{-1}$, and $ICont^{-1}$ in their proper versions (that is, in conjunction with $\neq$). New in the multidimensional theory are $BCont$ and $BCont^{-1}$, which are not realizable in the equidimensional RCC. This refinement of the containment relations can be used in conjunction with distinctions based on the relative dimension as demonstrated in Figure 9.15.

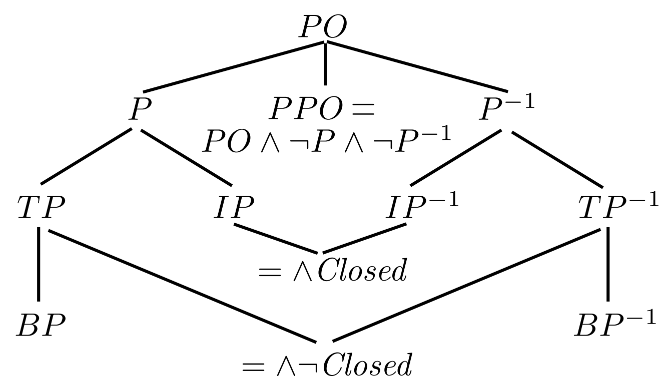


Figure 9.15: A refinement of the partial overlap relation PO from the lattice of basic relations in Figure 6.7 by the parthood equivalents of the relations in Figure 9.14.

be captured in $CODIB_{\downarrow}$.

In terms of theory relationships, the various implicitly or explicitly defined theories that build on the 9-intersection relations are (definably) interpreted in extensions of the more expressive theory $CODIB_{\downarrow}$. More precisely, for each theory T defined in [CDF98; CDFO93; Ege91; EH91; EM95; ME94; McK+05], there exists a theory T' in *CODIB* that is an extension of $CODIB_{\downarrow}$ such that the translation of every sentence provable from T to the language of *CODIB* is also provable from T'. Which concrete extension of $CODIB_{\downarrow}$ interprets a specific 9-intersection theory depends on the additional ontological assumptions of the specific 9-intersection theory. The 9-intersection theories vary in their ontological assumptions; for example, some posit that lines are simple, that regions have no holes, or that the codimension of any spatial entity is not greater than one. Because the theories are not explicitly defined as logical theories, those assumptions must be extracted from the context of each work. Therefore, a full integration of those external theories with the *CODIB* hierarchy that identifies the external theories' ontological assumptions

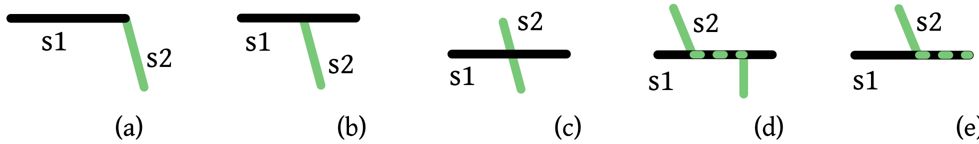


Figure 9.16: Five different ways how two named roads or highways may intersect: (a) one road ends where another starts (continuation as renamed road), (b) T-intersection, (c) X-intersection (crossing), (d) two road cross but share a section, and (e) two road start at the same point but split later. These five configurations are distinguishable from another in $\mathit{CODIB}_{\downarrow}$.
In (a) we have $\langle s1, s2\rangle \notin \mathbf{IO}_{\mathcal{M}}$, $\langle s1, s2\rangle \in \mathbf{BO}_{\mathcal{M}}$, and $\langle s1, s2\rangle, \langle s2, s1\rangle \notin \mathbf{IBC}_{\mathcal{M}}$.
In (b) we have $\langle s1, s2\rangle \notin \mathbf{IO}_{\mathcal{M}}$, $\langle s1, s2\rangle \notin \mathbf{BO}_{\mathcal{M}}$, and $\langle s1, s2\rangle \in \mathbf{IBC}_{\mathcal{M}}$ but $\langle s2, s1\rangle \notin \mathbf{IBC}_{\mathcal{M}}$.
In (c) we have $\langle s1, s2\rangle \in \mathbf{IO}_{\mathcal{M}}$, $\langle s1, s2\rangle \notin \mathbf{BO}_{\mathcal{M}}$, and $\langle s1, s2\rangle, \langle s2, s1\rangle \notin \mathbf{IBC}_{\mathcal{M}}$.
In all of (a)–(c) we also have $\langle s1, s2\rangle \in \mathbf{SC}_{\mathcal{M}}$, while in (d) and (e) we have $\langle s1, s2\rangle \in \mathbf{PO}_{\mathcal{M}}$. The extensions of *IO*, *BO*, *IBC* in (d) are the same as those in (c), and those in (e) are the same as those in (b).
This is not an exhaustive classification of how two roads (or two linear features in general) may intersect in a two-dimensional space. For an exhaustive classification, see [CDFO93; EM95; ME94].

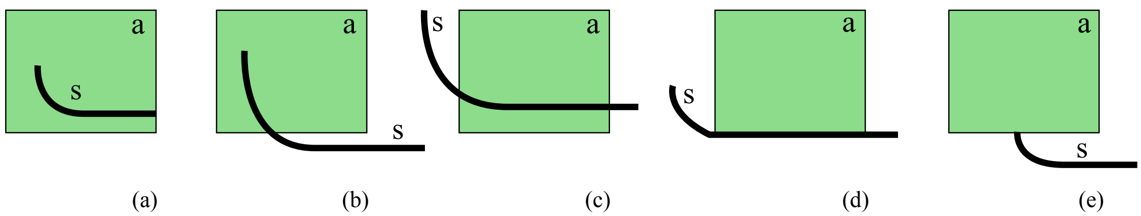


Figure 9.17: Five different ways how a road and a region (e.g., a park) may spatially relate to another.
(a) The road is in the park.
(b) The road enters the park.
(c) The road crosses the park.
(d) The road leads to the park without entering it.
(e) The road passes directly by the park.
These five configurations are distinguishable from another in $\mathit{CODIB}_{\downarrow}$. In (a)–(d) we have $\langle s, a\rangle \in \mathbf{Inc}_{\mathcal{M}}$, while in (e) $\langle s, a\rangle \in \mathbf{SC}_{\mathcal{M}}$. Moreover, in (a)–(c) $\langle s, a\rangle \in \mathbf{IO}_{\mathcal{M}}$, which is false in (d) and (e).
In both (a) and (e), the 'road ends (or begins) at the park boundary', that is $\langle s, a\rangle \in \mathbf{BO}_{\mathcal{M}}$, which is false in (b)–(d). In (a) and (b), the 'road ends in the park interior', that is $\langle a, s\rangle \in \mathbf{IBC}_{\mathcal{M}}$. The 'road crosses or straddles the park boundary' in (b)–(d), therefore we have $\langle s, a\rangle \in \mathbf{IBC}_{\mathcal{M}}$.
In (a), we additionally have $\langle s, a\rangle \in \mathbf{TCont}_{\mathcal{M}}$ but $\langle s, a\rangle \notin \mathbf{BCont}_{\mathcal{M}}$, which also implies $\langle s, a\rangle \notin \mathbf{IEC}_{\mathcal{M}}$ and $\langle s, a\rangle \notin \mathbf{BEC}_{\mathcal{M}}$. In all other configurations we have $\langle s, a\rangle \in \mathbf{IEC}_{\mathcal{M}}$ and $\langle a, s\rangle \in \mathbf{BEC}_{\mathcal{M}}$. Moreover, in all configurations $\langle a, s\rangle \in \mathbf{IEC}_{\mathcal{M}}$ and $\langle a, s\rangle \in \mathbf{BEC}_{\mathcal{M}}$.
This is not an exhaustive classification of how a road may spatially relate to a park. For an exhaustive classification, see [CDF98].

is still outstanding.

Those distinctions based on the interior, boundary, and exterior or an entity may still be insufficiently fine-grained for some desired mereotopological distinctions. Another criteria that leads to an even finer classification distinguishes whether the interior or boundary of an entity is in *full* or only in *partial* contact to a second entity. This effectively amends the binary classification of whether the interior or boundary of one entity is in contact to the interior or boundary of another entity by a three-valued measure: no contact, partial contact, and full contact. However, it is not simply a refinement of the relations *IO*, *IBC*, *BO*, *IEC*, *BEC*, and *EO* since the new relations are not specific to what something is connected

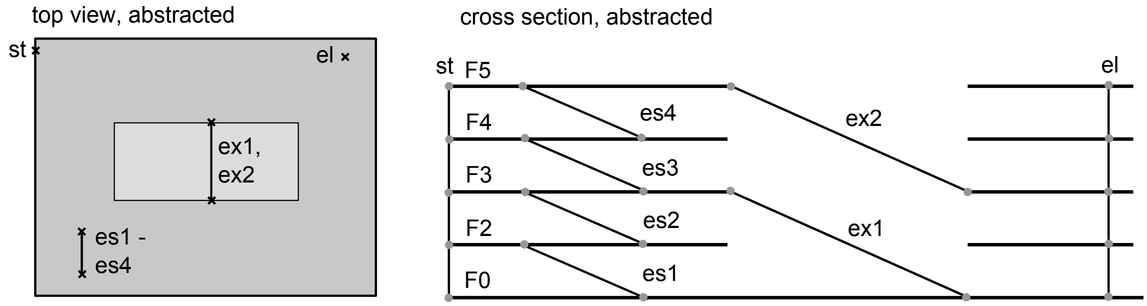


Figure 9.18: The abstract representation of the building from Figure 5.8 that can be captured as a model of $CODIB_{\updownarrow}$. We treat the building as three-dimensional, floors as two-dimensional, walls, staircases, escalators, and elevators as one-dimensional, and doors as zero-dimensional. We can then express that, for example, the staircase *st* is connected to the boundary of the floors (and the building), whereas the elevator (*el*) and the escalators ($es1-es4$, $ex1$, and $ex2$) meet each floor in the interior (and thereby in the interior of the building). Moreover, we can distinguish the lowest and the top floors as being contained in the boundary of the building, whereas the other floors are not contained in the boundary. Both distinctions are relevant when, e.g., evacuating the building: the top and bottom floor may provide easier access in an emergency (as main entrance or rooftop). Notice that the escalators or elevators may intersect floors in the floors' interiors because the floors are treated as two-dimensional entities being located in the three-dimensional building (and thereby having a codimension of one). Such cases were not treated in earlier classifications of topological relations.

to. Nevertheless these additional distinctions should be definable in $CODIB_{\downarrow}$ in a straightforward way using the primitive relations of $<_{\text{dim}}$, *Cont*, and *BCont*. A full investigation of this classification would be a worthwhile topic for future work (**Challenge 3**).

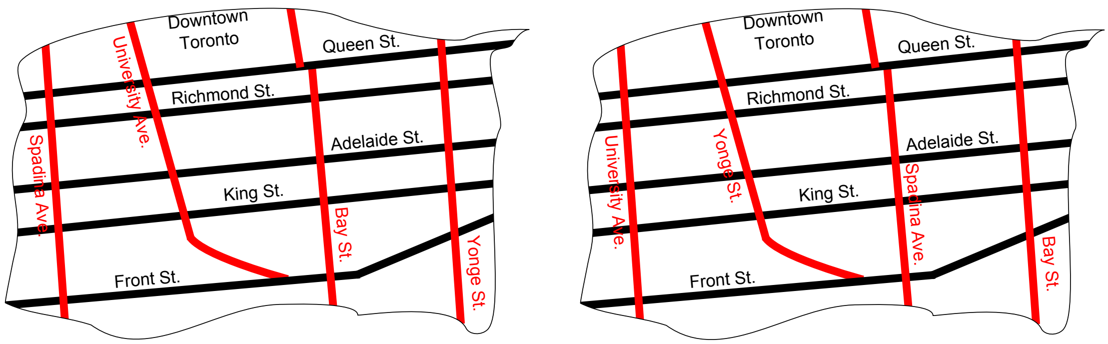
Chapter 10

Extension with betweenness: geometries[1]

The aim of this chapter is to partially bridge the gap between our multidimensional mereotopologies and classical geometries using the relationships we discussed in Chapter 2 between classes of structures and between theories. To start, we will explain in Section 10.1 what we mean by *classical geometries*, which include both Euclidean and non-Euclidean geometries, and give a short overview of theories of classical geometry and their relationships. In the remainder of the chapter we investigate how to extend theories from the *CODI* hierarchy to classical geometries. In Section 10.2 we definably extend *CODI* by concepts of 'points' and 'lines' and show that (1) all models of *CODI* have a substructure that is an incidence structure and is closed under incidence for points and lines and that (2) all incidence structures define models of *CODI* in a straightforward way. In a subsequent step, we introduce axioms that allow us to restrict models of *CODI* in a way that its incidence substructures correspond to incidence geometries such as linear or affine incidence geometries. We show this first for two-dimensional incidence geometries and then for three-dimensional incidence geometries with points incident with lines and/or planes.

To reconstruct more expressive kinds of classical geometries, we need a notion of "order", which is definable in models of neither *CODI* nor *CODIB* as Figure 10.1 demonstrates. The order is equally not definable in any structure in the class $\mathbb{M}_{\text{dense}}$, thus the class $\mathbb{M}_{\text{dense}}$ will be of no help any longer. But instead of extending the class of intended models, we simply describe the intended order relation and capture it by introducing a new primitive relation of order in Section 10.3. While classical geometries use a ternary relation of 'betweenness' to capture the order of points on straight lines, we relativize this notion of order in Section 10.3.1 to quaternary betweenness within an embedding spatial entity. This allows, in principle, to apply betweenness not only to points, but also to higher-dimensional spatial entities, such as curves, lines, areas, planes, or bodies. Such a general notion of 'betweenness' works independently of the dimension of space. The betweenness axioms closely resemble axiomatizations of ternary relations of betweenness known from, e.g., Hilbert's geometry [Hil71]. We will show that the traditional ternary betweenness relation in an incidence geometry is definable using our more general relation. To that extent, we combine in Section 10.3.2 our theory of quaternary betweenness with theories from the *CODI* hierarchy to define a theory of *ordered multidimensional mereotopology*, $OMT_{\downarrow}$. We will then show that all ordered incidence geometries that have an incidence structure consisting of points

[1]Parts of this chapter, in particular in the Sections 10.2.2 and 10.3, have been previously published as [HG11b].



(a) (b)

Figure 10.1: Two maps that have equivalent models of $CODI_{\downarrow}$ or $CODIB_{\downarrow}$ but distinct models of $OMT_{\downarrow}$. (a) shows an excerpt of a map of the city of Toronto, while (b) is a fake map in which the order of the north-side streets Spadina Ave., University Ave., Bay St., and Yonge St. has been changed. If we capture either map as a model of $CODI_{\downarrow}$ or $CODIB_{\downarrow}$ with only linear and point features (each street and each intersection being a domain entity), then the models are logically equivalent. This is because we have no relation that captures the ordering of points on a linear feature, or the ordering if linear features in an area.

being incident with lines and/or planes can be interpreted in an extension of $OMT_{\downarrow}$in the OMT hierarchy. While the language of OMT is thereby capable of reconstructing ordered incidence geometries, $OMT_{\downarrow}$ is axiomatically less restricted than even the weakest three-dimensional ordered incidence geometry used traditionally in geometry. Despite our focus on the three-dimensional case, nothing prevents the results from transferring to higher-dimensional geometries as well.

As part of our work in this chapter, we will identify a specific three-dimensional theory, namely OMT_{3d-g}, that is the natural qualitative equivalent of three-dimensional classical incidence geometries, it leaves out two key assumptions of classical theories (PL-A2, PLP-A2):

(a) two distinct points are incident with *at most* one line (PL-A2, part of the *line axiom*), and

(b) three noncollinear points are incident with *at most* one plane (PLP-A2, part of the *plane axiom*).

Leaving those two assumptions out essentially allows curved entities: two curved lines, line segments, planes, or plane segments (areas) may meet in multiple points without being identical. In the definition of an *ordered multidimensional mereotopology* we also weaken the classical notion of linear order, in which three collinear points are always orderable, to a setting where three distinct points (or three lines or line segments) may not be totally orderable, but only orderable with respect to a specific local context, the 'embedding' entity. Obviously, this lack of total orderability is a direct consequence of admitting that multiple distinct lines or planes are incident with a given set of two or three distinct points. Nevertheless, this weak theory of ordered multidimensional mereotopology is still expressive enough in its language to differentiate between the two maps in Figure 10.1.

Finally, before we go into the technical details, note that all theories in this chapter admit both discrete and continuous interpretations. Unlike much work in classical geometry, we do not assume that lines and higher-dimensional entities are dense sets of points, where lines, curves, planes, etc. have no endpoints. In other words, our theories admit models that are called *finite geometries*, see e.g., [Bat97].

10.1 Classical ordered incidence geometries

By classical geometries we mean Euclidean and non-Euclidean geometries, as generalized in what is called *neutral geometry* [Gre94; PJ65] or *absolute geometry* [BS60]. Each of those geometries relies on a fairly standard set of axioms of incidence and order, which we will review in this section. Two notable extensions of neutral geometry are Euclidean geometry and Lobačevskijan geometry. Note that we are not concerned with projective geometries at all.

Best-known among the earliest modern treatises of classical geometry is Hilbert's work [Hil71] that consists of three sets of axioms: set (I) for incidence, set (II) for order, and set (III) for congruence. We will focus on generalizing the sets (I) and (II). Once we have an axiomatic theory that generalizes those two sets in that it has an extension which is definably equivalent to the theory defined by Hilbert's two sets of axioms, we can readily reuse Hilbert's axiomatization of congruence—or, for that matter, any other equivalent axiomatization of congruence—to define full Euclidean geometry. Tarski equally axiomatized Euclidean geometry, but in a very different fashion. While Hilbert uses points, straight lines, and planes as primitive objects, Tarski's axiomatization [compare Tar59; TG99] is solely based upon points and sets of points. For our work here, we will stick with Hilbert's approach—despite it reliance on a larger set of primitive notions. Hilbert's first four groups of axioms, which concern incidence, order, parallelism, and congruence, can be axiomatized without the use of set theory. Moreover, Hilbert's axioms are easier expressed in extensions of our theories, which explicitly allow entities of different dimensions as first-class objects.

10.1.1 Incidence structures

First, we define a general incidence structure, generally following [Bue95, Chapter 3], though we deviate by defining incidence as an irreflexive and asymmetric relation. The irreflexive and asymmetric incidence relation naturally defines a reflexive and symmetric one, but simplifies the mapping to theories in the language of *CODI*.

Definition 10.1. *An incidence structure* $\mathfrak{I} = \langle \mathbf{X}, \mathbf{I}, *, \mathrm{t} \rangle$ *is a set* $\mathbf{X}$ *equipped with a surjective function* $\mathrm{t} : \mathbf{X} \to \mathbf{I}$ *into a set of types* $\mathbf{I}$ *and a binary, asymmetric, irreflexive relation* $*$*.*

We call $*$ the incidence relation. Unlike other definitions, the incidence relations considered here are not necessarily transitive. For our work here, we are mainly interested incidence structures that partition the domain into sets of equal type so that domain entities of equal type are never incident.

Definition 10.2. *A k-partite incidence structure is an incidence structure* $\mathfrak{I} = \langle \mathbf{X}, \mathbf{I}, *, \mathrm{t} \rangle$ *with* $k = |\mathbf{I}|$ *such that for all* $x, y \in \mathbf{X}$

$$\text{if } \mathfrak{I} \vDash x * y \text{ then } \mathfrak{I} \vDash \mathrm{t}(x) \neq \mathrm{t}(y).$$

$k = |\mathbf{I}|$ denotes the number of distinct types in the incidence structure; in geometric incidence structures $k - 1$ denotes the dimension of the space. Another way to formalize k-partite incidence structures is as structures $\langle \mathbf{P}, \mathbf{B}, * \rangle$ where the elements of $\mathbf{B}$, called blocks, are subsets of $\mathbf{P}$. In such a formalization, $\in$ is often used to denote the asymmetric incidence relation between points and lines so that $p \in B$ makes clear that p is a point and B is a block.

In the sequel, we are only interested in incidence structures that contain a distinguished type—the elements of which we call points—and in which only points are incident with non-points, a property formalized as axiom I.0a.

(I.0a) For any pair of entities x, y with $x * y$, x is a point and y is not a point.

We call such a structure a *point incidence structure*.

Definition 10.3. *A k-partite point incidence structure is a k-partite incidence structure* $\mathfrak{I} = \langle \mathbf{X}, \mathbf{I}, *, \mathrm{t} \rangle$ *with a distinguished nonempty set* $\mathbf{Pt} \subseteq \mathbf{X}$ *such that for all* $x, y \in \mathbf{X}$

$$\textit{if } \mathfrak{I} \models x * y \textit{ then } x \in \mathbf{Pt} \textit{ and } y \notin \mathbf{Pt}.$$

Bipartite and tripartite point incidence structure are the most commonly used incidence structures in geometry. Bipartite point incidence structures are often identified as structures $\langle \mathbf{Pt}, \mathbf{L}, * \rangle$ with two disjoint sets $\mathbf{Pt}$, called *points*, and $\mathbf{L}$, called *lines*. This is equivalent to a point incidence structure $\langle \mathbf{Pt} \cup \mathbf{L}, \{Pt, L\}, *, \mathrm{t} \rangle$ as defined earlier, where $\mathrm{t}(x) = \begin{cases} Pt \text{ if } x \in \mathbf{Pt} \\ L \text{ if } x \in \mathbf{L} \end{cases}$.

Analogously, we can define a tripartite point incidence structure as $\langle \mathbf{Pt}, \mathbf{L}, \mathbf{Pl}, * \rangle$. Notice that every k-partite point incidence structure with $k \geq 2$ has a bipartite or tripartite incidence substructure if $k \geq 2$ or $k \geq 3$, respectively.

10.1.2 Incidence geometries

We start by defining two-dimensional incidence geometries and afterwards generalize those to n-dimensional incidence geometries.

Two-dimensional incidence geometries

Any two-dimensional incidence geometry has an underlying bipartite point incidence structure. The basic incidence geometry with a bipartite point incidence structure is a *line space* [compare Bue95].

Definition 10.4. *A line space is a structure* $\mathfrak{I} = \langle \mathbf{Pt}, \mathbf{L}, * \rangle$ *comprised of a nonempty set of points* $\mathbf{Pt}$*, a set* $\mathbf{L}$ *of lines disjoint from* $\mathbf{Pt}$*, and a asymmetric, irreflexive incidence relation* $*$ *between points and lines that satisfies I.1.*

(I.1) For every line $l \in \mathbf{L}$ there exist two distinct points $p, q \in \mathbf{Pt}$ such that $p * l$ and $q * l$.

In other words, for every line $l \in \mathbf{L}$ there exist points $p, q \in \mathbf{Pt}$ with $p \neq q$ such that $p * l$ and $q * l$. In a k-dimensional incidence geometry with $k \geq 2$ we call points p, q, r collinear if and only if a line $l \in \mathbf{L}$ exists such that $p * l$, $b * l$, and $c * l$. Notice that a line space has an implicitly defined domain in $\mathbf{X} \supseteq \mathbf{pt} \cup \mathbf{L}$. The line spaces are axiomatized by the theory

$$IG_{\mathrm{2D}} = \{\text{I.0a, I.1}\}.$$

If a line space satisfies certain additional conditions, we call it a *semi-linear*, *linear*, or *affine space*. The subsequent definitions are again not our own, they can be found in various forms in standard references; our definitions are based on those by Batten [Bat97]. We introduce our own naming scheme for the axioms to group them not only by the primitive notions they axiomatize (those starting with 'I' are incidence axioms), but also to separate purely existential axioms (containing only existential quantifiers), which start with 'I.E', from the more substantial axioms.

Definition 10.5. *A semi-linear space is a line space that satisfies I.2a.*

(I.2a) For distinct $p, q \in \mathbf{Pt}$ there exists at most one line $l \in \mathbf{L}$ such that $p * l$ and $q * l$.

What we call a semi-linear space is often also referred to as a partial plane. Other names include near-linear [Bat97] or partial linear space [Bue95].

In a linear space, two distinct points uniquely define a line, which is postulated by the combination of I.2a and I.2b. Often, this combination is referred to as the *line axiom*.

Definition 10.6. *A linear space is a line space that satisfies I.2a and I.2b.*

(I.2b) For distinct $p, q \in \mathbf{Pt}$ there exists a line $l \in \mathbf{L}$ such that $p * l$ and $q * l$.

Clearly, every linear space is a semi-linear space, but not the converse. An affine space is a linear space that satisfies Euclid's parallel postulate (also know as the Playfair form): every point p not on a line l is incident with exactly one line m parallel to l, i.e., no point is incident with both l and m I.P. I.E1 is only postulated to exclude some trivial spaces with no or only one line.

Definition 10.7. *An affine space is a line space that satisfies I.2a, I.2b, I.P, I.E1.*

(I.P) **(Parallel Postulate)** A point $p \in \mathbf{Pt}$ not incident with a line $l \in \mathbf{L}$ is incident with exactly one line $m \in \mathbf{L}$ so that l and m are not incident with a common point.

(I.E1) There exist three noncollinear points $p, q, r \in \mathbf{Pt}$.

Affine spaces are also called affine planes [Bat97; Ewa71; Gre94]. The Euclidean plane $\mathbb{R}^2$ is probably the most well-known affine space.

The bipartite semi-linear, linear, and affine spaces we just introduced are often also called line geometries, point-line geometries, planar geometries, or two-dimensional geometries to distinguish them from their more prevalent three-dimensional versions.

Analogously to how semi-linear spaces generalize linear spaces, affine spaces can be generalized to semi-affine linear spaces [Dem62; VM09]. To do so, we weaken I.P so that *at most* one line through p exists that is parallel to l (I.Pa). The second part of I.P is then I.Pb. Observe that in the definition of an affine space, we could replace I.P by I.Pb: by I.1 and I.2a the parallel line m must be unique.

(I.Pa) A point $p \in \mathbf{Pt}$ not incident with a line $l \in \mathbf{L}$ is incident with at most one line $m \in \mathbf{L}$ so that l and m are not incident with a common point.

(I.Pb) A point $p \in \mathbf{Pt}$ not incident with a line $l \in \mathbf{L}$ is incident with a line $m \in \mathbf{L}$ so that l and m are not incident with a common point.

Definition 10.8. *A semi-affine space is a line space that satisfies I.2a, I.2b, I.Pa, I.E1.*

For the sake of completeness in generalizing affine spaces, we can also define a semi-affine semi-linear space by further omitting I.2b.

Definition 10.9. *A semi-affine semi-linear space is a line space that satisfies I.2a, I.Pa, I.E1.*

Semi-affine semi-linear spaces are interesting insofar as they do not force additional lines to exist, while still guaranteeing that no more than a single line exists through two distinct points and no more than one parallel line through a point not incident with a given line may exist.

Three-dimensional incidence geometries

The axioms I.1, I.2a, I.2b, I.Pa, I.Pb, and I.E1 are usually called *plane axioms* [BS60; Hil71] because they deal with points and lines on a single plane. If more than a single plane exist, the underlying k-partite point incidence structure must have $k \geq 3$. We can define when tripartite point incidence structures are incidence geometries analogue to bipartite point incidence structures being line spaces. The additional axioms I.3, I.4a, I.4b, I.5, I.6, and I.E2 are called the *space axioms*.

Definition 10.10. *A three-dimensional incidence geometry is a tripartite point incidence structure* $\langle \mathbf{Pt}, \mathbf{L}, \mathbf{Pl}, * \rangle$ *that satisfies I.0 – I.6 and I.E2.*

(I.3) A plane is incident with three noncollinear points.

(I.4a) Three noncollinear points are incident with a common plane.

(I.4b) Three noncollinear points are incident with at most one common plane.

(I.5) If two distinct points of a line are incident with a plane, then all points incident with the line are incident with the plane.

(I.6) If two planes are incident with a common point, they are incident with a second distinct common point.

(I.E2) There exist four non-coplanar points.

In the definition of a three-dimensional incidence geometry we intentionally left out the parallel axiom I.P; a three-dimensional incidence geometry that satisfies I.P is called an affine three-dimensional incidence geometry. In a k-dimensional incidence geometry with $k \geq 3$, we call points $p, q, r, s \in \mathbf{Pt}$ coplanar if and only if there exists a $x \in \mathbf{Pl}$ such that $p * x$, $q * x$, $r * x$, and $s * x$. I.E1 is provable for any three-dimensional incidence geometry, compare [BS60], and every three-dimensional incidence geometry has a linear line geometry. Our notion of a three-dimensional incidence geometry matches what Prenowitz and Jordan [PJ65, p. 138f.] call an *incidence geometry*.

We have the following correspondences to Hilbert's axiomatization [Hil71] of incidence that comprises the axioms (I_H.1) – (I_H.8). The parallel postulate I.P is not among Hilbert's axioms of incidence. Each of the axioms I.1 – I.6 and I.E2 is also equivalent to one of the nine incidence axioms used by Borsuk and Szmielew [BS60].

(I_H.1) corresponds to I.2b,

(I_H.2) corresponds to I.2a,

(I_H.3) corresponds to I.1 and I.E1,

(I_H.4) corresponds to I.3 and I.4a,

(I_H.5) corresponds to I.4b,

(I_H.6) corresponds to I.5,

(I_H.7) corresponds to I.6,

(I_H.8) corresponds to I.E2.

General incidence geometries

We can generalize this notion of a three-dimensional incidence geometry to incidence geometries that allow higher-dimensional geometries as well as lower-dimensional, in particular planar, geometries. More precisely, we can leave out I.6 which restricts the incidence geometry to a maximum of three dimensions. Equally, we can leave out I.E1 and I.E2, which require a minimum of two or three dimensions, respectively. This generalizes the definition of a two- or three-dimensional incidence geometry from Prenowitz and Jordan [PJ65, p. 138f.]. We obtain the following definition.

Definition 10.11. *An* <u>*incidence geometry*</u> *is a k-partite point incidence structure* $\mathfrak{I} = \langle \mathbf{X}, \mathbf{I}, *, \mathrm{t} \rangle$ *with three distinguished disjoint sets* $\mathbf{Pt}, \mathbf{L}, \mathbf{Pl} \subseteq \mathbf{X}$*, whose elements are called points, lines, and planes, respectively, such that for all* $x, y \in \mathbf{X}$*,*

- $x \in \mathbf{Pt}$ *and* $\mathfrak{I} \models \mathrm{t}(x) = \mathrm{t}(y)$*, then* $y \in \mathbf{Pt}$*,*
- $x \in \mathbf{L}$ *and* $\mathfrak{I} \models \mathrm{t}(x) = \mathrm{t}(y)$*, then* $y \in \mathbf{L}$*,*
- $x \in \mathbf{Pl}$ *and* $\mathfrak{I} \models \mathrm{t}(x) = \mathrm{t}(y)$*, then* $y \in \mathbf{Pl}$*;*

and all elements in $\mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl}$ *satisfy I.0 – I.5.*

Incidence geometries are axiomatized by the theory

$$IG = \{\mathrm{I.0} - \mathrm{I.5}\},$$

which includes I.0b to ensure that the sets $\mathbf{Pt}, \mathbf{L}, \mathbf{Pl}$ are indeed disjoint.

(I.0a) $x * y \to Pt(x) \wedge \neg Pt(x)$ (only points are incident with non-points)

(I.0b) $[\neg Pt(x) \vee \neg L(x)] \wedge [\neg Pt(x) \vee \neg Pl(x)] \wedge [\neg L(x) \vee \neg Pl(x)]$ (Pt, L, and Pl are disjoint)

(I.1) $L(l) \to \exists p, q[p \neq q \wedge p * l \wedge q * l]$ (for every line two distinct points incident with the line exist)

(I.2a) $Pt(p) \wedge Pt(q) \wedge p \neq q \wedge L(l) \wedge L(m) \wedge p * l \wedge q * l \wedge p * m \wedge q * m \to l = m$
(two points are incident with at most one line)

(I.2b) $Pt(p) \wedge Pt(q) \wedge p \neq q \to \exists l[L(l) \wedge p * l \wedge q * l]$
(two distinct points are incident with some common line)

(I.3) $Pl(x) \to \exists p, q, r\big[p \neq q \neq r \neq p \wedge p * x \wedge q * x \wedge r * x \wedge \forall l[L(l) \to \neg[p * l \wedge q * l \wedge r * l]]\big]$
(a plane is incident with three distinct, noncollinear points)

(I.4a) $Pt(p) \wedge Pt(q) \wedge Pt(r) \wedge \forall l\big[L(l) \to \neg[p * l \wedge q * l \wedge r * l]\big] \to \exists x[Pl(x) \wedge p * x \wedge q * x \wedge r * x]$
(three noncollinear points are incident with a common plane)

(I.4b) $Pt(p) \wedge Pt(q) \wedge Pt(r) \wedge \forall l\big[L(l) \to \neg[p*l \wedge q*l \wedge r*l]\big] \wedge Pl(x) \wedge Pl(y) \wedge p*x \wedge q*x \wedge r*x \wedge p*y \wedge q*y \wedge r*y \to x = y$ (three noncollinear points are incident with at most one common plane)

(I.5) $L(l) \wedge Pl(x) \wedge p * l \wedge q * l \wedge p \neq q \wedge p * x \wedge q * x \to \forall r[r * l \to r * x]$ (if two distinct points of a line are incident with a plane, then all points incident with the line are incident with the plane)

Axiom Set 10.1: Axioms I.0 – I.5 of *IG*.

10.1.3 Ordered incidence geometries

Ordered incidence geometry first emerged in Pasch's *Vorlesungen über Neuere Geometrie* [Pas88] as a rigorous, i.e., axiomatic generalization of Euclidean geometry that leaves out congruence as a primitive notion. For that reason, Coxeter called ordered incidence geometry quite fittingly "geometry without measurements" [Pam11]. The idea of using betweenness as an additional primitive relation for geometry was quickly taken up by Peano and Hilbert [Hil71], so that *betweenness* became a generally accepted primitive concept besides those of points, lines, incidence, and congruence. Since then, various kinds of *ordered incidence geometries* (sometimes also called *ordered incidence structures*) have been proposed and studied. Pambuccian [Pam11] offers a very comprehensive account of the various axiomatizations of ordered geometry, translating them into a unified first-order logical framework.

We choose the notion of an ordered incidence geometry as defined by Prenowitz and Jordan [PJ65, p. 217], which builds upon our earlier definition of a three-dimensional incidence geometry (Definition 10.10), which we also adapted from [PJ65]. The original definition is restricted to two- and three-dimensional ordered incidence geometry, but easily extends to higher-dimensional incidence geometries. The resulting theory allows entities of higher dimensions to exist, but does not axiomatize order between higher-dimensional entities; while the Pasch axiom establishes an order between line segment, no equivalent axioms for, e.g., two-dimensional areas in a three-dimensional space are included. This is a general limitation of classical ordered incidence geometries: they are usually restricted to maximal three dimensions. Other approaches to geometry that fix this problem have been proposed by Nuut and Hashimoto [Has58]; they define what is called an *n-dimensional betweenness geometry*. A recent overview including notes on the historical development of n-dimensional betweenness geometry is given by Lumiste [Lum07]. Though we do not study the relationship to betweenness geometry in detail, it appears to be an extension of weak ordered incidence geometry as defined in a moment.

Definition 10.12. *An <u>ordered incidence geometry</u> $\langle \mathbf{X}, \mathbf{I}, *, \mathrm{t}, \mathbf{B}\rangle$ is an incidence geometry $\langle \mathbf{X}, \mathbf{I}, *, \mathrm{t}\rangle$ with distinguished disjoint sets $\mathbf{Pt}, \mathbf{L}, \mathbf{Pl} \subseteq \mathbf{X}$ equipped with a ternary relation $\mathbf{B} : \mathbf{Pt} \times \mathbf{Pt} \times \mathbf{Pt}$, called betweenness, that satisfies O.1 – O.6.*

(O.1) (Symmetry Property) *$B(a, b, c)$ implies $B(c, b, a)$.*

(O.2) (Anticyclic Property) *$B(a, b, c)$ implies the falsity of $B(b, c, a)$.*

(O.3) (Linear Coherence) *a, b, and c are distinct and collinear points iff $B(a, b, c)$, $B(b, c, a)$, or $B(c, a, b)$.*

(O.4) (Separation Property) *Let p colline with, and be distinct from a, b, c. Then $B(a, p, b)$ implies $B(b, p, c)$ or $B(a, p, c)$ but not both.*

(O.5) (Existence) *If a and b are distinct points, then there exist x, y, z such that $B(x, a, b)$, $B(a, y, b)$, $B(a, b, z)$.*

(O.6) (Pasch Axiom) *Let L coplane with and not contain points a, b, c. Then if L intersects $\overleftrightarrow{ab}$, it intersects $\overleftrightarrow{bc}$ or $\overleftrightarrow{ac}$ but not both.*

We altered O.3 to reflect the intended interpretation of $B(a, b, c)$ as 'point b is in between points a and c', which is implied by the chosen notation in [PJ65]. Equally, we restricted the antecedent of O.5 to points a and b.

The notation $\overleftrightarrow{ab}$ in O.6 denotes bounded line segments, which we have not yet defined; they are however definable in various ways, compare any of [BS60; Gre94; Hil71; PJ65] and our discussion in Section 10.3.5. Line segments will not play a central role in our exposition, thus we completely leave out O.6 from our studies. Once we formally define line segments in *CODI*, we can express O.6 as well.

We are interested in both discrete and continuous ordered incidence geometries, hence we omit O.5, which rules out any finite or discrete geometries. This results in a weaker kind of ordered incidence geometry defined as follows, which will form the basis for reconstructing classical geometries as extensions of *CODI*.

(O.1) $B(p,q,r) \to B(r,q,p)$ (B symmetric)

(O.2) $B(p,q,r) \to \neg B(q,r,p)$ (B anticyclic)

(O.3) $p \neq q \wedge p \neq r \wedge q \neq r \wedge \exists l[L(l) \wedge p * l \wedge q * l \wedge r * l] \leftrightarrow B(p,q,r) \vee B(q,r,p) \vee B(r,p,q)$ (linear coherence)

(O.4) $L(l) \wedge p * l \wedge q * l \wedge r * l \wedge s * l \wedge s \neq p \wedge s \neq q \wedge s \neq r \wedge B(p,s,q) \to [B(q,s,r) \wedge \neg B(p,s,r)] \vee [\neg B(q,s,r) \wedge B(p,s,r)]$ (total separability)

Axiom Set 10.2: Axioms O.1 – O.4 of *WOIG*.

Definition 10.13. *A weak ordered incidence geometry* $\langle \mathbf{X}, \mathbf{I}, *, \mathrm{t}, \mathbf{B} \rangle$ *is an incidence geometry* $\langle \mathbf{X}, \mathbf{I}, *, \mathrm{t} \rangle$ *with distinguished disjoint sets* $\mathbf{Pt}, \mathbf{L}, \mathbf{Pl} \subseteq \mathbf{X}$ *equipped with a ternary relation* $\mathbf{B} : \mathbf{Pt} \times \mathbf{Pt} \times \mathbf{Pt}$*, called betweenness, that satisfies O.1 – O.4.*

Clearly, every ordered incidence geometry is also a weak ordered incidence geometry. Depending on the involved incidence geometry, we call the resulting weak ordered incidence geometry a two-, three-, or k-dimensional weak ordered incidence geometry.

We axiomatize weak ordered incidence geometries by the theory

$$WOIG = \{\text{I.0 – I.5, O.1 – O.4}\}.$$

An equivalent two-dimensional version can be axiomatized by omitting I.3 – I.5. A nontrivial finite model of *WOIG* is given in Figure 10.2.

Hilbert's four axioms of order [Hil71]—we refer to them as $\mathrm{O_H}.1$ – $\mathrm{O_H}.4$—are terser and differ significantly from O.1 – O.6. Hilbert relies heavily on the Pasch axiom to prove some of the properties axiomatized by O.1 – O.6, in particular separability O.4. Hilbert includes the Pasch axiom O.6 as $\mathrm{O_H}.4$, and O.1 and O.3 are together equivalent to his axiom $\mathrm{O_H}.1$. $\mathrm{O_H}.2$ is a weaker form of O.5, while $\mathrm{O_H}.3$ is a weaker form of O.2. However, we cannot directly generalize his axiomatization to a weak ordered incidence geometry, because once we omit the Pasch axiom, many properties we want to include are no longer provable.

Notice that our chosen axiomatization *WOIG* interlinks incidence and order notions, that is, the axioms O.1 – O.4 do not form an interesting self-contained theory of betweenness. For example, they are not sufficient to prove the two transitivity properties of betweenness (O.7, O.8).

(O.7) **(Inner Transitivity)** If $B(x,a,b)$ and $B(a,b,y)$ then $B(x,a,y)$.

(O.8) **(Outer Transitivity)** If $B(x,a,b)$ and $B(a,y,b)$ then $B(x,a,y)$.

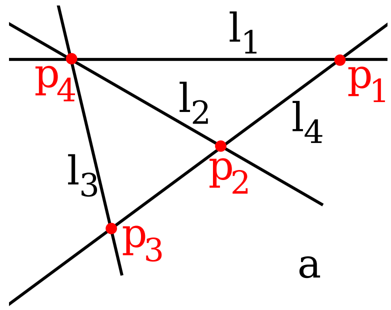


Figure 10.2: A spatial configuration that matches a finite model of *WOIG* consisting of four points p_1, p_2, p_3, p_4, four lines l_1, l_2, l_3, l_4, and a plane a. All points are incidence with the plane. In this model, p_2 is in between p_1 and p_3 i.e., $B(p_1, p_2, p_3)$. This model has been found by the model generator Paradox3 from the theory *WOIG*.

However, O.7 and O.8 are theorems of weak ordered incidence geometry.

Lemma 10.1. *WOIG* $\models$ *{O.7, O.8}*

Proof. Assume $B(x, a, b) \wedge B(a, b, y)$.
Then there exists a line l incident with x, a, and b by O.3. Equally, there exists a line m incident with a, b, and y. By I.2b, $l = m$. Thus O.7 follows from the weaker version O.7′. Analogously, to prove O.8 it suffices to prove O.8′.

(O.7′) $B(x, a, b) \wedge B(a, b, y) \wedge L(l) \wedge a * l * b * l \wedge x * l \wedge y * y \rightarrow B(x, a, y)$

(O.8′) $B(x, a, b) \wedge B(a, y, b) \wedge L(l) \wedge a * l * b * l \wedge x * l \wedge y * y \rightarrow B(x, a, y)$

O.7′ and O.8′ can be proved automatically. □

For this reason, stand-alone axiomatization of betweenness such as the one by Huntington and Kline [HK17] usually include O.7 and O.8, while axiomatizations of (weak) ordered incidence geometries do not need those properties as axioms. For example, in the set {A, B, C, D, 1, 2} of independent axioms for betweenness from [HK17] the postulates 1 and 2 correspond to O.7 and O.8. A and C correspond to O.1 and O.2, and B and D are the two directions of O.3. O.4 is provable from C and 7, where 7 is provable from A, B, C, and 2. See [HK17] for the postulates and the proofs.

10.2 *CODI*'s relationship to incidence geometries

As first step in our pursuit to show that particular extensions of *CODI* faithfully interpret classical geometries, we show in this section that *CODI* faithfully interprets two- and three-dimensional incidence geometries. This, in turn, requires us to establish that all models of CODI define incidence structures through the incidence relation.

10.2.1 *CODI* faithfully interprets point incidence structures

We now show that *CODI*, the theory of containment and linear dimension, is a direct abstraction of (geometric) incidence structures, though the theory *CODI* has much richer models, containing many relations and functions not relevant for incidence structures. We first show that any model of *CODI* contains a substructure that is a k-partite incidence structure in the following way: The incidence relation $*$ is defined by the extension of the relation *Inc* while the number of equivalence classes of entities of identical dimension determine k in the resulting incidence structure, that is, entities of identical dimension have identical type $I_i \in \mathbf{I}$. This confirms that *CODI*'s incidence relation *Inc* as defined in Chapter 6 is indeed an incidence relation in the mathematical sense, that is, *Inc* defines a relation that satisfies the properties of an incidence relation within an incidence structure as defined in Definition 10.1.

Theorem 10.1. *Any model $\mathcal{M}$ of CODI with domain $\mathbf{M}$ defines an incidence structure $\mathfrak{I} = \langle \mathbf{M}, \mathbf{I}, *, \dim \rangle$ such that for all $x, y \in \mathbf{M}$,*

$$\langle x, y \rangle \in *_{\mathfrak{I}} \iff \langle x, y \rangle \in \mathbf{Inc}_{\mathcal{M}} \textit{ and } \langle x, y \rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}.$$

Proof. Let $\mathcal{M}$ be an arbitrary model of *CODI*.
The definition of the relation $*$ forces it to be asymmetric (by Inc-T1, D-A1) and irreflexive (by D-A2).

Because $\leq_{\dim}$ is a transitive relation, we can find an order over the finite set of domain entities in $\mathbf{M}$ such that $\mathbf{M} = \{x_1, x_2, x_3, \dots\}$ and for all $x_i, x_j \in \mathbf{M}$ with $i \leq j$ we have $x_i \leq_{\dim} x_j$. We then define $\dim(x_n)$ for all $x_n \in \mathbf{M}$ recursively as a step function:

$$\dim(x_i) = \begin{cases} 0 & \text{if } i = 1 \\ \dim(x_{i-1}) & \text{if } x_i =_{\dim} x_{i-1} \\ \dim(x_{i-1}) + 1 & \text{if } x_i >_{\dim} x_{i-1} \end{cases}$$

The set $\mathbf{I} = \{0, \dots, k\}$ defines a set of types. Notice that k is usually much smaller than n because many types are reused. It is easy to see that dim is then a function from $\mathbf{M}$ into $\mathbf{I}$, which is surjective because every $i \in \mathbf{I}$ is mapped to from some x_j.

The so-defined structure $\langle \mathbf{M}, \mathbf{I}, *, \dim \rangle$ is thereby an incidence structure. □

Importantly, the function dim preserves the intended meaning of relative dimension, that is, the number of distinct types in $\mathbf{I}$ corresponds to the dimensionality of the space: $\mathbf{MinDim}_{\mathcal{M}}$ contains the entities (called flats in incidence structures) of dimension 0, $\{x \in \mathbf{M} \mid$ there exists a $y \in \mathbf{MinDim}_{\mathcal{M}}$ with $\langle y, x \rangle \in (\prec_{\mathbf{dim}})_{\mathcal{M}}\}$ the flats of dimension 1, etc. Using numbers 0 to k for the types turns out to be a convenient choice, $\dim(x)$ corresponds then to the usual numeric dimension of x as long as the intended structure that corresponds to a model $\mathcal{M}$ is in the class $\mathbb{M}_{\text{dense}}$ (compare Section 9.1). If this is not the case, Theorem 10.1 still works fine, just the actual intended dimension of entities may be larger than the type assigned to it in the proof of Theorem 10.1.

Our focus will be on point incidence structures, which can also be constructed from models of *CODI*.

Corollary 10.1. *Any model $\mathcal{M}$ of CODI with domain $\mathbf{M}$ defines a k-partite point incidence structure $\mathfrak{I} = \langle \mathbf{M}, \mathbf{I}, *, \dim \rangle$ with $\mathbf{Pt} = \mathbf{MinDim}_{\mathcal{M}}$ as distinguished set such that for all $x, y \in \mathbf{M}$,*

$$\langle x, y \rangle \in *_{\mathfrak{I}} \iff \langle x, y \rangle \in \mathbf{Inc}_{\mathcal{M}} \textit{ and } x \in \mathbf{MinDim}_{\mathcal{M}}.$$

As the converse of Corollary 10.1, any point incidence structure corresponds to some model of *CODI* in a natural way. A similar construction is very difficult to find for general incidence structures because the intended interpretation of *Cont* is far from obvious within an arbitrary incidence structure.

Theorem 10.2. *Any k-partite point incidence structure $\mathfrak{I} = \langle \mathbf{X}, \mathbf{I}, *, \dim \rangle$ with a distinguished set* $\mathbf{Pt}$ *defines a model $\mathcal{M}$ of CODI such that*

1. *$Dom(\mathcal{M}) = \mathbf{X}$,*
2. *$\mathbf{ZEX}_{\mathcal{M}} = \emptyset$,*
3. *for all $x \in \mathbf{Pt}$, $\langle x, y \rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$ iff $y \notin \mathbf{Pt}$,*
4. *$\langle x, y \rangle \in \mathbf{Cont}_{\mathcal{M}} \iff x = y$ or $\langle x, y \rangle \in *_{\mathfrak{I}}$.*

Proof. Let $\langle \mathbf{X}, \mathbf{I}, *, \dim \rangle$ be an arbitrary k-partite point incidence structure. We can represent it as $\langle \mathbf{Pt}, \mathbf{B_2}, \mathbf{B_3}, \ldots, \mathbf{B_k}, * \rangle$ such that $\mathbf{X} = \mathbf{Pt} \cup \bigcup_{2 \leq i \leq k} \mathbf{B_i}$ and for all i and all $x, y \in \mathbf{B_i}$ we have $\dim(x) = \dim(y)$.

In addition to the conditions (1)–(4) we choose for all $x, y \in \mathrm{Dom}(\mathcal{M})$,

5. for all $x \notin \mathbf{Pt}$, $\langle x, y \rangle \in (<_{\mathbf{dim}})_{\mathcal{M}}$ iff $x \in \mathbf{B_i}$ and $y \in \mathbf{B_j}$ with $i < j$,

to completely define $(<_{\mathbf{dim}})_{\mathcal{M}}$ together with condition (3).
To verify that $\mathcal{M}$ is a model of *CODI* we must verify the axioms D-A1 – D-A6, C-A1 – C-A4, and CD-A1.

D-A1 – D-A3 are satisfied by conditions (3) and (5). D-A4 and D-A5 are trivially satisfied by condition (2) and (4) because $\mathbf{Cont}(x, x)$ for all $x \in \mathbf{X}$. D-A6 is satisfied because $\mathbf{MinDim}_{\mathcal{M}} = \mathbf{Pt}_{\mathcal{M}}$ is nonempty by Definition 10.3. C-A1 and C-A4 are trivially satisfied by condition (2), C-A2 and C-A3 are satisfied by the explicit inclusion of $x = y$ in condition (4). CD-A1 is trivially satisfied by the construction in conditions (3), (4), and (5).

Thus any finite k-partite point incidence structure defines a model of *CODI*. □

It is easy to verify that *CODI* extends the theory of point incidence structures with the mapping of the incidence relation $*$ from Theorem 10.1. Note that any model $\mathcal{M}$ of *CODI* that is constructed in Theorem 10.2 is implicitly an expansion of the point incidence structure; we preserve the incidence relation as

$$\langle x, y \rangle \in \big(\mathbf{Inc}_{\mathcal{M}} \cap (<_{\mathbf{dim}})_{\mathcal{M}}\big) \iff \langle x, y \rangle \in *_{\mathfrak{I}}$$

and also the set $\mathbf{I}$ and the type function dim within the extension $(<_{\mathbf{dim}})_{\mathcal{M}}$. Thus *CODI* faithfully interprets the theory of point incidence structures by Theorem 2.6.

But not all models of *CODI* can be obtained through the expansion in Theorem 10.2. Clearly, *CODI* has a more expressive language than the theory of point incidence structures and thus *CODI* is not interpretable in the theory of point incidence structures.

10.2.2 $CODI_{\mathrm{pl}}$ faithfully interprets two-dimensional incidence geometries

Now we show how the *line axiom* together with the other *plane axioms* can be used to reconstruct the (finite) two-dimensional incidence geometries known as line spaces. For this purpose, we first define two classes of maximal entities, which we call points, *Pt*, and lines, *L* (Pt-D, L-D), lines being of one dimension greater than points (L-D). We also restrict points to indivisible entities using CD-E1; it ensures that points are actually points and not point sets. Then points are both minimal and maximal in their dimension.

Note that points and lines are only maximal in the intended geometrical interpretation. Not in every model of *CODI* we would think of points as maximal in their dimension; usually only the set of all points would be maximal in its dimension. Likewise, lines as defined in L-D are not necessarily thought of as maximal in their dimension in any model of *CODI*: the sum of two lines would contain both lines or a branching curve would be greater than any non-branching curve therein. What we really want to express is that points and lines are maximal self-connected entities in their dimension, as we will do in Section 10.3 where we require maximal entities to be self-connected (OMT-A3). But the results in this and the subsequent section can be obtained without the use of the mereological closure operations that let us define self-connectedness. In that sense, all results in the present section do not rely on points and lines (and later also planes) being maximal. We only require the following consequence from PL-E1: any point incident with a line is contained in the line (follows already from CD-E1) and any point or line incident with a plane is contained in the plane. This consequence of PL-E1 allows us to use $Cont(x, y) \land x <_{\text{dim}} y$ as the asymmetric incidence relation instead of having to extract the incidence relation from *Inc*, which would also be possible, though more cumbersome.

To avoid nontrivial incidence structures, it is commonly assumed that every line contains at least two distinct points (PL-A1) as postulated by I.1. The two definitions together with PL-A1, PL-E1, and CD-E1 define an extension of *CODI*, namely

$$CODI_{\text{pl}} = CODI \cup \{\text{PL-A1, Pt-D, L-D, PL-E1, CD-E1}\}.$$

This is the basic theory we will work with throughout this chapter.

(Pt-D) $Pt(x) \leftrightarrow Max(x) \land MinDim(x)$
(points are maximal in their dimension and of lowest nonzero dimension)

(L-D) $L(x) \leftrightarrow Max(x) \land \forall y[Pt(y) \rightarrow y \prec_{\text{dim}} x]$
(lines are maximal in their dimension and of one dimension higher than points)

(PL-A1) $L(x) \rightarrow \exists y, z[Pt(y) \land Pt(z) \land Cont(y, x) \land Cont(z, x) \land y \neq z]$
(**Line Existence Axiom**: a line contains at least two distinct points; I.1)

Axiom Set 10.3: Definitions Pt-D and L-D and axiom PL-A1 of $CODI_{\text{pl}}$.

(PL-E1) $Max(x) \land Max(y) \land Inc(x, y) \land x <_{\text{dim}} y \rightarrow Cont(x, y)$ (incidence between maximal entities requires the lower-dimensional entity to be contained in the higher-dimensional entity)

Axiom Set 10.4: Extension axiom PL-E1 of $CODI_{\text{pl}}$.

It is easy to verify that all points are of identical dimension (PL-T1), that all lines are of identical dimension (PL-T2), and that points and lines are disjoint classes (PL-T3).

(PL-T1) $Pt(x) \land Pt(y) \rightarrow x =_{\text{dim}} y$ (points are of uniform dimension)

(PL-T2) $L(x) \land L(y) \rightarrow x =_{\text{dim}} y$ (lines are of uniform dimension)

(PL-T3) $\neg Pt(x) \lor \neg L(x)$ (points and lines are disjoint classes)

Lemma 10.2. $CODI_{\text{pl}} \models$ *{PL-T1 – PL-T3}*

Points and lines can be interpreted in the usual geometric sense in a spatial configuration but other interpretations are also feasible, e.g., as two-dimensional regions and four-dimensional space-time objects—but only if the intended interpretation is taken from the class $\mathbb{M}$ and not from the more restricted class $\mathbb{M}_{\text{dense}}$ introduced in the previous chapter.

Apart from points and lines, other entities of same or differing dimension can still exist, but are irrelevant for the construction of line spaces, which we will construct from arbitrary models of *CODI* in our next theorem. Containment of points in lines can be expressed in *CODI* using the containment relation *Cont* or the incidence relation *Inc*, in particular in the presence of PL-E1.

Theorem 10.3. *Any model $\mathcal{M}$ of $CODI_{\text{pl}}$ with domain $\mathbf{M}$ defines a substructure $\mathfrak{I} = \langle \mathbf{Pt}_\mathcal{M}, \mathbf{L}_\mathcal{M}, * \rangle$ that is a line space such that for all $x, y \in \mathbf{M}$,*

$$\langle x, y \rangle \in *_\mathfrak{I} \iff \langle x, y \rangle \in \mathbf{Cont}_\mathcal{M} \text{ and } x \in \mathbf{Pt}_\mathcal{M} \text{ and } y \in \mathbf{L}_\mathcal{M}.$$

Proof. Let $\mathcal{M}$ be an arbitrary model of $CODI_{\text{pl}}$.

By the definition of points (Pt-D), the definition of *MinDim* (D-D6), and by the existence of an entity of minimal dimension (D-A6), there exists some entity $x \in \mathbf{Pt}_\mathcal{M}$, hence $\mathbf{Pt}_\mathcal{M}$ is nonempty.

Moreover, $*$ is asymmetric and irreflexive because $\mathbf{Pt}_\mathcal{M} \cap \mathbf{L}_\mathcal{M} = \emptyset$ (PL-T3).

By PL-A1 for every $l \in \mathbf{L}_\mathcal{M}$ there must exist at least two distinct points $p, q \in \mathbf{Pt}_\mathcal{M}$ such that $\langle p, l \rangle, \langle q, l \rangle \in \mathbf{Cont}_\mathcal{M}$. Hence $\langle p, l \rangle, \langle q, l \rangle \in *_\mathfrak{I}$ and every line is thus incident with at least two distinct points. Thus I.1 is satisfied.

Hence the structure $\langle \mathbf{Pt}_\mathcal{M}, \mathbf{L}_\mathcal{M}, * \rangle$ is a line space.

$\langle \mathbf{Pt}_\mathcal{M}, \mathbf{L}_\mathcal{M}, * \rangle$ with domain $\mathbf{Pt} \cup \mathbf{L}$ is a substructure of $\mathcal{M}$ if the model is represented as $\mathcal{M} = \langle \mathbf{X}, \mathbf{Pt}_\mathcal{M}, \mathbf{L}_\mathcal{M}, \mathbf{Cont}_\mathcal{M}, (<_{\mathbf{dim}})_\mathcal{M}$. Then $*_\mathfrak{I} \cup \{\langle x, x \rangle : x \in \mathbf{Pt} \cup \mathbf{L}\}$ is the restriction of $\mathbf{Cont}_\mathcal{M}$ to the domain $\mathbf{Pt} \cup \mathbf{L}$ and $\{\langle x, y \rangle : x \in \mathbf{Pt} \text{ and } y \in \mathbf{L}\}$ is the trivial restriction of $(<_{\mathbf{dim}})_\mathcal{M}$ to the domain $\mathbf{Pt} \cup \mathbf{L}$. □

Because the model $\mathcal{M}$ must satisfy PL-E1, the definition of the incidence relation $*$ is equivalent to

$$\langle x, y \rangle \in *_\mathfrak{I} \iff \langle x, y \rangle \in \mathbf{Inc}_\mathcal{M} \text{ and } x \in \mathbf{Pt}_\mathcal{M} \text{ and } y \in \mathbf{L}_\mathcal{M},$$

which more clearly reflects that we can use the symmetric incidence relation *Inc* defined in $CODI_{\text{pl}}$ to define the asymmetric incidence relation $*$ of the corresponding line space. Thereby, PL-E1 restricts incidence among maximal entities to containment.

We can also prove the converse of Theorem 10.3, namely that every line space can definably expanded to a model of $CODI_{\text{pl}}$ in a straightforward way.

Theorem 10.4. *A line space $\langle \mathbf{Pt}, \mathbf{L}, * \rangle$ can be definably expanded to a model $\mathcal{M}$ of $CODI_{\text{pl}}$ with $\mathbf{M} = \mathbf{Pt} \cup \mathbf{L}$, $\mathbf{Pt}_\mathcal{M} = \mathbf{Pt}$, and $\mathbf{L}_\mathcal{M} = \mathbf{L}$ such that for all $x, y \in \mathbf{M}$,*

$$\langle x, y \rangle \in \mathbf{Cont}_\mathcal{M} \iff x = y \text{ or } \langle x, y \rangle \in *_\mathfrak{I}.$$

Proof. Let $\langle \mathbf{Pt}, \mathbf{L}, * \rangle$ be an arbitrary line space. Notice that it implicitly has a domain $\mathbf{X} = \mathbf{Pt} \cup \mathbf{L}$. We define the structure $\mathcal{M}$ as follows:

1. $\mathbf{M} = \mathbf{X} = \mathbf{Pt} \cup \mathbf{L}$,

2. $\mathbf{ZEX}_{\mathcal{M}} = \emptyset$,
3. $\mathbf{Pt}_{\mathcal{M}} = \mathbf{Pt}$,
4. $\mathbf{L}_{\mathcal{M}} = \mathbf{L}$,
5. $\langle x, y\rangle \in (<_{\mathbf{dim}})_{\mathcal{M}} \iff x \in \mathbf{Pt} \wedge y \in \mathbf{L}$,
6. $\langle x, y\rangle \in \mathbf{Cont}_{\mathcal{M}} \iff x = y$ or $\langle x, y\rangle \in *_{\mathfrak{I}}$.

To verify that $\mathcal{M}$ is a model of $\mathit{CODI}_{\text{pl}}$ we must verify the axioms D-A1 – D-A6, C-A1 – C-A4, CD-A1, and PL-A1.

Similarly to the proof of Theorem 10.2, the axioms D-A1 – D-A6, C-A1 – C-A4, CD-A1 are satisfied. D-A1 – D-A3 are satisfied by (5). D-A4, D-A5, C-A1, and C-A4 are trivially satisfied by (2), D-A6 is satisfied because $\mathbf{MinDim}_{\mathcal{M}} = \mathbf{Pt}_{\mathcal{M}} = \mathbf{Pt} \neq \emptyset$ by Definition 10.4.

C-A2 is satisfied by (3), (4), and (6). The antecedent of C-A3 (transitivity of *Cont*) can never be satisfied and thus C-A3 is trivially satisfied. CD-A1 is satisfied by the construction in (3) – (6).

It remains to prove PL-A1. Let l be an arbitrary entity in $\mathbf{L}_{\mathcal{M}} = \mathbf{L}$. Because $\langle \mathbf{Pt}, \mathbf{L}, *\rangle$ is a line space, i.e., satisfies I.1, l must be incident with at least two distinct points $p, q \in \mathbf{Pt} = \mathbf{Pt}_{\mathcal{M}}$, i.e., $\langle p, l\rangle, \langle q, l\rangle \in *_{\mathfrak{I}}$. Hence by (6) we have $\langle p, l\rangle, \langle q, l\rangle \in \mathbf{Cont}_{\mathcal{M}}$ as well as $p \neq q$. This satisfies the consequent of PL-A1. □

It is easy to show that the theory of line spaces, IG_{2D}, is interpreted in the theory $\mathit{CODI}_{\text{pl}}$. All we need to do is to prove that I.0a and I.1 are satisfied in any model of $\mathit{CODI}_{\text{pl}}$, which is easy to see because we explicitly included PL-A1 to mirror I.1, while I.0a is implicitly satisfied for the definition of $*_{\mathfrak{I}}$. Then $\mathit{CODI}_{\text{pl}}$ faithfully interprets IG_{2D} by Theorem 2.6 because every line space is definably equivalent to a structure $\langle \mathbf{Pt}, \mathbf{L}, \mathbf{Cont}_{\mathcal{M}}\rangle$, which then is expandable to a model of $\mathit{CODI}_{\text{pl}}$—all according to Theorem 10.4. Clearly, $\mathit{CODI}_{\text{pl}}$ has a more expressive language than the theory of line spaces and thus no interpretation in the other direction is possible.

We restrict $\mathit{CODI}_{\text{pl}}$ further to obtain

$$\mathit{CODI}_{\text{pl}-\text{slin}} = \mathit{CODI}_{\text{pl}} \cup \text{PL-A2},$$
$$\mathit{CODI}_{\text{pl}-\text{lin}} = \mathit{CODI}_{\text{pl}-\text{slin}} \cup \text{PL-A3},$$
$$\mathit{CODI}_{\text{pl}-\text{aff}} = \mathit{CODI}_{\text{pl}-\text{lin}} \cup \{\text{PL-A4, PL-A5}\}.$$

These theories axiomatize semi-linear, linear, and affine spaces, respectively. $\mathit{CODI}_{\text{pl}-\text{lin}}$ and $\mathit{CODI}_{\text{pl}-\text{aff}}$ assume the *line axiom*, i.e., that two distinct points uniquely define a line (by PL-A2 and PL-A3). We again include references to the axioms of incidence geometry from Section 10.1 in parentheses.

Note that we can generate models of $\mathit{CODI}_{\text{pl}-\text{aff}}$ in which two lines are called parallel in the sense of PL-A5 and I.P in that they do not intersect, but are not parallel in the stronger Euclidean sense. See Figure 10.3 for such a model.

A remark on the different kinds of axioms is in order. PL-A1 and PL-A3 are existential axioms: they force two distinct points to exist for every line and a line to exist for every two distinct points. Equally, PL-A4 and PL-A5 are of existential nature, requiring more lines to arise from existing points and requiring that at least three distinct points exist. In this sense, all of PL-A1, PL-A3, PL-A4, and PL-A5 require additional spatial entities to exist. PL-A2 is of a very different kind: it prohibits

(PL-A2) $L(l) \wedge L(m) \wedge Pt(p) \wedge Pt(q) \wedge p \neq q \wedge Cont(p,l) \wedge Cont(p,m) \wedge Cont(q,l) \wedge Cont(q,m) \rightarrow l = m$
(**Line Axiom, Part I**: two distinct points are contained in at most one line; I.2a)

(PL-A3) $Pt(p) \wedge Pt(q) \wedge p \neq q \rightarrow \exists l[L(l) \wedge Cont(p,l) \wedge Cont(q,l)]$
(**Line Axiom, Part II**: two distinct points are contained in some common line; I.2b)

(PL-A4) $\exists p,q,r\big[Pt(p) \wedge Pt(q) \wedge Pt(r) \wedge p \neq q \neq r \neq p \wedge \forall l[L(l) \rightarrow (\neg Cont(p,l) \vee \neg Cont(q,l) \vee \neg Cont(r,l))]\big]$ (**Line Dimension Axiom**: three distinct noncollinear points, i.e., which are not contained in any single line, exist; I.E1)

(PL-A5) $L(l) \wedge Pt(p) \wedge \neg Cont(p,l) \rightarrow \exists m[L(m) \wedge Cont(p,m) \wedge \neg C(l,m)]$ (**Parallel Axiom**: a point not contained in a line l is contained in a line m disconnected from ('parallel to') l, I.P)

Axiom Set 10.5: Axioms PL-A2 – PL-A5 of $CODI_{\text{pl-aff}}$.

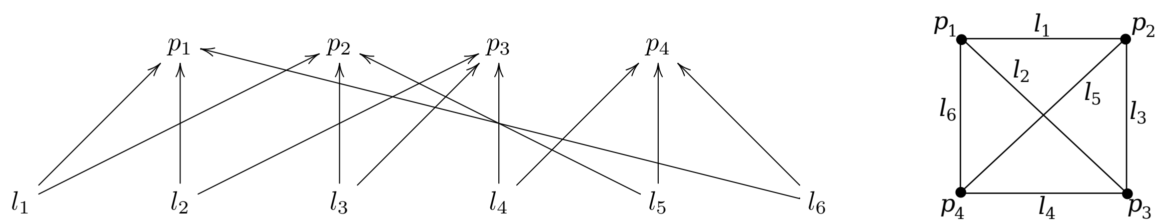


Figure 10.3: The incidence structure of the smallest model of $CODI_{\text{pl-aff}}$ and a corresponding spatial configuration. Notice that l_2 and l_5 do not intersect in the center, thus they are "parallel" lines. This model is planar, i.e., realizable in a 2D space if l_5 is routed outside the rectangle formed by l_1, l_3, l_4, and l_6.

extra lines in that two distinct points may only be contained in a single line. Or, put differently, in a semi-linear space, two distinct lines can only intersect in one point, though they may not intersect at all. Thereby, PL-A2 effectively eliminates models with curved lines: every model of $CODI_{\text{pl-slin}}$ can be realized as a spatial configuration in which all lines, i.e., all entities in the extension $\mathbf{L}_\mathcal{M}$, are straight lines. Therefore, all models of extensions of $CODI_{\text{pl-slin}}$, including $CODI_{\text{pl-lin}}$ or $CODI_{\text{pl-aff}}$, can be spatially interpreted as containing only straight lines.

Next, we will show that the models of $CODI_{\text{pl-slin}}$, $CODI_{\text{pl-lin}}$, and $CODI_{\text{pl-aff}}$ have natural substructures—the same substructures that we already proved to be line spaces—that are semi-linear, linear, and affine spaces, respectively.

Theorem 10.5. *Let $\mathcal{M}$ be a model of $CODI_{\text{pl-slin}}$ ($CODI_{\text{pl-lin}}$, $CODI_{\text{pl-aff}}$) with domain $\mathbf{M}$. Then the substructure $\mathfrak{I} = \langle \mathbf{Pt}_\mathcal{M}, \mathbf{L}_\mathcal{M}, * \rangle$ with*

$$\langle x,y\rangle \in *_\mathfrak{I} \iff \langle x,y\rangle \in \mathbf{Cont}_\mathcal{M} \text{ and } x \in \mathbf{Pt}_\mathcal{M} \text{ and } y \in \mathbf{L}_\mathcal{M}$$

for all $x, y \in \mathbf{Pt}_\mathcal{M} \cup \mathbf{L}_\mathcal{M}$ is a semi-linear (linear, affine) space.

Proof. We first prove that the structure $\langle \mathbf{Pt}_\mathcal{M}, \mathbf{L}_\mathcal{M}, * \rangle$ of an arbitrary model $\mathcal{M}$ of $CODI_{\text{pl-slin}}$ is a semi-linear space and then show that the structure $\langle \mathbf{Pt}_\mathcal{M}, \mathbf{L}_\mathcal{M}, * \rangle$ is a linear or affine space if $\mathcal{M}$ is a model of $CODI_{\text{pl-lin}}$ or $CODI_{\text{pl-aff}}$.

$CODI_{\text{pl-slin}}$**:** $\langle \mathbf{Pt}_\mathcal{M}, \mathbf{L}_\mathcal{M}, * \rangle$ is a semi-linear space.

Let $\mathcal{M}$ be an arbitrary model of $\mathit{CODI}_{\mathrm{pl-slin}}$ and let $\mathcal{I} = \langle \mathbf{Pt}_{\mathcal{M}}, \mathbf{L}_{\mathcal{M}}, * \rangle$ be the substructure that is a line space and that must exist according to Theorem 10.3.

Let l be an arbitrary line in the line space.

Then $l \in \mathbf{L}_{\mathcal{M}}$. By PL-A1 there exists distinct $p, q \in \mathbf{Pt}_{\mathcal{M}}$ such that $\langle p, l\rangle, \langle q, l\rangle \in \mathbf{Cont}_{\mathcal{M}}$. Thus $\langle p, l\rangle, \langle q, l\rangle \in *_{\mathfrak{I}}$ by the definition of the incidence relation. Hence I.1 is satisfied: there exists two distinct points p and q such that $p * l$ and $q * l$.

Now let $p, q \in \mathbf{Pt}_{\mathcal{M}}$ be distinct points in the line space. Suppose—contrary to I.2a—that there exist two distinct lines l and m such that $\langle p, l\rangle, \langle q, l\rangle, \langle p, m\rangle, \langle q, m\rangle \in *_{\mathfrak{I}}$. Then $l, m \in \mathbf{L}_{\mathcal{M}}$ and thus $\langle p, l\rangle, \langle q, l\rangle, \langle p, m\rangle, \langle q, m\rangle \in \mathbf{Cont}_{\mathcal{M}}$, thereby contradicting PL-A2, an axiom of $\mathit{CODI}_{\mathrm{pl-slin}}$. Hence our supposition that such two distinct lines can exist was false, and there exist at most one line incident with p and q, satisfying I.2a.

Hence, the substructure $\langle \mathbf{Pt}_{\mathcal{M}}, \mathbf{L}_{\mathcal{M}}, * \rangle$ obtained from an arbitrary model of $\mathit{CODI}_{\mathrm{pl-slin}}$ is a semi-linear space.

$\mathit{CODI}_{\mathrm{pl-lin}}$: $\langle \mathbf{Pt}_{\mathcal{M}}, \mathbf{L}_{\mathcal{M}}, * \rangle$ is a linear space.

Now let $\mathcal{M}$ also satisfy PL-A3, i.e., $\mathcal{M}$ is an arbitrary model of $\mathit{CODI}_{\mathrm{pl-lin}}$. We already proved that $\langle \mathbf{Pt}_{\mathcal{M}}, \mathbf{L}_{\mathcal{M}}, * \rangle$ satisfies I.1 and I.2a. It remains to show that I.2b is satisfied: for points p and q a line with $p * l$ and $q * l$ exists, which follows immediately from PL-A3.

Hence, the substructure $\langle \mathbf{Pt}_{\mathcal{M}}, \mathbf{L}_{\mathcal{M}}, * \rangle$ obtained from an arbitrary model of $\mathit{CODI}_{\mathrm{pl-lin}}$ is a linear space.

$\mathit{CODI}_{\mathrm{pl-aff}}$: $\langle \mathbf{Pt}_{\mathcal{M}}, \mathbf{L}_{\mathcal{M}}, * \rangle$ is an affine space.

Now let $\mathcal{M}$ also satisfy PL-A4 and PL-A5, i.e., $\mathcal{M}$ is an arbitrary model of $\mathit{CODI}_{\mathrm{pl-aff}}$. By PL-A4, three distinct points exist that are not incident with a single line, thus proving I.E1. Now we show that I.P is satisfied. Recall that it suffices to prove I.Pb. We assume the antecedent of I.Pb is satisfied: Let p be a point and l be a line of the line space so that $\langle p, l\rangle \notin (*_{\mathbf{dim}})_{\mathcal{M}}$ is false. Then $p \in \mathbf{Pt}_{\mathcal{M}}, \in \mathbf{L}_{\mathcal{M}}$, and $\langle p, l\rangle \notin \mathbf{Cont}_{\mathcal{M}}$—satisfying the antecedent of PL-A5. By PL-A5, there must exist a line $m \in \mathbf{L}_{\mathcal{M}}$ such that $\langle p, m\rangle \in \mathbf{Cont}_{\mathcal{M}}$ and thus $\langle p, m\rangle \in *_{\mathfrak{I}}$. Moreover, l and m cannot share a point by $\neg C(l, m) \rightarrow \forall p[\neg \mathit{Cont}(p, l) \vee \neg \mathit{Cont}(p, m)]$. Hence, the consequent of I.Pb is also satisfied and hence I.P is satisfied.

Consequently, the substructure $\mathcal{I} = \langle \mathbf{Pt}_{\mathcal{M}}, \mathbf{L}_{\mathcal{M}}, * \rangle$ obtained from an arbitrary model of $\mathit{CODI}_{\mathrm{pl-aff}}$ is an affine space.

□

For the converse, we can prove that every semi-linear, linear, or affine space can be definably expanded to a model of $\mathit{CODI}_{\mathrm{pl-slin}}$, $\mathit{CODI}_{\mathrm{pl-lin}}$, or $\mathit{CODI}_{\mathrm{pl-aff}}$, respectively.

Theorem 10.6. *Any semi-linear (linear, affine) space $\mathfrak{I} = \langle \mathbf{Pt}, \mathbf{L}, * \rangle$ can be definably expanded to a model $\mathcal{M}$ of $\mathit{CODI}_{\mathrm{pl-slin}}$ ($\mathit{CODI}_{\mathrm{pl-lin}}$, $\mathit{CODI}_{\mathrm{pl-aff}}$) with $\mathbf{M} = \mathbf{Pt} \cup \mathbf{L}$, $\mathbf{Pt}_{\mathcal{M}} = \mathbf{Pt}$, and $\mathbf{L}_{\mathcal{M}} = \mathbf{L}$ such that for all $x, y \in \mathbf{M}$,*

$$\langle x, y\rangle \in \mathbf{Cont}_{\mathcal{M}} \iff x = y \text{ or } \langle x, y\rangle \in *_{\mathfrak{I}}.$$

Proof. Recall that any semi-linear, linear, or affine space is a line space. Hence by Theorem 10.4, there exists a corresponding model $\mathcal{M}$ of $\mathit{CODI}_{\mathrm{pl}}$ such that for all $x, y \in \mathbf{M}$,

$$\langle x, y\rangle \in \mathbf{Cont}_{\mathcal{M}} \iff x = y \text{ or } \langle x, y\rangle \in *_{\mathfrak{I}}.$$

It remains to show that if the line space is semi-linear (linear, affine), the resulting model $\mathcal{M}$ of $CODI_{\text{pl}}$ also satisfies PL-A2 ({PL-A2, PL-A3}, {PL-A2 – PL-A5}). PL-A2, PL-A3, PL-A4, and PL-A5 follow immediately from I.2a, I.2b, I.E1, and I.P analogue to the proofs in Theorem 10.5. □

Theorems 10.5 and 10.6 imply that semi-linear, linear, and affine spaces are equivalent to classes of substructures of the theories $CODI_{\text{pl-slin}}$, $CODI_{\text{pl-lin}}$, and $CODI_{\text{pl-aff}}$, respectively. Again, we can directly relate the theories: $CODI_{\text{pl-slin}}$, $CODI_{\text{pl-lin}}$, and $CODI_{\text{pl-aff}}$ faithfully interpret the theories of semi-linear, linear, and affine line space, respectively.

The Euclidean plane $\mathbb{R}^n$ as an affine space is thus also a substructure of a model of $CODI_{\text{pl-aff}}$. In other words, the Euclidean plane is faithfully interpretable in $CODI_{\text{pl-aff}}$. In fact, we could extend $CODI_{\text{pl-aff}}$ to a complete theory that has $\mathbb{R}^2$ as its only model (up to elementary equivalence) by forcing an empty extension for all relations of *CODI* that are irrelevant to the Euclidean plane. Likewise, higher-dimensional Euclidean spaces are faithfully interpretable in suitable extensions of $CODI_{\text{pl-aff}}$ as we will sketch out next.

10.2.3 $CODI_{\text{plp-lin}}$ faithfully interprets incidence geometries

We can extend $CODI_{\text{pl}}$ to higher-dimensional equivalents of semi-linear, linear, or affine spaces if we define higher-dimensional maximal entities analogue to L-D. For example, three-dimensional planes can be defined as in Pl-D. We define the extension of *CODI* by definitions of points, lines, and planes as

$$CODI_{\text{plp}} = CODI_{\text{pl}} \cup \text{Pl-D}.$$

Then we can express the *space axioms* of three-dimensional incidence geometry as PLP-A1 – PLP-A4 and as the extension axioms PLP-E2 and PLP-E3, which we will discuss subsequently. Again, we include the corresponding incidence axioms in parentheses.

(Pl-D) $Pl(x) \leftrightarrow Max(x) \wedge \exists y, z[y \prec_{\text{dim}} x \wedge z \prec_{\text{dim}} y \wedge Pt(z)]$
(planes are maximal in their dimension and of two dimensions higher than points)

(PLP-A1) $Pt(p) \wedge Pt(q) \wedge Pt(r) \wedge p \neq q \wedge p \neq r \wedge q \neq r \wedge \forall l[L(l) \rightarrow \neg Cont(p,l) \vee \neg Cont(q,l) \vee \neg Cont(r,l)] \rightarrow \exists x[Pl(x) \wedge Cont(p,x) \wedge Cont(q,x) \wedge Cont(r,x)]$ (**Plane Axiom, Part I**: three points that are not contained in a common line are contained in a common plane, I.4a)

(PLP-A2) $Pt(p) \wedge Pt(q) \wedge Pt(r) \wedge p \neq q \wedge p \neq r \wedge q \neq r \wedge \forall l[L(l) \rightarrow \neg Cont(p,l) \vee \neg Cont(q,l) \vee \neg Cont(r,l)] \wedge Pl(x) \wedge Pl(y) \wedge Cont(p,x) \wedge Cont(q,x) \wedge Cont(r,x) \wedge Cont(p,y) \wedge Cont(q,y) \wedge Cont(r,y) \rightarrow x = y$
(**Plane Axiom, Part II**: three distinct points that are not contained in a common line are contained in at most one common plane, I.4b)

(PLP-A3) $Pl(x) \rightarrow \exists p,q,r\big[Pt(p) \wedge Pt(q) \wedge Pt(r) \wedge p \neq q \wedge p \neq r \wedge q \neq r \wedge Cont(p,x) \wedge Cont(q,x) \wedge Cont(r,x) \wedge \forall l[L(l) \rightarrow \neg Cont(p,l) \vee \neg Cont(q,l) \vee \neg Cont(r,l)]\big]$ (**Plane Existence Axiom**: a plane contains three points that are not contained in a single line; I.3)

(PLP-A4) $Pl(x) \wedge L(l) \wedge Pt(p) \wedge Pt(q) \wedge p \neq q \wedge Cont(p,l) \wedge Cont(q,l) \wedge Cont(p,x) \wedge Cont(q,x) \rightarrow Cont(l,x)$ (**Line-Plane Intersection Axiom**: if a plane contains two distinct points of a line, it contains the line; I.5)

Axiom Set 10.6: Definition Pl-D and axioms PLP-A1 – PLP-A4 of $CODI_{\text{plp-aff}}$.

It is easy to verify that in $CODI_{\text{plp}}$ all planes are of identical dimension (PLP-T1), and that points and lines are disjoint from planes (PLP-T2).

(PLP-T1) $Pl(x) \wedge Pl(y) \rightarrow x =_{\text{dim}} y$ (planes are of uniform dimension)

(PLP-T2) $[\neg Pt(x) \vee \neg Pl(x)] \wedge [\neg L(x) \vee \neg Pl(x)]$ (planes are disjoint from points and lines)

Lemma 10.3. *$CODI_{\text{plp}} \models$ {PLP-T1, PLP-T2}*

We define extension of *CODI* to theories of semi-linear, linear, and affine incidence geometry, respectively, as

$$CODI_{\text{plp-slin}} = CODI_{\text{pl-slin}} \cup CODI_{\text{plp}} \cup \{\text{PLP-A1} - \text{PLP-A4}\},$$
$$CODI_{\text{plp-lin}} = CODI_{\text{pl-lin}} \cup CODI_{\text{plp-slin}},$$
$$CODI_{\text{plp-aff}} = CODI_{\text{pl-aff}} \cup CODI_{\text{plp-slin}}.$$

The next theorem will show that any model of $CODI_{\text{plp-lin}}$ is indeed a linear incidence geometry.

Theorem 10.7. *Any model $\mathcal{M}$ of $CODI_{\text{plp-lin}}$ with domain $\mathbf{M}$ defines an incidence geometry $\langle \mathbf{M}, \mathbf{I}, *, \dim\rangle$ with distinguished sets $\mathbf{Pt} = \mathbf{Pt}_{\mathcal{M}}$, $\mathbf{L} = \mathbf{L}_{\mathcal{M}}$, and $\mathbf{Pl} = \mathbf{Pl}_{\mathcal{M}}$ such that for all $x, y \in \mathbf{M}$,*

$$\dim(x) = \dim(y) \iff \langle x, y\rangle \in (=_{\mathbf{dim}})_{\mathcal{M}}, \text{ and}$$
$$\langle x, y\rangle \in *_{\mathfrak{I}} \iff \langle x, y\rangle \in \mathbf{Cont}_{\mathcal{M}} \text{ and } x \in \mathbf{Pt}_{\mathcal{M}} \text{ and } y \in \mathbf{L}_{\mathcal{M}} \cup \mathbf{Pl}_{\mathcal{M}}.$$

Proof. Any model of $CODI_{\text{plp-lin}}$ is a model of *CODI* and thus defines a k-partite point incidence structure (see Corollary 10.1) as constructed in Theorem 10.1.
Recall that the relation $=_{\text{dim}}$ partitions the domain $\mathbf{M}$ naturally into classes of entities of equal dimension, hence by choosing a minimal set of types $\mathbf{I}$, dim defines a surjective function; it is already a function by the first condition in the theorem statement. Moreover, $*$ can only hold between points and non-points—as required by I.0a. Thereby, $*$ is also asymmetric and irreflexive.

It remains to prove that the elements in the sets $\mathbf{Pt}$, $\mathbf{L}$, and $\mathbf{Pl}$ satisfy I.0 – I.5. This is straightforward, analogue to the proofs of Theorems 10.3 and 10.5. In particular, I.3 – I.5 follow from PLP-A1 – PLP-A4, while I.1 – I.2 follow again from PL-A1 – PL-A3. I.0b follows from $\mathbf{Pt}_{\mathcal{M}}$, $\mathbf{L}_{\mathcal{M}}$, and $\mathbf{Pl}_{\mathcal{M}}$ being disjoint for any model $\mathcal{M}$ of $CODI_{\text{plp-lin}}$. Then I.0a immediately follows by the definition of $*_{\mathfrak{I}}$ in our statement of the theorem. □

Notice that the entities not in the set $\mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl}$ are in no incidence relation in the constructed incidence geometry, hence the incidence geometry is not a substructure of the model $\mathcal{M}$. If we are only interested in the points, lines, and planes, then we can obtain a corresponding three-dimensional incidence geometry $\langle \mathbf{X}, \mathbf{I}, *, \dim\rangle$ such that $\mathbf{X} = \mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl}$, with the type function and the incidence relation defined as before but restricted to elements in $\mathbf{X}$. This would be a substructure of $\mathcal{M}$.

As the converse of Theorem 10.7, $CODI_{\text{plp-lin}}$ axiomatizes incidence geometries as defined in Definition 10.11, that is, any incidence geometry is definably expandable to a model of $CODI_{\text{plp-lin}}$.

Theorem 10.8. *Any incidence geometry $\mathfrak{I} = \langle \mathbf{X}, \mathbf{I}, *, \dim\rangle$ with distinguished sets $\mathbf{Pt}, \mathbf{L}, \mathbf{Pl} \subseteq \mathbf{X}$ can be definably expanded to a model $\mathcal{M}$ of $CODI_{\text{plp-slin}}$ ($CODI_{\text{pl-lin}}$, $CODI_{\text{pl-aff}}$) with $\mathbf{M} = \mathbf{X}$, $\mathbf{Pt}_{\mathcal{M}} = \mathbf{Pt}$,*

$\mathbf{L}_\mathcal{M} = \mathbf{L}$, *and* $\mathbf{Pl}_\mathcal{M} = \mathbf{Pl}$ *such that for all* $x, y \in \mathbf{M}$,

$$\langle x,y\rangle \in \mathbf{Cont}_\mathcal{M} \iff x = y \textit{ or } \langle x,y\rangle \in *_\mathfrak{I} \textit{ or } \big(x \in \mathbf{L}, y \in \mathbf{Pl} \textit{ and there exist distinct } p,q \in \mathbf{Pt} \textit{ with } \langle p,x\rangle, \langle q,x\rangle, \langle p,y\rangle, \langle q,y\rangle \in *_\mathfrak{I}\big).$$

Proof. We define a structure $\mathcal{M}$ as follows

1. $\mathbf{M} = \mathbf{X}$,
2. $\mathbf{ZEX}_\mathcal{M} = \emptyset$,
3. $\mathbf{Pt}_\mathcal{M} = \mathbf{Pt}$,
4. $\mathbf{L}_\mathcal{M} = \mathbf{L}$,
5. $\mathbf{Pl}_\mathcal{M} = \mathbf{Pl}$,

and for all $x, y \in \mathbf{X}$,

6. $\langle x,y\rangle \in (<_\mathbf{dim})_\mathcal{M} \iff \big(x \in \mathbf{Pt} \text{ and} y \notin \mathbf{Pt}\big) \text{ or } \big(x \in \mathbf{L} \text{ and } y \notin (\mathbf{Pt} \cup \mathbf{L})\big) \text{ or } \big(x \in \mathbf{Pl} \text{ and } y \notin (\mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl})\big)$,
7. $\langle x,y\rangle \in \mathbf{Cont}_\mathcal{M} \iff x = y \text{ or } \langle x,y\rangle \in *_\mathfrak{I} \text{ or } \big(x \in \mathbf{L},\ y \in \mathbf{Pl} \text{ and there exist distinct } p,q \in \mathbf{Pt} \text{ with } \langle p,x\rangle, \langle q,x\rangle, \langle p,y\rangle, \langle q,y\rangle \in *_\mathfrak{I}\big)$.

We maintain the relative dimension ordering for all entities not in $\mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl}$ from Theorem 10.1(5) defined as

8. for all $x \notin \mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl}$, $\langle x,y\rangle \in (<_\mathbf{dim})_\mathcal{M} \iff x \in \mathbf{B_i}$ and $y \in \mathbf{B_j}$ with $i < j$,

where the incidence structure $\mathfrak{I}$ is represented as $\langle \mathbf{Pt}, \mathbf{L}, \mathbf{Pl}, \mathbf{B_4}, \ldots, \mathbf{B_k}, *\rangle$ with the domain $\mathbf{X} = \mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl} \cup \bigcup_{4 \le i \le k} \mathbf{B_i}$.

Showing that the so-defined structure $\mathcal{M}$ satisfies all axioms of $CODI_{\text{plp}-\text{lin}}$ is analogue to the proofs of Theorems 10.4 and 10.6. In particular, the axioms C-A2, C-A3, and CD-A1 are affected and must be re-verified. C-A2 and CD-A1 are still satisfied because the new case added in (7) is asymmetric: for any $x \in \mathbf{L}$ and any $y \in \mathbf{Pl}$ we have $\langle x,y\rangle \in (<_\mathbf{dim})_\mathcal{M}$ by (6) and thus $\langle x,y\rangle \notin \mathbf{Cont}_\mathcal{M}$.

The antecedent of C-A3 with distinct x, y, z only applies when $x \in \mathbf{Pt}_\mathcal{M}$, $y \in \mathbf{L}_\mathcal{M}$, and $z \in \mathbf{Pl}_\mathcal{M}$—otherwise we cannot have $\langle x,y\rangle, \langle y,z\rangle \in \mathbf{Cont}_\mathcal{M}$. Suppose $\langle y,z\rangle \in \mathbf{Cont}_\mathcal{M}$, $y \in \mathbf{L}_\mathcal{M}$, and $z \in \mathbf{Pl}_\mathcal{M}$. Because $\langle y,z\rangle \in *_\mathfrak{I}$ requires $y \in \mathbf{Pt}$ (by Definition 10.3), this is only possible if the last condition of (7) is satisfied: there exist two points $p, q \in \mathbf{Pt}$ such that $\langle p,y\rangle, \langle q,y\rangle, \langle p,z\rangle, \langle q,z\rangle \in *_\mathfrak{I}$. But then by I.5 all points, including x, that are incident with y are also incident with z and thus $\langle x,z\rangle \in \mathbf{Cont}_\mathcal{M}$.

The axioms PL-A1 – PL-A3 are direct translation of I.1, I.2a, and I.2b as already shown in Theorem 10.4 and 10.6. The axioms PLP-A1 – PLP-A3 are provably direct translations of I.3, I.4a, I.4b, while PLP-A4 follows similar to the proof of C-A3 from the last condition of (7) together with I.5. □

From Theorem 10.7 it immediately follows that $CODI_{\text{plp}-\text{lin}}$ interprets the theory of incidence structures, *IG*, because the translations of all axioms of *IG* are provable from $CODI_{\text{plp}-\text{lin}}$. $CODI_{\text{plp}-\text{lin}}$ faithfully interprets *IG* because every model of *IG* can be expanded to a model of $CODI_{\text{plp}-\text{lin}}$ as we just proved.

We can further extend $\mathit{CODI}_{\mathrm{plp-lin}}$ or any other extension of $\mathit{CODI}_{\mathrm{plp-slin}}$ by any of PLP-E1, PLP-E2, or PLP-E3 to restrict the model's dimensionality. For example, Hilbert's theory of incidence [Hil71] is the extension of our notion of incidence geometry; it is equivalent to $\mathit{CODI}_{\mathrm{plp-lin}} \cup$ {PL-A4, PLP-E1 − PLP-E3}, that is, its models have exactly three dimensions.

(PLP-E1) $Pt(x) \vee L(x) \vee Pl(x)$ (every entity is either a point, line, or plane)

(PLP-E2) $Pl(x) \wedge Pl(y) \wedge x \neq y \wedge Pt(p) \wedge Cont(p,x) \wedge Cont(p,y) \rightarrow \exists q[Pt(q) \wedge p \neq q \wedge Cont(q,x) \wedge Cont(q,y)]$ (two planes that share a point share a second distinct point; I.6)

(PLP-E3) $\exists p,q,r,s\big[Pt(p) \wedge Pt(q) \wedge Pt(r) \wedge Pt(s) \wedge p \neq q \wedge p \neq r \wedge p \neq s \wedge q \neq r \wedge q \neq s \wedge r \neq s \wedge \forall x[Pl(x) \rightarrow \neg Cont(p,x) \vee \neg Cont(q,x) \vee \neg Cont(r,x) \vee \neg Cont(s,x)]\big]$ (**Plane Dimension Axiom**: there exist four distinct points that are not contained in a single plane; I.E2)

Axiom Set 10.7: Extension axioms PLP-E1 − PLP-E3 of $\mathit{CODI}_{\mathrm{plp}}$.

Analogously to how we extended *CODI* to theories that faithfully interpret particular classes of incidence geometries with incidence only between points and lines or planes, it is fairly straightforward to construct an extension of *CODI* that faithfully interprets n-dimensional incidence geometries with finite $n > 3$ in which the higher-dimensional entities are equally well-behaved as lines and planes. We will only sketch the required axioms here because we will not work with those higher-dimensional incidence geometries subsequently. The sketched axioms could be more formally stated as axiom schemata with a parameter i for the dimensionality. To interpret semi-linear, linear, or affine n-dimensional point incidence structures we would extend the theories $\mathit{CODI}_{\mathrm{pl-slin}}$, $\mathit{CODI}_{\mathrm{pl-lin}}$, or $\mathit{CODI}_{\mathrm{pl-aff}}$, respectively, by the following axioms for every $i < n$:

1. Define i-dimensional entities of maximal dimension (analogue to Pl-D);
2. Postulate that any $i+1$ distinct points not contained in a common $(i-1)$-dimensional entity are contained in a unique i-dimensional entity (analogue to PLP-A1, PLP-A2);
3. Postulate that any i-dimensional entity contains at least $i+1$ distinct points not contained in a common $(i-1)$-dimensional entity (analogue to PLP-A3);
4. Postulate that any i-dimensional entity that contains i distinct points not contained in a common entity of $i-2$ dimensions, must contain the $(i-1)$-dimensional entity spanned by the points (analogue to PLP-A4).

As necessary, we can add analogue versions of PLP-E1, PLP-E2 and PLP-E3 restricting the dimensionality. While PLP-E1 and PLP-E3 easily generalize—PLP-E3 to 'there exist $i+1$ distinct points not contained in a common i-dimensional entity'—it is not clear what a generalization of PLP-E2 looks like.

Such higher-dimensional incidence geometries can be axiomatized more elegantly using the notion of a 'block' with points being zero-dimensional blocks, lines one-dimensional blocks, etc. For more detail on the use of blocks in incidence geometry see, e.g., [Bue95]. We will not pursue such an axiomatization in this thesis. Even showing that $\mathit{CODI}_{\mathrm{plp-lin}}$ axiomatizes any k-dimensional *incidence geometry* as defined in Definition 10.11 is a quite tedious extension of the proof in Theorem 10.8. For our work here, it suffices to observe that two- and three-dimensional incidence geometries are extensions of the theories $\mathit{CODI}_{\mathrm{pl-slin}}$ or $\mathit{CODI}_{\mathrm{plp-lin}}$, respectively.

10.2.4 A mereotopological generalization of incidence geometries

Up to now, we have shown in this section how to extend *CODI* to classical two- and three-dimensional incidence geometries as well as to equivalents that allow entities of higher dimensions. Because the weakest of the resulting theories already assume PL-A2, i.e., that two points can be contained in at most one line, all lines are implicitly treated as straight lines. If we leave PL-A2 out, we can generalize classical geometries with semi-linear spaces as weakest theory, to mereotopologies, the weakest being *CODI*'s definitional extension to *CODI* ∪ {Pt-D, L-D, Pl-D}. For example, $CODI_{\mathrm{pl}}$ is the mereotopological equivalent of semi-linear spaces: once we add PL-A2, we obtain the geometric theory of semi-linear spaces. We can define the mereotopological equivalent of two-dimensional incidence geometries as $CODI_{\mathrm{pl}}$ ∪ PL-A3 and of two-dimensional affine incidence geometries as $CODI_{\mathrm{pl}}$ ∪ {PL-A3, PL-A4, PL-A5′}, where PL-A5′ captures one part, namely I.Pb, of the parallel axiom.

(PL-A5′) $L(l) \wedge Pt(p) \wedge \neg Cont(p,l) \rightarrow \exists m\big[L(m) \wedge Cont(p,m) \wedge \neg C(l,m)\big]$
(point p not contained in line l is contained in some line m disconnected from l)

Axiom Set 10.8: Extension axiom PL-A5′ of $CODI_{\mathrm{pl}}$.

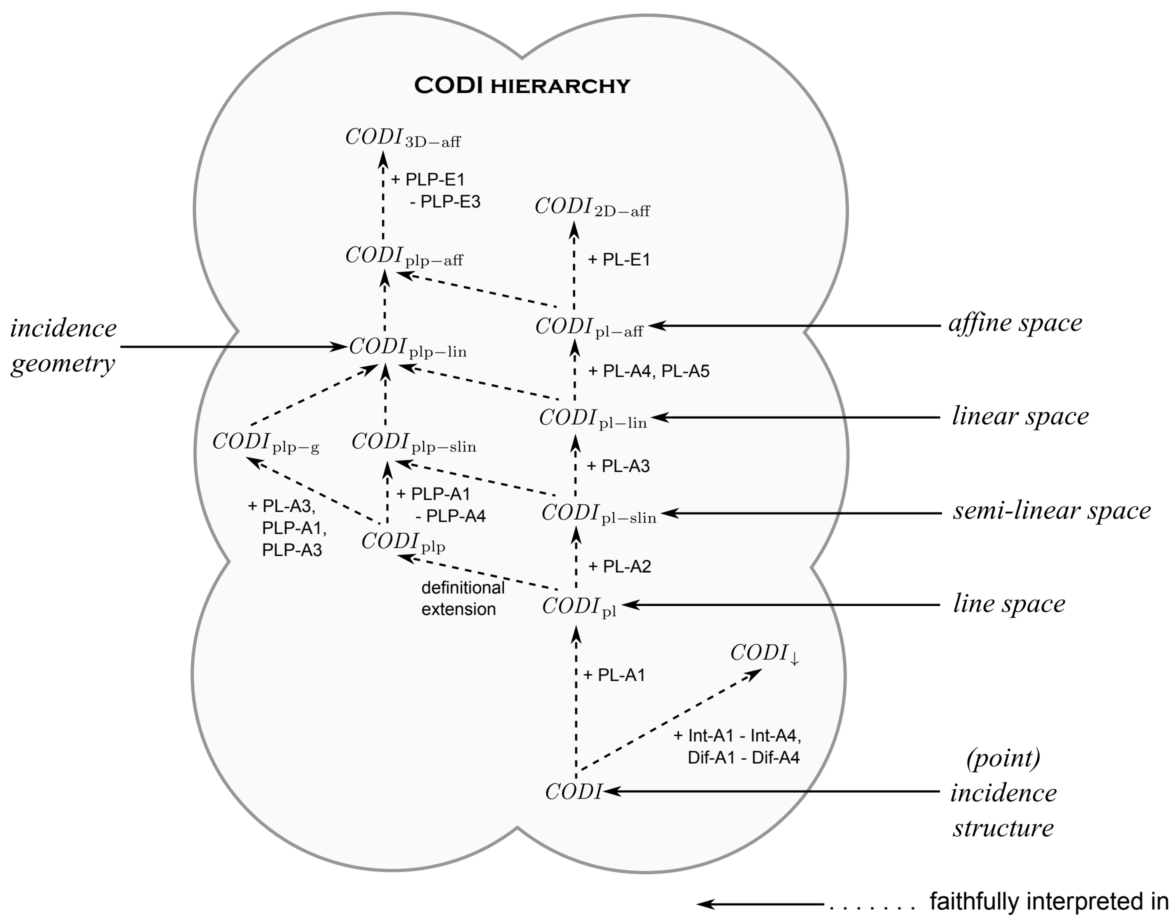


Figure 10.4: The theories introduced in Section 10.2 and their relationship to incidence structures and two-dimensional incidence geometries (planar or line spaces/geometries).

Most importantly, these constructions verify that the theory $CODI_{\rm pl}$ is a mereotopological generalization of semi-linear spaces. Equivalently,

$$CODI_{\rm plp-g} = CODI_{\rm plp} \cup \{\text{PL-A3, PLP-A1, PLP-A3}\},$$

which is weaker than $CODI_{\rm plp-lin}$ but incomparable to $CODI_{\rm plp-slin}$ as illustrated in Figure 10.4, can be considered the mereotopological generalization of three-dimensional incidence geometry. In every model of it, every line contains at least two distinct points and every plane contains at least three distinct noncollinear points, but more than one line may exist through two distinct points and more than one plane may exist through three distinct noncollinear points.

As a final remark, observe that all theories presented in this section allow both discrete, and thus finite, as well as continuous incidence geometries. The relationship among the various theories from this section and their correspondences to incidence structures and two-dimensional incidence geometries is illustrated in Figure 10.4.

10.3 Betweenness in multidimensional space

We have demonstrated that the theory of linear dimension and containment, *CODI*, is a dimension-independent first-order axiomatization of mereotopology and a generalization of incidence geometry. We now propose an extension by 'betweenness'—a qualitative spatial relation of relative positions that (1) avoids using implied references as necessary for cardinal directions [Fra96] or orientations [Fre92] and (2) avoids using specific numeric dimensions and thus fits into our general dimension-independent approach. e.g., a point can be in between two other points on a line; equally, a line can be in between two other lines within a region (or on a plane).

Betweenness is commonly used in everyday descriptions of space, in particular when describing street networks in a city. When sketching directions in a city, order among streets or buildings is among the information that is most reliably preserved as small empirical studies indicate [WL12; WS09]. Without betweenness, e.g., a model of a grid network of streets is invariant under permutations of parallel streets, see Figure 10.1 for an example. Other commonly used non-mereotopological spatial relations, in particular convexity but also the concepts of line segments, rays, or half-planes can be defined in terms of betweenness if both the betweenness relation and the incidence structure are sufficiently restricted, see our discussion in Section 10.3.5.

10.3.1 Relativized betweenness

Ternary betweenness relations have been studied as part of many geometries [Hil71; TG99; Veb04] and also as independent systems [HK17]. Pambuccian [Pam11] gives an excellent and extremely broad overview of the various axiomatizations of order that have been used in combination with incidence geometry, tracing the work on betweenness back to Moritz Pasch [Pas88]. Here we are not concerned with the differences between the various axiomatizations of betweenness. Instead, we are concerned solely with developing an axiomatization of betweenness that works in our multidimensional setting, i.e., that not only works for points and lines, but also for higher-dimensional entities and that can be axiomatized independent of the concrete numeric dimensions in question. To achieve this, we generalize Huntington and Kline's set of independent postulates A, B, C, D, 1, and 2 for strict betweenness on

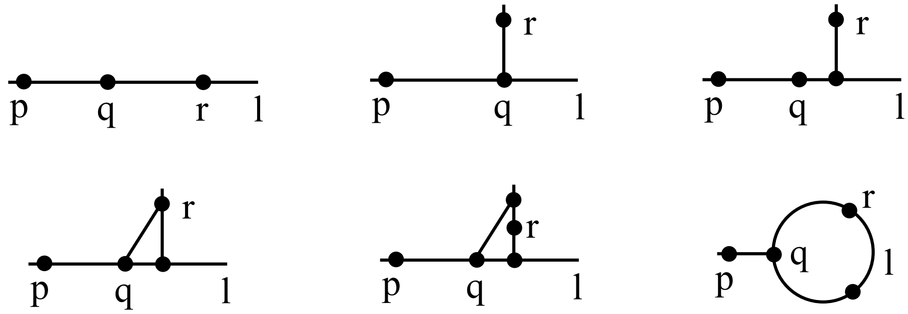


Figure 10.5: Spatial configurations with points on a simple or complex line in which $Btw(l, p, q, r)$ holds.

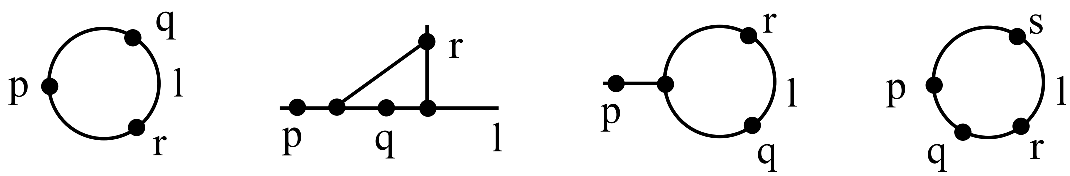


Figure 10.6: Spatial configurations with points on simple or complex line in which $Btw(l, p, q, r)$ is violated.

an undirected line [HK17] to a quaternary between relation $Btw(r, a, b, c)$ meaning 'among the entities contained in r, b is strictly in between a and c'. The intended topological interpretation of $Btw(r, a, b, c)$ is borrowed from the Jordan-Curve-Theorem: Any continuous set (i.e., consisting of a single connected piece) contained in r and containing both a and c must include some point of b. In other words, b divides r into two subsets—one containing a and the other containing c. Some examples and counterexamples of a point being in between two other points on a line are given in Figures 10.5 and 10.6. See Figure 10.7(a) for an example why betweenness must be relativized to an embedding entity, resulting in the quaternary relation. We include the names of the axioms from [HK17] in parentheses as reference, those were mapped at the end of Section 10.1.3 to our set of order axioms O.1 – O.4 used for the theory of weak ordered incidence geometries.

In higher-dimensional cases betweenness is not always a total order, e.g., intersecting lines in a plane cannot be ordered, compare Figure 10.7(b). A real map in which three streets may be ordered differently within different regions is given in Figure 10.9. Therefore the theory

$$BTW = \{\text{B-A1} - \text{B-A6}\}$$

omits the postulate B of total orderability (compare O.3 or OMT-E1 further down) as well as the postulates 4 – 8 from [HK17]. In the presence of the orderability, the ternary version of separability [compare PJ65] is provable. But because postulate B does not always hold in multidimensional space, we must include B-A6 as axiom, stating that x separates a and b in r if x is in between a and b in r, with the consequence that any fourth entity y in r cannot be between a and x and between x and b at the same time. Note that B-A6 was missing from our earlier axiomatization in [HG11b].

We can easily show that B-A6 is independent of the remaining axioms.

Lemma 10.4. *{B-A1 – B-A5}* $\nvDash$ *B-A6*

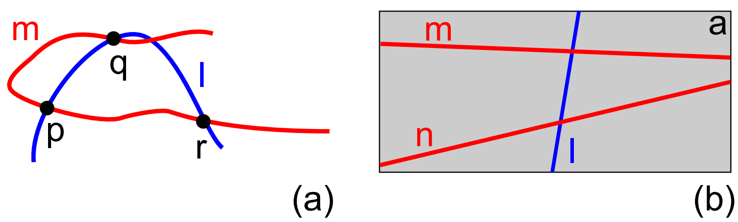


Figure 10.7: Two examples of non-orderability in multidimensional space.
In (a) three points located on two distinct lines are ordered differently on each line. This spatial configuration is captured by a model $\mathcal{M}$ of $CODI_{\downarrow}$ with domain $\mathbf{M} = \{\text{ze}, p, q, r, l, m\}$ an the extensions $\mathbf{Cont}_{\mathcal{M}} = \{\langle p,l\rangle, \langle p,m\rangle, \langle q,l\rangle, \langle q,m\rangle, \langle r,l\rangle, \langle r,m\rangle\}$, $(<_{\mathbf{dim}})_{\mathcal{M}} = \{\langle p,l\rangle, \langle r,l\rangle, \langle q,l\rangle, \langle q,m\rangle, \langle r,l\rangle, \langle r,m\rangle\}$, and $\mathbf{ZEX}_{\mathcal{M}} = \{\text{ze}\}$. In $\mathcal{M}$ we have $Btw(l,p,q,r)$ and $Btw(m,q,p,r)$; hence p, q, and r are not in any specific order—only in a specific order with respect to one entity, l or m, they are contained in. This may happen for, e.g., three cities located along one river (l being a river with q being downriver of p and r being further downriver) but that are connected by a road m in a different order.
In (b), three lines contained in a region cannot be ordered at all. This happens, e.g., for streets, compare Figure 10.1 in the introduction of this chapter. This is also possible in geometries, but their betweenness relation applies only to points anyway.

(B-A1)	$Btw(r,a,b,c) \rightarrow a \neq b \neq c \neq a$	(strong ≅ irreflexive; D)
(B-A2)	$Btw(r,a,b,c) \rightarrow Btw(r,c,b,a)$	(outer symmetry; A)
(B-A3)	$Btw(r,a,b,c) \rightarrow \neg Btw(r,a,c,b)$	(strict betweenness ≅ acyclic; C)
(B-A4)	$Btw(r,x,a,b) \wedge Btw(r,a,b,y) \rightarrow Btw(r,x,a,y)$	(outer transitivity; 1)
(B-A5)	$Btw(r,x,a,b) \wedge Btw(r,a,y,b) \rightarrow Btw(r,x,a,y)$	(inner transitivity; 2)
(B-A6)	$Btw(r,a,x,b) \rightarrow \neg Btw(r,a,x,c) \vee \neg Btw(r,b,x,c)$	(separability)

Axiom Set 10.9: Axioms B-A1 – B-A6 of *BTW*.

Proof. Consider a model $\mathcal{M}$ with domain $\mathbf{M} = \{r,a,b,c,d\}$ and with the extension

$$\mathbf{Btw}_{\mathcal{M}} = \{\langle r,a,b,c\rangle, \langle r,a,b,d\rangle, \langle r,c,b,a\rangle, \langle r,c,b,d\rangle, \langle r,d,b,a\rangle, \langle r,d,b,c\rangle\}.$$

It is easy to see that $\mathcal{M}$ satisfies B-A1, B-A2, B-A3. Moreover, it can be checked that the antecedent of neither B-A4 nor B-A5 is ever satisfied, so both axioms are vacuously true.

However, $Btw(r,a,b,c)$ is in $\mathbf{Btw}_{\mathcal{M}}$, but both $Btw(r,c,b,d)$ and $Btw(r,a,b,d)$ are also in $\mathbf{Btw}_{\mathcal{M}}$, thereby violating B-A6. □

From B-A2 and B-A3 it follows that for arbitrary a,b,c no more than one can be in between the other two (B-T1). Moreover, B-T2, which is the quaternary equivalent of Huntington's postulate 3, is provable. It is simply an alternative way to state inner transitivity (B-A5) and follows from the two kinds of transitivity (B-A4, B-A5) together with symmetry (B-A2). No other of Huntington's postulates 4 – 8 are provable.

(B-T1)	$Btw(r,a,b,c) \rightarrow \neg Btw(r,b,a,c)$	(only one of a,b,c can be in between the other two)
(B-T2)	$Btw(r,x,a,b) \wedge Btw(r,a,y,b) \rightarrow Btw(r,x,y,b)$	(inner transitivity; 3)

Lemma 10.5. *BTW* ⊨ *{B-T1, B-T2}*

Figure 10.8: An excerpt of a map of the city of Toronto with many examples of quaternary betweenness relations. Within the region denoted as 'Westend' (the yellow region bounded by Bathurst Street in the east and Royal York Rd. in the west), Dufferin Street is in between Parkside Drive and Bathurst Street. Also, the road formed by Keele Street together with Parkside Drive is in between Dufferin Street and Jane Street in the Westend, though Keele Street alone is not in between Dufferin Street and Jane Street. However, in the upper-half map bounded by Bloor Street to the south, Keele Street is in between Jane Street and Dufferin Street. Likewise, the point feature Bloor Station is in between Jane Street and Dufferin Street in the upper-half map bounded by Bloor Street. Moreover, Dundas Street is in between Bloor Station and Parkside Drive, but not in between Bloor Street and Parkside Drive in the map. Other examples of betweenness are: The Humber River between Royal York Street and Dufferin Street in the Westend, or High Park in between Parkside Drive and the Humber River within the lower-half map bounded by Bloor Street to the north. High Park is not between Parkside Drive and the Humber River, or for that matter, between Dufferin Street and the Humber River within the Westend.

The axiom B-A3 strictly rules out cyclic orders, that is x, y, and z cannot be located in a cycle within r, just as O.2 rules out cyclic orders in ordered incidence geometry. For example in the rightmost configuration in Figure 10.6 none of p, q, or r is in between the others—as already remarked by Pasch [Pas88]. Equally, in the leftmost configuration in Figure 10.6, none of p, q, r, or s is in between any other two. This matches the intended topological interpretation of *Btw* that an entity b is in between a and c in r only if every continuous subset of r connecting a and c must be in contact with b. This is clearly not in the case in the rightmost and leftmost configurations in Figure 10.6: for any pair of points we can always find an arc that does not pass through a given third point. One can define *separation* relations that deal with undirected cyclic orders to express, e.g., that q lies in between p and r by saying that the pair of points p, r is separated by the pair of points q, s [compare Hun35; HR32; Pas88]. Once we relativize such a quaternary separation relation to the general multidimensional setting, we obtain a five-place relation, whose thorough investigation goes beyond the scope of this thesis. Here it suffices to simply be aware that our generalized betweenness relation only works for non-cyclic spatial configurations; its intended interpretation is unsuitable for cyclic orders. As long as r does not have any loops or holes, this is never a problem. Even if r has a holes, y may be in between x and z within r, see Figure 10.10(a) for an example.

10.3.2 Ordered multidimensional mereotopology

If we want to extend our multidimensional mereotopology *CODI* with the betweenness theory *BTW*, we have to be careful about their interaction. As we already discussed, betweenness only holds for entities contained in a common entity, which must be self-connected (OMT-A1). Moreover, every self-

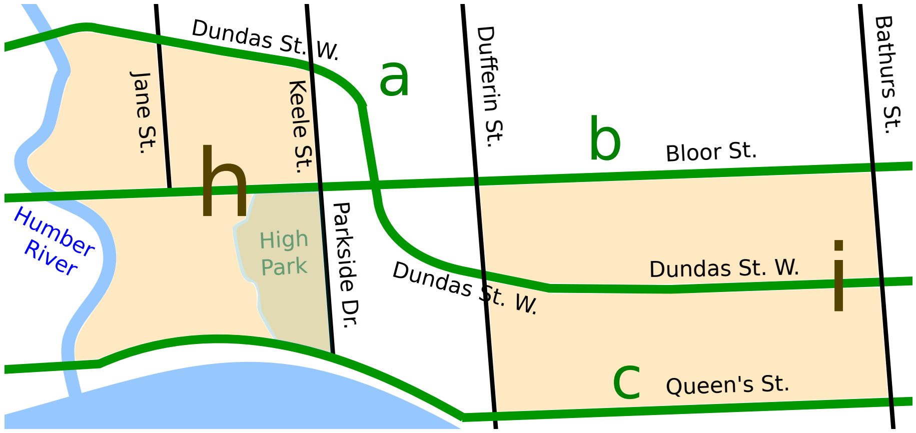


Figure 10.9: An example of a map of the city of Toronto, in which Dundas Street and Bloor Street are not orderable in the region h + i, no matter what the third entity would be. However, we do have *Btw*(h, Dundas St., Bloor St., Queen St.) as well as *Btw*(i, Bloor St., Dundas St., Queen St.)

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Figure 10.10: Betweenness in holed regions. In (a) the lines l, m, and n are ordered within a, while in (b) they are not ordered within a. That is, in (a) $Btw(a, l, m, n)$ is intended to hold while in (b) it is not intended to hold.

connected entity contained in r that connects x and z must intersect y, i.e., be in contact to y (OMT-A2). This is equivalent to the ideas of the Jordan curve theorem and the Pasch axiom as found in ordered incidence geometry. Because *Con* can only be defined in $CODI_{\downarrow}$ but not in *CODI*, we require $CODI_{\downarrow}$ as weakest underlying theory of containment and dimension. Deviating from our original proposal of a *mereotopology with betweenness* called *BMT* in [HG11b], we allow betweenness for entities of arbitrary dimensions here—as long as the three entities in betweenness relation are contained in the embedding entity, that is, their dimension is not greater than the dimension of the embedding entity (by CD-A1). This gives an even more general notion of betweenness; allowing not only three points ordered on a line or three (nonintersecting) lines ordered within a region, but also uses such as a line being between two points in a region, or a line segment being in between a point and another line segment on a line, compare Figure 10.12.

Finally, because we now operate in $CODI_{\downarrow}$, we can require maximal entities to be self-connected (OMT-A3). In particular, points, lines, and planes will then be self-connected if defined as in the previous section. OMT-A2 and OMT-A3 were missing from our original axiomatization in [HG11b].

We define the theory of containment, dimension, and betweenness—also called *ordered multidimensional mereotopology* due to its similarity to ordered incidence geometry which we will formalize in a moment—as

$$OMT_{\downarrow} = CODI_{\downarrow} \cup BTW \cup \{\text{OMT-A1} - \text{OMT-A3}\}.$$

(OMT-A1) $Btw(r, x, y, z) \rightarrow Con(r) \wedge Cont(x, r) \wedge Cont(y, r) \wedge Cont(z, r)$
('betweenness' only among entities contained in a common self-connected entity)

(OMT-A2) $Btw(r, x, y, z) \wedge Cont(v, r) \wedge Con(v) \wedge C(v, x) \wedge C(v, z) \rightarrow C(v, y)$
(if y is in between x and z within r, then every self-connected entity v contained in r and connected to x and z is also connected to y)

(OMT-A3) $Max(x) \rightarrow Con(x)$ (maximal entities are self-connected)

Axiom Set 10.10: Axioms OMT-A1 – OMT-A3 of $OMT_{\downarrow}$.

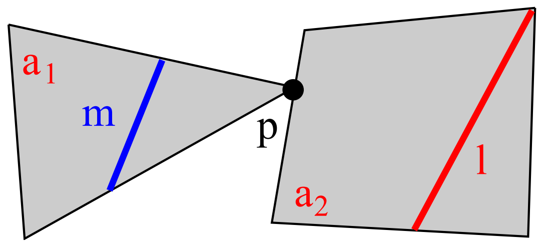

Figure 10.11: Example of betweenness in which the entity in between two others is two dimensions smaller than the embedding entity. Assume that $a = a_1 + a_2$, $\mathbf{Btw}(a, l, p, m)$, and $p \not\prec_{\mathbf{dim}} a$. This is only possible if any two complementary parts a_1 and a_2 of a only meet in p. Otherwise, there would always exist another continuous subset of a that connects l and m. This motivates OMT-A4.

In $OMT_{\downarrow}$ we can prove that all of r, x, y, and z in betweenness relation are nonzero regions (OMT-T1).

(OMT-T1) $Btw(r, x, y, z) \rightarrow \neg ZEX(r) \wedge \neg ZEX(x) \wedge \neg ZEX(y) \wedge \neg ZEX(z)$

(only nonzero regions can be in betweenness relation)

Lemma 10.6. $OMT_{\downarrow} \models$ *OMT-T1*

As a special case of quaternary betweenness, if the middle entity y has a dimension that is more than one lower than the embedding's entity dimension, the embedding entity r is not internally self-connected (as defined in Section 9.2)—the entity y must be the only entity shared by two complementary parts of r, compare Figure 10.11. OMT-A4 states the contrapositive of this idea. However, since the correct interpretation of $ICon$ cannot be ensured in $CODI_{\downarrow}$ but only in $CODIB_{\downarrow}$, it is sensible to use OMT-A4 only in combination with $CODIB_{\downarrow}$ and extensions thereof. We can define the theory

$$OMTB_{\downarrow} = OMT_{\downarrow} \cup CODIB_{\downarrow} \cup \{\text{OMT-A4}\}.$$

(OMT-A4) $Btw(r, x, y, z) \wedge ICon(r) \rightarrow y \prec_{\text{dim}} r \vee y =_{\text{dim}} r$ (an entity is between two other in an internally self-connected entity r is of the same or the next lower dimension as r)

Axiom Set 10.11: Axiom OMT-A4 of $OMTB_{\downarrow}$.

10.3.3 $OMT_{\downarrow}$'s relation to ordered incidence geometry

We have introduced a new theory $OMT_{\downarrow}$ by extending our multidimensional mereotopology by a notion of order relativized to an embedding entity. Now we show that the name *ordered multidimensional mereotopology* for the theory $OMT_{\downarrow}$ is justified, because $OMT_{\downarrow}$ is axiomatically less restricted than weak ordered incidence geometry (as introduced in Definition 10.13) but is still expressive enough to be restricted to weak ordered incidence geometry without introducing new primitive concepts. In other words, the theory axiomatizating n-dimensional weak ordered incidence geometries is definably equivalent to a theory in the *OMT* hierarchy. We will prove this only for weak ordered incidence geometries restricted to points being incident with lines or planes, axiomatized as *WOIG*. As we have seen in Section 10.1.3, all ordered incidence geometries extend weak ordered incidence geometry. Since it is well-understood how other classical three-dimensional geometries, such as affine ordered geometry, neutral geometry, Euclidean, or Lobačevskijan geometry, can be constructed as extensions of ordered incidence geometry (see e.g., Pambuccian's overview [Pam11]), interpreting *WOIG* achieves our goal of relating ordered multidimensional mereotopology to classical geometries.

For the reconstruction of weak ordered incidence geometry in the *OMT* hierarchy we use the theory

$$OMT_{\downarrow-\mathrm{plp}-\mathrm{lin}} = OMT_{\downarrow} \cup CODI_{\mathrm{plp}-\mathrm{lin}}.$$

$OMT_{\downarrow-\mathrm{plp}-\mathrm{lin}}$ is the ordered multidimensional mereotopology extended by axioms that constrain *CODI* in such a way that every model has a substructure that is an incidence geometry (defined as in Definition 10.11) as we showed in Theorem 10.7. To reconstruct ordered incidence geometries, we must restrict our general betweenness relation *Btw* to a linear interpretation that satisfies total orderability as required in ordered incidence geometries by O.3. For this purpose we introduce OMT-E1.

(OMT-E1) $L(r) \wedge Pt(x) \wedge Pt(y) \wedge Pt(z) \wedge Cont(x,r) \wedge Cont(y,r) \wedge Cont(z,r) \wedge \neg C(x,y) \wedge \neg C(x,z) \wedge \neg C(y,z) \rightarrow [Btw(r,x,y,z) \vee Btw(r,x,z,y) \vee Btw(r,y,x,z)]$
(three disconnected points contained in a line are orderable)

Axiom Set 10.12: Axiom OMT-E1 of $OMT_{\mathrm{3D}-\mathrm{lin}}$.

We define the theory in the *OMT* hierarchy that commonly interprets all ordered incidence geometries as

$$OMT_{\mathrm{3D}-\mathrm{lin}} = OMT_{\downarrow-\mathrm{plp}-\mathrm{lin}} \cup \text{OMT-E1}.$$

Now we prove that any model of $OMT_{\mathrm{3D}-\mathrm{lin}}$ defines a structure that is a weak ordered three-dimensional incidence geometry.

Theorem 10.9. *Any model* $\mathcal{M}$ *of* $OMT_{\mathrm{3D}-\mathrm{lin}}$ *defines a weak ordered incidence geometry* $\mathfrak{I} = \langle \mathbf{X}, \mathbf{I}, *, \dim, \mathbf{B} \rangle$ *with distinguished sets* $\mathbf{Pt} = \mathbf{Pt}_{\mathcal{M}}$*,* $\mathbf{L} = \mathbf{L}_{\mathcal{M}}$*, and* $\mathbf{Pl} = \mathbf{Pl}_{\mathcal{M}}$*, and with* $\mathbf{X} = \mathbf{M} \supseteq \mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl}$*, a type function* $\dim : \mathbf{X} \to \mathbf{I}$*, and an incidence relation* $*$ *such that for all* $x, y \in \mathbf{X}$*,*

$$\dim(x) = \dim(y) \iff \langle x,y \rangle \in (=_{\mathbf{dim}})_{\mathcal{M}}, \text{ and}$$
$$\langle x,y \rangle \in *_{\mathfrak{I}} \iff \langle x,y \rangle \in \mathbf{Cont}_{\mathcal{M}} \text{ and } x \in \mathbf{Pt}_{\mathcal{M}} \text{ and } y \in \mathbf{L}_{\mathcal{M}} \cup \mathbf{Pl}_{\mathcal{M}},$$

and a ternary betweenness relation B such that for all $a, b, c \in \mathbf{X}$,

$$\langle x, y, z\rangle \in \mathbf{B}_{\mathfrak{I}} \iff x, y, z \in \mathbf{Pt}_{\mathcal{M}} \text{ and there exists a } r \in \mathbf{L}_{\mathcal{M}} \text{ such that } \langle r, x, y, z\rangle \in \mathbf{Btw}_{\mathcal{M}}.$$

Proof. In Theorem 10.7 we already proved that any model of $CODI_{\text{plp-lin}}$ defines an incidence geometry $\langle \mathbf{X}, \mathbf{I}, *, \dim\rangle$ that satisfies all the conditions that pertain to the incidence geometry substructure of $\langle \mathbf{X}, \mathbf{I}, *, \dim, \mathbf{B}\rangle$. Because $OMT_{\text{3D-lin}}$ extends $CODI_{\text{plp-lin}}$, any model $\mathcal{M}$ of $OMT_{\text{3D-lin}}$ also defines such an incidence geometry.

It remains to prove that the axioms O.1 – O.4 are satisfied for the above definition of the ternary betweenness relation B.

(O.1): $B(a, b, c) \to B(c, b, a)$.
Assume the antecedent $\mathbf{B}(a, b, c)$ holds for arbitrary $a, b, c \in \mathbf{X}$.
Then $\mathbf{Btw}(l, a, b, c)$ for some $l \in \mathbf{L}_{\mathcal{M}}$ and $a =_{\dim} b =_{\dim} c \prec_{\dim} l$ by Pt-D, L-D. Then we deduce $\mathbf{Btw}(\mathbf{l}, \mathbf{c}, \mathbf{b}, \mathbf{a})$ by B-A2 and thus $\mathbf{B}(c, b, a)$, the desired consequent.

(O.2): $B(a, b, c) \to \neg B(b, c, a)$.
Assume the antecedent $\mathbf{B}(a, b, c)$ holds for arbitrary $a, b, c \in \mathbf{X}$.
Then $\mathbf{Btw}(l, a, b, c)$ for some l with $l \in \mathbf{L}_{\mathcal{M}}$ and $a =_{\dim} b =_{\dim} c \prec_{\dim} l$. For any such l, $\neg\mathbf{Btw}(l, a, c, b)$ follows from B-A3 and $\neg\mathbf{Btw}(l, b, c, a)$ by B-A2. Hence, for no $l \in \mathbf{L}_{\mathcal{M}}$ and $a =_{\dim} b =_{\dim} c \prec_{\dim} r$ we have $\mathbf{Btw}(r, b, c, a)$ and thus $\neg\mathbf{B}(b, c, a)$, the desired consequent.

(O.3): $a \neq b \wedge a \neq c \wedge b \neq c \wedge \exists l[L(l) \wedge a * l \wedge b * l \wedge c * l] \leftrightarrow B(a, b, c) \vee B(b, c, a) \vee B(c, a, b)$.
We prove each direction of the implication separately.

Direction (a): $a \neq b \wedge a \neq c \wedge b \neq c \wedge \exists l[L(l) \wedge a * l \wedge b * l \wedge c * l] \to B(a, b, c) \vee B(b, c, a) \vee B(c, a, b)$.
Assume the antecedent $a \neq b \wedge a \neq c \wedge b \neq c \wedge \exists l[L(l) \wedge a * l \wedge b * l \wedge c * l]$ holds for arbitrary $a, b, c \in \mathbf{X}$ and some $l \in \mathbf{X}$.
By the mapping of the incidence relation, we have $a, b, c \in \mathbf{Pt}$ and $\mathbf{Cont}(a, l)$, $\mathbf{Cont}(b, l)$, $\mathbf{Cont}(c, l)$. Moreover, $\neg\mathbf{C}(\mathbf{a}, \mathbf{b})$, $\neg\mathbf{C}(\mathbf{a}, \mathbf{c})$, and $\neg\mathbf{C}(\mathbf{b}, \mathbf{c})$ by C-D and the fact that a, b, c are indivisible points (by CD-E1). Finally, $l \in \mathbf{L}$.

Altogether, these properties satisfy the antecedent of OMT-E1, so that we can conclude $\mathbf{Btw}(l, a, b, c) \vee \mathbf{Btw}(l, a, c, b) \vee \mathbf{Btw}(l, b, a, c)$. By our definition of $\mathbf{B}_{\mathfrak{I}}$ we immediately conclude that one of $\mathbf{B}(a, b, c)$, $\mathbf{B}(a, c, b)$, or $\mathbf{B}(b, a, c)$ must hold—our desired consequence.

Direction (b): $B(a, b, c) \vee B(b, c, a) \vee B(c, a, b) \to a \neq b \wedge a \neq c \wedge b \neq c \wedge \exists l[L(l) \wedge a * l \wedge b * l \wedge c * l]$.
Assume $\mathbf{B}(a, b, c) \vee \mathbf{B}(b, c, a) \vee \mathbf{B}(c, a, b)$ holds for arbitrary $a, b, c \in \mathbf{X}$. Then by the mapping of $\mathbf{B}_{\mathfrak{I}}$, we have $a =_{\dim} b =_{\dim} c \prec_{\dim} l$ and $\mathbf{Btw}(l, a, b, c) \vee \mathbf{Btw}(l, a, c, b) \vee \mathbf{Btw}(l, b, a, c)$ for some $l \in \mathbf{L}_{\mathcal{M}}$. Then by B-A1, $a \neq b \wedge a \neq c \wedge b \neq c$. Moreover, by OMT-A3 $l \in \mathbf{Con}_{\mathcal{M}}$ and thus by OMT-A1 we obtain $\mathbf{Cont}(a, l)$, $\mathbf{Cont}(b, l)$, and $\mathbf{Cont}(c, l)$. And finally, by CD-E1, L-D only points are of a lower dimension than lines, thus $a, b, c \in \mathbf{Pt}_{\mathcal{M}}$. Then, by the mapping of the incidence relation $*$, we get $\langle a, l\rangle, \langle b, l\rangle, \langle c, l\rangle \in *_{\mathfrak{I}}$, thereby satisfying the desired consequence:$a \neq b \wedge a \neq c \wedge b \neq c \wedge l \in \mathbf{L} \wedge a * l \wedge b * l \wedge c * l$.

(O.4): $L(l) \wedge a * l \wedge b * l \wedge c * l \wedge p * l \wedge p \neq a \wedge p \neq b \wedge p \neq c \wedge B(a, p, b) \to [B(b, p, c) \wedge \neg B(a, p, c)] \vee [\neg B(b, p, c) \wedge B(a, p, c)]$.
Assume $l \in \mathbf{L}$, $\langle a, l\rangle, \langle b, l\rangle, \langle c, l\rangle, \langle p, l\rangle \in *_{\mathfrak{I}}$, $p \neq a$, $p \neq b$, $p \neq c$, and $\mathbf{B}(a, p, b)$.

Then by the mapping of the incidence relation $*$ from the statement of the theorem, we must have $a, b, c, p \in \mathbf{Pt}_{\mathcal{M}}$ as well as $\langle a, l\rangle, \langle b, l\rangle, \langle c, l\rangle, \langle p, l\rangle \in \mathbf{Cont}_{\mathcal{M}}$. Moreover, as in the proof of direction $\rightarrow$ of O.3, we must have $\neg\mathbf{C}(p, a)$, $\neg\mathbf{C}(p, b)$, and $\neg\mathbf{C}(p, c)$. Hence by OMT-E1, the points p, b, and c must be orderable within l, i.e., one of $\mathbf{Btw}(l, p, b, c)$, $\mathbf{Btw}(l, b, c, p)$, or $\mathbf{Btw}(l, c, p, b)$ holds. We consider these three cases separately.

Case (a): Assume $\mathbf{Btw}(l, p, b, c)$.

By our other assumption $\mathbf{B}(a, p, b)$, some line m such that $\mathbf{Btw}(m, a, p, b)$ exists. By OMT-A1 and B-A1, we get $\mathbf{Cont}(p, m)$, $\mathbf{Cont}(b, m)$, and $p \neq b$ and from $\mathbf{Btw}(m, a, p, b)$ we get $\mathbf{Cont}(p, l)$, $\mathbf{Cont}(b, l)$, and $p \neq b$. Applying PL-A2 (only one line can contain two distinct points) lets us immediately conclude $l = m$. We thus have $\mathbf{Btw}(l, p, b, c)$ and $\mathbf{Btw}(l, a, p, b)$, which imply $\mathbf{Btw}(l, a, p, c)$ by B-A4. Then $\mathbf{B}(a, p, c)$ follows from the mapping of $\mathbf{B}_{\mathfrak{I}}$.

From $\mathbf{Btw}(l, p, b, c)$ we also get $\mathbf{B}(p, b, c)$ and thus $\neg\mathbf{B}(b, p, c)$ by O.2, which we proved earlier. Hence $\neg\mathbf{B}(b, p, c) \wedge \mathbf{B}(a, p, c)$ is satisfies and thus O.4 is satisfied in this case.

Case (b): Assume $\mathbf{Btw}(l, b, c, p)$.

Analogue to Case (a) we obtain $\mathbf{Btw}(l, b, c, p)$ and $\mathbf{Btw}(l, a, p, b)$. Together they imply $\mathbf{Btw}(l, a, p, c)$ by B-A5. The remainder of the proof is analogue to Case (a), so that O.4 is satisfied in this case.

Case (c): Assume $\mathbf{Btw}(l, c, p, b)$.

Then $\mathbf{Btw}(l, b, p, c)$ by B-A2 and thus $\mathbf{B}(b, p, c)$ by the mapping of $\mathbf{B}_{\mathfrak{I}}$.

Suppose we had $\mathbf{B}(a, p, c)$.
Recall that we initially also assumed $\mathbf{B}(a, p, b)$. Then a line m exists such that $\mathbf{Btw}(m, a, p, b)$. Together with $\mathbf{Btw}(l, b, p, c)$, $l =$ follows by PL-A2. By B-A2, the symmetric equivalents $\mathbf{Btw}(l, c, p, b)$ and $\mathbf{Btw}(l, a, p, b)$ hold as well.

Now by our supposition $\mathbf{B}(a, p, c)$ and PL-A2, we would conclude $\mathbf{Btw}(l, a, p, c)$. This would imply $\neg\mathbf{Btw}(l, a, p, b)$ or $\mathbf{Btw}(l, c, p, b)$, in contradiction to our earlier findings $\mathbf{Btw}(l, c, p, b)$ and $\mathbf{Btw}(l, a, p, b)$. Thus our supposition $\mathbf{B}(a, p, c)$ was wrong and we must have $\neg\mathbf{B}(a, p, c)$. Together with $\mathbf{B}(b, p, c)$ O.4 holds then in this case as well.

The cases are trivially exhaustive by OMT-E1, hence in any case O.4 is satisfied.

We have proven that O.1 – O.4 are satisfied for B as ternary betweenness relation and that $\langle \mathbf{X}, \mathbf{I}, *, \dim\rangle$ is an incidence geometry. Hence, $\langle \mathbf{X}, \mathbf{I}, *, \dim, \mathbf{B}\rangle$ is a weak ordered incidence geometry. □

For the converse direction, we prove that any weak ordered incidence geometry defines a model of $OMT_{\text{3D}-\text{lin}}$ in the expected way.

Theorem 10.10. *Any weak ordered incidence geometry $\langle \mathbf{X}, \mathbf{I}, *, \dim, \mathbf{B}\rangle$ defines a model $\mathcal{M}$ of $OMT_{\text{3D}-\text{lin}}$ with $\mathbf{M} = \mathbf{Pt} \cup \mathbf{L} \cup \mathbf{Pl} \cup \{\text{ze}\}$, $\mathbf{Pt}_{\mathcal{M}} = \mathbf{Pt} \subseteq \mathbf{X}$, $\mathbf{L}_{\mathcal{M}} = \mathbf{L} \subseteq \mathbf{X}$, $\mathbf{Pl}_{\mathcal{M}} = \mathbf{Pl} \subseteq \mathbf{X}$, and $\text{ze} \notin \mathbf{X}$ such that for all $x, y, z, v \in \mathbf{M}$,*

$$\langle x, y\rangle \in \mathbf{Cont}_{\mathcal{M}} \iff x = y \text{ or } \langle x, y\rangle \in *_{\mathfrak{I}} \text{ or } \big(x \in \mathbf{L}, y \in \mathbf{Pl} \text{ and there exist distinct } p, q \in \mathbf{Pt} \text{ with } \langle p, x\rangle, \langle q, x\rangle, \langle p, y\rangle, \langle q, y\rangle \in *_{\mathfrak{I}}\big), \text{ and}$$

$$\langle r, x, y, z\rangle \in \mathbf{Btw}_{\mathcal{M}} \iff x, y, z \in \mathbf{Pt}, r \in \mathbf{L}, \text{ and } \langle x, r\rangle, \langle y, r\rangle, \langle z, r\rangle \in *_{\mathfrak{I}} \text{ and } \langle x, y, z\rangle \in \mathbf{B}_{\mathfrak{I}}.$$

Proof. Because $\langle \mathbf{X}, \mathbf{I}, *, \dim \rangle$ is an incidence geometry, it defines a model $\mathcal{M}$ of $CODI_{\text{plp}-\text{lin}}$ by Theorem 10.8 with the desired extension of *Cont*. We extend $\mathcal{M}$ to a structure $\mathcal{M}' = \mathcal{M} \cup \mathbf{Btw}_{\mathcal{M}}$ wherein $\mathbf{Btw}_{\mathcal{M}}$ is defined as above and wherein $\mathbf{ZEX}_{\mathcal{M}} = \{\text{ze}\}$ with ze $\notin \mathbf{X}$. We must show that $\mathcal{M}'$ is a model of $OMT_{3\text{D}-\text{lin}}$. Because we left the extension of all relations apart from *ZEX* and *Btw* unchanged, all axioms of $CODI_{\text{plp}-\text{lin}}$ are still satisfied—it requires to only check that ze behaves correctly, which is straightforward since it is in no relation to any other entity in the domain.

Hence it suffices to prove all axioms of $OMT_{3\text{D}-\text{lin}}$ that are not axioms of $CODI_{\text{plp}-\text{lin}}$; these are: Int-A1 – Int-A4, Dif-A1 – Dif-A4, Z-A1, B-A1 – B-A6, OMT-A1 – OMT-A3, and OMT-E1. For the axioms B-A1 – B-A6 and OMT-E1 we only refer to the corresponding axioms in weak ordered incidence geometries; the details are essentially the reverse of the proofs for O.1 – O.4 in Theorem 10.9.

(Z-A1): $\exists x[ZEX(x)]$.

By ze $\in \mathbf{ZEX}_{\mathcal{M}}$.

(OMT-A3): $Max(x) \rightarrow Con(x)$.

By the definition of the domain $\mathbf{X}$, all entities except for ze are maximal in their dimension, i.e., no entity can be a proper part of some entity. Hence no entity can have a proper part, that is all entities are minimal as well. Thus $Con(x)$ (by Con-D) is trivially satisfied for all $x \in \mathbf{X}$.

(OMT-A1): $Btw(r, x, y, z) \rightarrow Con(r) \wedge Cont(x, r) \wedge Cont(y, r) \wedge Cont(z, r)$.

By the mapping of $\mathbf{Btw}_{\mathcal{M}}$, $\mathbf{Btw}(r, x, y, z)$ requires $r \in \mathbf{L}$, $x, y, z \in \mathbf{Pt}$, and $\langle x, r\rangle, \langle y, r\rangle, \langle z, r\rangle \in *_{\mathfrak{I}}$. Together those are mapped to $\mathbf{Cont}(x, r)$, $\mathbf{Cont}(y, r)$, and $\mathbf{Cont}(z, r)$, while $r \in \mathbf{Con}_{\mathcal{M}}$ follows by OMT-A3.

(OMT-A2): $Btw(r, x, y, z) \wedge Cont(v, r) \wedge Con(v) \wedge C(v, x) \wedge C(v, z) \rightarrow C(v, y)$.

Recall that the domain $\mathbf{X}$ contains only points, lines, planes, and the zero region, and by the definition of $\mathbf{Btw}_{\mathcal{M}}$, only points can be between one another within lines. Then $\mathbf{Cont}(v, r)$ can only hold when v is a point or $v = r$, since only one line can exist through a point by I.2a. If $v = r$, $\mathbf{C}(r, y)$ follows trivially $y * r$, which we obtain from the definition of $\mathbf{Btw}(r, x, y, z)$.

Suppose v is a point. Then v must be connected to x, i.e., it must share an entity with x that is contained in v. That can only be v itself, because no nonmaximal entities of the dimension of points exists in the domain $\mathbf{X}$. Hence, v cannot be a point.

(Int-A1) – (Int-A4): Follow directly from the fact that all nonzero entities are minimal (compare proof of OMT-A3). Hence the intersection between two distinct points is the zero entity; the intersection of a point with a line or plane is the point itself; the intersection of two distinct lines is the zero region or a point; the intersection of a line with a plane is either zero, a point, or the line itself; and the intersection of two distinct planes is either empty, a point, or a line (recall that we did not include PLP-A2 as an axiom; allowing for higher-dimensional embedding spaces).

Then all of (Int-A1) – (Int-A4) are immediately provable.

(Dif-A1) – (Dif-A4): Also follow also from the fact that all nonzero entities are minimal (compare proof of OMT-A3). The difference $x - y$ where x is of a greater dimension than y is x itself (as by definition), and the difference $x - y$ between two distinct equidimensional entities is x. The difference $x - y$ where x is a point and y a line or point is zero of the point is incident with the line or plane, and the point x otherwise. If x is a line and y a plane, the difference is zero if two distinct points incident with the line are incident with the plane (by PLP-A4), and the line x otherwise.

Then all of (Dif-A1) – (Dif-A4) are immediately provable.

(B-A1): $Btw(r, a, b, c) \to a \neq b \neq c \neq a$.
Follows from the direction $\leftarrow$ of the implication in O.3.

(B-A2): $Btw(r, a, b, c) \to Btw(r, c, b, a)$.
Follows from O.1.

(B-A3): $Btw(r, a, b, c) \to \neg Btw(r, a, c, b)$.
Follows from O.2.

(B-A4): $Btw(r, x, a, b) \wedge Btw(r, a, b, y) \to Btw(r, x, a, y)$.
Follows from O.7 which we proved for weak ordered incidence geometries in Lemma 10.1.

(B-A5): $Btw(r, x, a, b) \wedge Btw(r, a, y, b) \to Btw(r, x, a, y)$.
Follows from O.8 which we proved for weak ordered incidence geometries in Lemma 10.1.

(B-A6): $Btw(r, a, x, b) \to \neg Btw(r, a, x, c) \vee \neg Btw(r, b, x, c)$.
Follows from O.4.

(OMT-E1): $L(r) \wedge Pt(x) \wedge Pt(y) \wedge Pt(z) \wedge Cont(x, r) \wedge Cont(y, r) \wedge Cont(z, r) \wedge \neg C(x, y) \wedge \neg C(x, z) \wedge \neg C(y, z) \to [Btw(r, x, y, z) \vee Btw(r, x, z, y) \vee Btw(r, y, x, z)]$.
Follows from the direction $\to$ of the implication in O.3.

Consequently, the extended model $\mathcal{M} \cup \mathbf{Btw}_{\mathcal{M}'}$ with $\mathbf{M'} = \mathbf{M} \cup \{\text{ze}\}$ satisfies all axioms of $OMT_{3\text{D}-\text{lin}}$. □

Thus, every weak ordered incidence geometry with its domain extended by a zero entity is definably equivalent to a model of $OMT_{3\text{D}-\text{lin}}$. As an example of such a model, we have used the model finder Paradox3 to verify that the model of *WOIG* given in Figure 10.2 with an extra zero entity is indeed a model of $OMT_{3\text{D}-\text{lin}}$. The input is specified in omt/consistency/omt_3d_lin_nontrivial.clif. As future work, we could automate or at least partially automate the proofs of the axioms of $OMT_{3\text{D}-\text{lin}}$ from the axiomatization of weak ordered incidence geometry, *WOIG* (**Question 4**).

All more restricted incidence geometries also define models of $OMT_{3\text{D}-\text{lin}}$. For example, once we define line segments we could express the Pasch axiom (compare I.6 in Section 10.1.3) in an extension of $OMT_{3\text{D}-\text{lin}}$. Together with an axiom enforcing that all lines are dense orders of points with no first or last point, we can then reconstruct ordered incidence geometries—defined in Definition 10.12—as a theory in the hierarchy *OMT*. The latter axiom is the one that enforces the models to be continuous geometries, while $OMT_{3\text{D}-\text{lin}}$ admits both discrete geometries, and thus also finite geometries, and continuous geometries as models.

Again, we can express the relationship between weak ordered incidence geometries and $OMT_{3\text{D}-\text{lin}}$ also purely in terms of theory interpretations: the theory $OMT_{3\text{D}-\text{lin}}$ interprets the theory *WOIG* because the translations of all axioms of *WOIG* into the language of *OMT* are provable from $OMT_{3\text{D}-\text{lin}}$ as Theorem 10.9 essentially shows.

$OMT_{3\text{D}-\text{lin}}$ does not faithfully interpret *WOIG* because

$$OMT_{3\text{D}-\text{lin}} \models \exists x \forall y [\neg Inc(x, y)]$$

because a zero region exists, whereas

$$WOIG \not\models \exists x \forall y[\neg Inc(x,y)].$$

WOIG extended by the sentence I.Z will, however, be faithfully interpreted by OMT_{3d-lin} because then every model of *WOIG* $\cup$ I.Z can be definably expanded to a model of OMT_{3d-lin}. This is a simple adaption of Theorem 10.10.

(I.Z) $\forall y\big[\neg(\text{ze} * y) \wedge [\dim(y) = \dim(\text{ze}) \rightarrow y = \text{ze}]\big]$
(an entity ze not incident with any other entity and of its own type exists)

Axiom Set 10.13: Extension axioms I.Z of *WOIG*.

Even though Theorems 10.9 and 10.10 apply to k-partite ordered incidence geometries with any finite k, incidence and betweenness are restricted to the first three partitions **Pt**, **L**, and **Pl**. We have not verified the adequacy of the multidimensional betweenness relation for capturing higher-dimensional spaces. How to adequately verify whether $OMT_{\downarrow}$ is a suitable model for higher-dimensional ordered incidence geometry is left as a challenge for the future (**Challenge 6**). One approach to tackle this problem requires altering the definition of a weak ordered incidence geometry to include axioms governing the order among higher-dimensional entities in a way analogue to how general n-dimensional betweenness geometry has been defined in [Has58]. Then we could try to construct an appropriate extension of $OMT_{\downarrow}$ that interprets such an n-dimensional betweenness geometry.

10.3.4 Definability of other notions of spatial order

While our quaternary betweenness relation is extremely general, often the only necessary kind of betweenness is the one among points, as in all the (weak) ordered incidence geometries we considered or as in the interpretation of betweenness used for points on oriented curves by Kulik and Eschenbach in [KE99]. In fact, we should be able to reconstruct the theory from [KE99] as an extension of $OMT_{\downarrow}$, since the primitives in [KE99] are incidence, points, oriented curves, and a ternary relation of precedence of two points on a oriented curve.

However, to capture an order among nonintersecting lines or curves, such as the order over a set of parallel streets, the broad interpretation of our multidimensional betweenness relation comes handy. For example, we can express that

$$Btw(\text{Toronto}, \text{SpadinaAve.}, \text{UniversityAve.}, \text{YongeSt.})$$

holds in the map in Figure 10.1(a) without having to explicitly reference the order over street intersections along a common intersecting street, such as King Street. Especially in finite models, such kind of betweenness may not be definable: suppose not all street intersections in the map in Figure 10.1(a) are modelled as points due to incomplete information or information abstraction, then $Btw(\text{Toronto}, \text{SpadinaAve.}, \text{UniversityAve.}, \text{YongeSt.})$ may not be definable in terms of the order over points along a line.

In continuous ordered incidence geometries, betweenness among points is sufficient to define other higher-dimensional notions of order, such as the betweenness for line segments that is captured by the

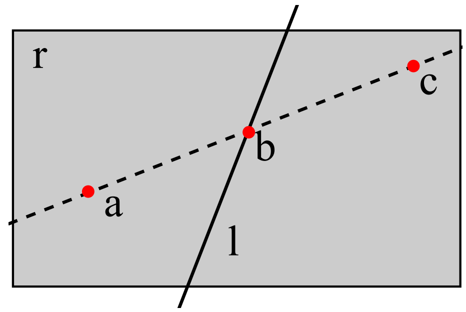

Figure 10.12: In $OMT_{\downarrow}$ we can capture the relation of a line l being between the two points a and c directly as $\mathbf{Btw}(r, a, l \cdot r, c)$ or as $\mathbf{Btw}(r, a, l, c)$ if r is a plane containing l. In classical ordered incidence geometry we have to state this in a round-about way: 'there exists a point b collinear with a and bc and incident with l such that $\mathbf{B}(a, b, c)$, compare [BS60]. Notice that this classical definition depends on the *line axiom*, I.2, whereas our definition does not depend upon the corresponding axioms PL-A2 and PL-A3. The trade-off is that *Btw* is needed as a primitive, quaternary relation.

Pasch axiom. This definability relies on the fact that all lines are straight, i.e., that any two points uniquely define a line. Borsuk and Szmielew [BS60], for example, define a line l to be in between two points a and c if and only if the unique line defined by a and c intersects l in a point b such that $B(a, b, c)$, see Figure 10.12 for an example. In our multidimensional version, we can express this more directly as $Btw(r, a, l, c)$, meaning that in the plane r, l lies between a and c, i.e., l separates a from c.

Our chosen intended interpretation of $Btw(r, a, b, c)$ as 'any self-connected entity wholly within r that intersects a and c also intersects b' is topologically rather strict. In human descriptions of space, we use the preposition 'between' often in less strict senses. For example, in the map in Figure 10.8 we would want to say that Keele Street is in between Dufferin Street and Jane Street in the area called 'Westend'—despite its inconsistency with the strict interpretation. Many such weaker notions of betweenness are definable relations in $OMT_{\downarrow}$. For example, we could define *weak betweenness* for three streets a, b, and c within a common region r as: street b is in between street a and c if and only if

- within no part of r, $a \cdot v$ is in between $b \cdot v$ and $c \cdot v$ or $c \cdot v$ is in between $a \cdot v$ and $b \cdot v$ and
- there exists some part v of r such that $b \cdot v$ is in between $a \cdot v$ and $c \cdot v$ within v.

Axiomatically, we could express this relation through the formula

$$\forall v[P(v, r) \rightarrow \neg Btw(v, b \cdot v, a \cdot v, c \cdot v) \wedge \neg Btw(r, a \cdot v, c \cdot v, b \cdot v)] \wedge \exists v[P(v, r) \wedge Btw(v, a \cdot v, b \cdot v, c \cdot v)].$$

Other specialized kinds of betweenness that are definable include notions such as *enclosure* (e.g., any entity x nontangentially contained in y is enclosed by the boundary of y if such a boundary exists), betweenness as the order of the intersections with a common entity (e.g., Parkside Drive is in between Jane Street and Dufferin Street along Bloor Street in Figure 10.8.), or betweenness as an order over parts (e.g., lines not fully contained in a region r can still be ordered based on the parts, i.e., line segments, that are contained in r). A more comprehensive treatise of these different notions of betweenness is left as future work (**Challenge 7**).

10.3.5 Definability of notions of convexity

Traditionally, convexity, convex or continuous line segments, and higher-dimensional equivalents such as convex polygons or polytopes, are geometric notions that are tightly coupled to betweenness. In linear spaces, it is well known that standard geometric convexity is definable, see for example [Cop98]. Likewise, a sufficiently restricted notion of betweenness lets us define the geometric notion of a line segments.

In $CODI_{\updownarrow}$ or any other theory in the *CODI* hierarchy, we cannot define whether the composite manifold represented by some domain entity is convex or not in a particular intended structure. Though we can give a definition of a relation of convex in the language of *CODI*, any structure in the class of intended models in which some composite manifold is not convex is represented by a model of $CODI_{\updownarrow}$ which can also be interpreted as a structure in which the composite manifold is convex. In other words, the language of the *CODI* hierarchy is simply not expressive enough to define the intended relation of convexity, just as equidimensional meoreotopologies are not expressive enough to define convexity [Coh+97a; Pra99]. For this reason, the basic RCC theory, for example, has been supplemented by a primitive unary function conv that assign each entity to its convex hull [Coh+97a; RCC92]. We will use a similar notion of convex hull, denoted as ch in Chapter 11.

Now that we have extended *CODI* by an additional primitive notion of betweenness resulting in the hierarchy *OMT*, can we define convexity of manifolds in the language of *OMT*? The answer is not straightforward, it is an exercise in reverse mathematics: what axioms are necessary to define the intended notion of convexity of manifolds? Since we weakened the axioms of ordered incidence geometry, we can no longer take for granted that convex line segments and convex regions in general are definable as in, e.g., linear ordered incidence geometries [BTBI87]. It turns out that definability of convexity in the language of *OMT* depends on the specific theory, not just the primitive language of its hierarchy. In linear ordered incidence geometries, the geometric notion of convexity for entities is easily definable as

$$\text{convex}(r) \leftrightarrow \forall x, y, z[x * r \wedge z * r \wedge B(x, y, z) \rightarrow y * r].$$

Commonly, this is expressed using the defined concept of a line segment as "a figure [i.e., a spatial entity that is not a point] is convex if and only if for any points x and y contained in the figure, then the line segment $\overleftrightarrow{xy}$ is also contained in the figure" [BS60]. Line segments are defined as sets of points, denoted by $\overleftrightarrow{xz}$, such that a point y is in the segment $\overleftrightarrow{xz}$ if and only if y is in between x and z. But such a definition depends on axioms of ordered incidence geometry that are not all assumed or provable in $OMT_{\downarrow}$. Even though we can give definitions of convexity and line segments, there are models of $OMT_{\downarrow}$ in which entities are deemed convex by that definition even though they may represent intended structures in which the corresponding entities are not convex. To restrict the definition of convex to the intended interpretation, we require the following three assumptions made by ordered incidence geometry. Firstly, it assumes that two points uniquely define a straight line segment, which is not true unless we include the *line axiom* (PL-A2, PL-A3). Secondly, it assumes points to form total dense orders: any pair of distinct points forces other points before, in between, and after the pair to exist on the same line, and all points orderable on a single line must be orderable in the entire space O.3. Only once we restrict $OMT_{\downarrow}$ to $OMT_{\mathrm{3D-lin}}$ we enforce total orderability of points by including OMT-E1. Thirdly, ordered incidence geometry usually assumes a three-dimensional space, i.e., the extension axioms PLP-E1 – PLP-E3 must be satisfied. As a result the following two different notions of convexity for manifolds in $\mathbb{R}^n$ coincide.

Let $\mathfrak{M}^n$ be a complex manifold and let ${\rm MF_1}^m \in \mathfrak{M}^n$ be an atomic or composite manifold with $m \leq n$.

Convexity of an entity in its own dimension: We say that MF_1 is *convex in its own dimension* if and only if for any manifold ${\rm MF_2}^m \in \mathfrak{M}^n$ with $\Sigma\mathrm{MF}_1 \subseteq \Sigma\mathrm{MF}_2$, any two points $x, y \in {\rm MF_1}^m$, and any line segment $\overleftrightarrow{xy} \in \mathrm{MF}_2$, we have $\overleftrightarrow{xy} \in \mathrm{MF}_1$. The idea is the following: an entity is *convex in its own dimension* if it can be flattened out by topologically-invariant transformations into a convex set of points. For example, a two-dimensional area with a hole that is contained in some larger, two-dimensional area without that hole would not be convex in its own dimension, no matter whether it is curved or not, because it will always maintain a genus of 1 within that larger area. But a piece of paper that is folded in three-dimensional space is convex as long as it is rectangular and thus convex when flattened out.

Convexity in space: We say that MF_1 is *complex in the space* if and only if for any manifold ${\rm MF_2}^i \in \mathfrak{M}^n$ with $i \leq n$ and $\Sigma\mathrm{MF}_1 \subseteq \Sigma\mathrm{MF}_2$, any two points $x, y \in {\rm MF_1}^m$, and any line segment $\overleftrightarrow{xy} \in \mathrm{MF}_2$, we have $\overleftrightarrow{xy} \in \mathrm{MF}_1$. The idea here is that entities of codimension > 0 that are curved in a higher dimension are no longer convex, because there exist line segments in the higher-dimensional space that connect two points in the curved entity but that are not wholly contained in the curved entity. For example, a curve located in a two-dimensional space is not convex, its convex hull would be a two-dimensional area.

The first interpretation may be closer to the spirit of our general axiomatization. For example, any single point but also any set of points is always convex, a one-dimensional curve is only nonconvex if it disconnected, while the convex hull of a two-dimensional area is the convex hull in its embedding plane,

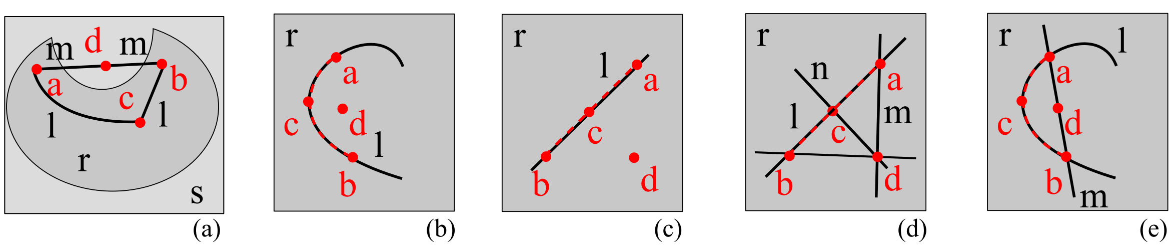

Figure 10.13: Examples of the different notions of convexity.
Let the spatial configuration in (a) be represented as a model of $OMT_\downarrow$ with domain $\mathbf{M} = \{a, b, c, d, l, m, r, v, v - r, \mathrm{ze}\}$. Then the two-dimensional region r is nonconvex in its dimension, because the line segment $\overleftrightarrow{ab}$ defined by points a and b that are contained in r, contains a third point d in a region x that is of the same dimension as r, but d is not contained in r. However, the linear feature l is convex in its dimension, because any line segment with a and b as endpoints that is in some linear feature containing l, is completely contained in l. But l is not convex in the space because there exists a straight line segment m, that is contained in v but that is not entirely contained in l.
Let the spatial configuration in (b) be represented as a model $\mathcal{M}$ of $OMT_\downarrow$ with domain $\mathbf{M} = \{a, b, c, d, l, r, \mathrm{ze}\}$. Then l is convex in its dimension and convex in space. It is convex in space because the only entity (apart from l itself) that contains l is r, whose only extra point d is not on a line with a and b. Effectively, the model $\mathcal{M}$ equally captures the configurations (b) and (c)—it can simply not distinguish them.
If in a model of $OMT_\downarrow$ the line axiom is satisfied, that is, every pair of points uniquely define a line, the model is realizable using only straight lines as in configuration in (d). If more than one line for a pair of given points may exist as in (e), we have no way to tell which of the two line segments $\overleftrightarrow{ab}$ is straight or curved—only one can be straight.

not in three- or higher-dimensional space. The second interpretation closely resembles how convex hulls are often used in two- or three-dimensional geometry: the convex hull of a on-dimensional curved line segment in a two-dimensional space (like on a piece of paper) would be two-dimensional, that is, of a higher dimension than the entity itself. Equally, in this interpretation three points in a three-dimensional space have a convex hull that is of zero dimensions if they coincide, one dimension if they are collinear, and two-dimensions otherwise. Four points may have a three-dimensional convex hull.

To formalize these two notions within the general theory $OMT_{\downarrow}$ requires a much deeper analysis beyond what can be achieved in a short thesis section. In particular, these notions of convexity must be relativized to the embedding entity just as we relativized betweenness to an embedding entity. Moreover, we need to give an adequate definition of line segments (see LS-D) in $OMT_{\downarrow}$ that accounts for the possibility that two points are contained in more than a single distinct line. Here we can just give an outline how such a definition could work. The use of $OMTB_{\downarrow}$ would precisify the axiomatization by allowing us to state that an endpoint is boundary-contained in a line part and not just contained as expressed in EP-D. The axioms LP-A1, LP-A2 and definitions LP-D, EPt-D. LS-D are very preliminary, they have not yet received the scrutiny as our previous axioms and are only included as a starting point for future work (**Question 6**). They are closely related to how [KE99] define curve segments. Notice that many other concepts, such as boundaries are definable if a relation of convexity is available as a primitive or defined relation. For more details on the expressiveness of convexity, see [Coh95; Dav06; Pra99].

(LP-D) $LP(l) \rightarrow \exists x[MinDim(x) \wedge x \prec_{\text{dim}} l]$ (linear part: essentially a one-dimensional feature)

(EPt-D) $EPt(l,p) \rightarrow LP(l) \wedge Pt(p) \wedge Cont(p,l)$ (p is an endpoint of the linear part l)

(LP-A1) $L(l) \rightarrow \neg\exists p[EPt(l,p)]$ (lines have no endpoints)

(LP-A2) $EPt(l,p) \wedge EPt(l,q) \wedge p \neq q \rightarrow \forall x[EPt(l,x) \rightarrow p = x \vee q = x]$
(linear parts have at most two endpoints; allows for line segments, rays, and lines)

(LS-D) $LS(l) \leftrightarrow LP(l) \wedge \exists p,q\big[EPt(l,p) \wedge EPt(l,q) \wedge p \neq q \wedge \forall x[Btw(l,a,x,b) \wedge Pt(x) \leftrightarrow Pt(x) \wedge Cont(x,l) \wedge x \neq p \wedge x \neq q]\big]$ (a line segment is a linear part with two distinct endpoints p and q such that all other points x in between p and q are contained in the line segment l and any point x contained in l is in between p and q within l)

Axiom Set 10.14: Definitions LP-D, EPt-D, LS-D and axioms LP-A1, LP-A2 as extension of $OMT_{\downarrow}$.

Note that in any extension of $OMT_{\downarrow}$ that allows discrete models, we additionally have to deal with the problem that the typical continuous understanding of convexity is not definably because line segments between two points may contain no other points and thus could be straight or curved. However, this is a minor problem, because such a model is homeomorphic to a space in which all minimal line segments (those that do not contain any points except the endpoints) are straight. Then an entity is convex if it is convex in its homeomorphic entity in which all minimal line segments are straight. We can apply this discrete understanding of convexity to both of the above two notions of convexity by only considered the points in the domain.

10.4 Summary

In this chapter we investigated how the multidimensional mereotopology that we introduced in earlier chapters is related to classical geometries. As our first result, Theorem 10.1 shows that a model $\mathcal{M}$ of *CODI* always has a substructure over the model's domain **M** that is a k-partite incidence structure—and also a k-partite point incidence structure by Corollary 10.1, where the extension $\mathbf{Inc}_{\mathcal{M}} \cap (<_{\mathbf{dim}})_{\mathcal{M}}$ defines the incidence structure's asymmetric, irreflexive incidence relation $*$. In the converse direction, Theorem 10.2 establishes that$k$-partite point incidence structures can be expanded to models of *CODI* in a natural way. Consequently, *CODI* faithfully interprets the theory of k-partite point incidence structures.

In a next step, we provided definitions of points and lines in *CODI* and extended *CODI* such that every line contains at least two points. Each model of the resulting theory $CODI_{\mathrm{pl}}$ has a substructure that is a line space (Theorem 10.3)—a two-dimensional incidence geometry—and every line space can be expanded to a model of $CODI_{\mathrm{pl}}$ (Theorem 10.4). The result extends to special classes of line spaces, namely semi-linear, linear, and affine spaces, which correspond to substructures of $CODI_{\mathrm{pl-slin}}$, $CODI_{\mathrm{pl-lin}}$, $CODI_{\mathrm{pl-aff}}$, respectively (Theorem 10.5 and 10.6). In terms of theories, the theories of line spaces and of semi-linear, linear, affine line spaces are faithfully interpreted in the theories $CODI_{\mathrm{pl}}$ ($CODI_{\mathrm{pl-slin}}$, $CODI_{\mathrm{pl-lin}}$, $CODI_{\mathrm{pl-aff}}$), respectively. We further extended $CODI_{\mathrm{pl}}$ to $CODI_{\mathrm{plp}}$, which also defines planes. Based on $CODI_{\mathrm{plp}}$, we lifted the two-dimensional results to three-dimensional incidence geometries. In particular, all linear incidence geometries (which contain a three-dimensional incidence structure) can be expanded to models of $CODI_{\mathrm{plp-lin}}$ (Theorem 10.8) and models of $CODI_{\mathrm{plp-lin}}$ have substructures that are incidence geometries (Theorem 10.7). Again, we can state that the theory $CODI_{\mathrm{plp-lin}}$ faithfully interprets the theory *IG* of incidence geometries. In the final part of Section 10.2, we identified the theory $CODI_{\mathrm{plp-g}}$ as a natural qualitative abstraction of three-dimensional incidence geometry in the primitive language of *CODI*.

Classical geometries are usually constructed from three primitive relations: incidence, betweenness, and congruence. While congruence introduces a metric—a reason why extensions of equidimensional mereotopologies with notions of congruence are capable of expressing full Euclidean geometry as shown in [BGM96; BM10]—betweenness does not. In only imposes an order over points or other, higher-dimensional entities and is thereby still of qualitative nature. Order plays a crucial role when representing spatial configurations, such as street maps, qualitatively without loosing critical knowledge. Therefore, it is an important and worthwhile undertaking to study notions of order in multidimensional mereotopology. It required us to first find a suitable multidimensional notion of order that works regardless whether the involved entities are curved or not. As such a relation, we proposed in Section 10.3 a quaternary primitive relation *Btw*, which relativizes the traditional ternary betweenness relation found in geometries to a reference entity, the local context. We adopted many of the axioms of geometric betweenness to the more general multidimensional setting, but had to leave out or weaken other axioms to suit the more general setting. As result, we obtained the theory *BTW*, which resides in a hierarchy of its own.

Subsequently, we axiomatized the interaction between *BTW* and the *CODI* theories to come up with $OMT_{\downarrow}$ as the basic theory of ordered multidimensional mereotopology in Section 10.3.2. See Figure 10.14 for a complete depiction of *OMT* hierarchy. In Section 10.3.3, we then established the relationship between $OMT_{\mathrm{3D-lin}}$, a combination of $OMT_{\downarrow}$ with $CODI_{plp-lin}$, and weak ordered three-dimensional incidence geometries: every weak ordered incidence geometry can be expanded to a model of $OMT_{\mathrm{3D-lin}}$ if the domain is supplemented by a zero entity (Theorem 10.10) and any model of $OMT_{\mathrm{3D-lin}}$

has a substructure that is a weak ordered incidence geometry (Theorem 10.9). This extends the earlier correspondence between $CODI_{\mathrm{plp-lin}}$ and three-dimensional incidence geometries by proving that the order required for weak ordered incidence geometries is definable using the multidimensional order defined in $OMT_{\mathrm{3D-lin}}$. Moreover, we established the following theory relationship: the theory of weak ordered incidence geometries, *WOIG*, is interpretable in $OMT_{\mathrm{3D-lin}}$. Faithful interpretability is only prevented by the missing zero region, which can be fixed by supplementing *WOIG* by the axiom I.Z. Generally speaking, the ordered multidimensional mereotopology $OMT_{\mathrm{3D-lin}}$ indeed reconstructs ordered incidence geometry in a more expressive language, which is equally capable of defining a qualitative analogue of ordered incidence geometry. The essential differences between ordered incidence geometry and ordered mereotopology $OMT_{\downarrow}$ is captured by the axioms necessary to extend $OMT_{\downarrow}$ to $OMT_{\mathrm{3D-lin}}$: OMT-E1 (three points are always orderable), the line axioms PL-A1 – PL-A3, the space axioms PLP-A1 – PLP-A4, and CD-E1 (indivisibility of points). An overview of the various theories that we introduced in this chapter to extend *CODI* towards geometries is given in Figure 10.14, while their relationships to incidence structures, incidence geometries, and ordered incidence geometries are summarized graphically in Figure 10.15.

As far we as we know, our previous work from [HG11b] and the work presented in this chapter, which builds on our previous work but significantly extends it, have been the first enquiries into ordered multidimensional mereotopology as a qualitative analogue to ordered incidence geometries. As often, this poses more questions than it answers. Important unanswered questions concern the completeness of the chosen axiomatization: are there other axioms of general betweenness that are not yet provable in $OMT_{\downarrow}$? Missing axioms may particularly arise from order between higher-dimensional entities, which we studied only on a preliminary level. Other important open questions concern the definability of closely related notions, in particular of the many relations humans use in everyday descriptions that involve some kind of spatial order, and of the closely associated geometrical notions of line segments, higher-dimensional equivalents, and convex regions. With regards to those questions, we offered some discussion in Sections 10.3.4 and 10.3.5, respectively, which can serve as starting points for future investigations (**Question 7**).

A separate issue concerns the integration of incidence geometries and ordered incidence geometries with the *CODI* and *OMT* theories. At the moment, we have faithful interpretations in one direction, meaning the *CODI* or *OMT* theories are logical extension (with the appropriate definitions) of (ordered) incidence geometries. However, eventually we would like to establish definable equivalences: which *CODI* and *OMT* theories are definably equivalent to the (ordered) incidence geometries? This would provide a much stronger integration of geometries with our qualitative spatial theories. We leave this task as future work (**Question 5**).

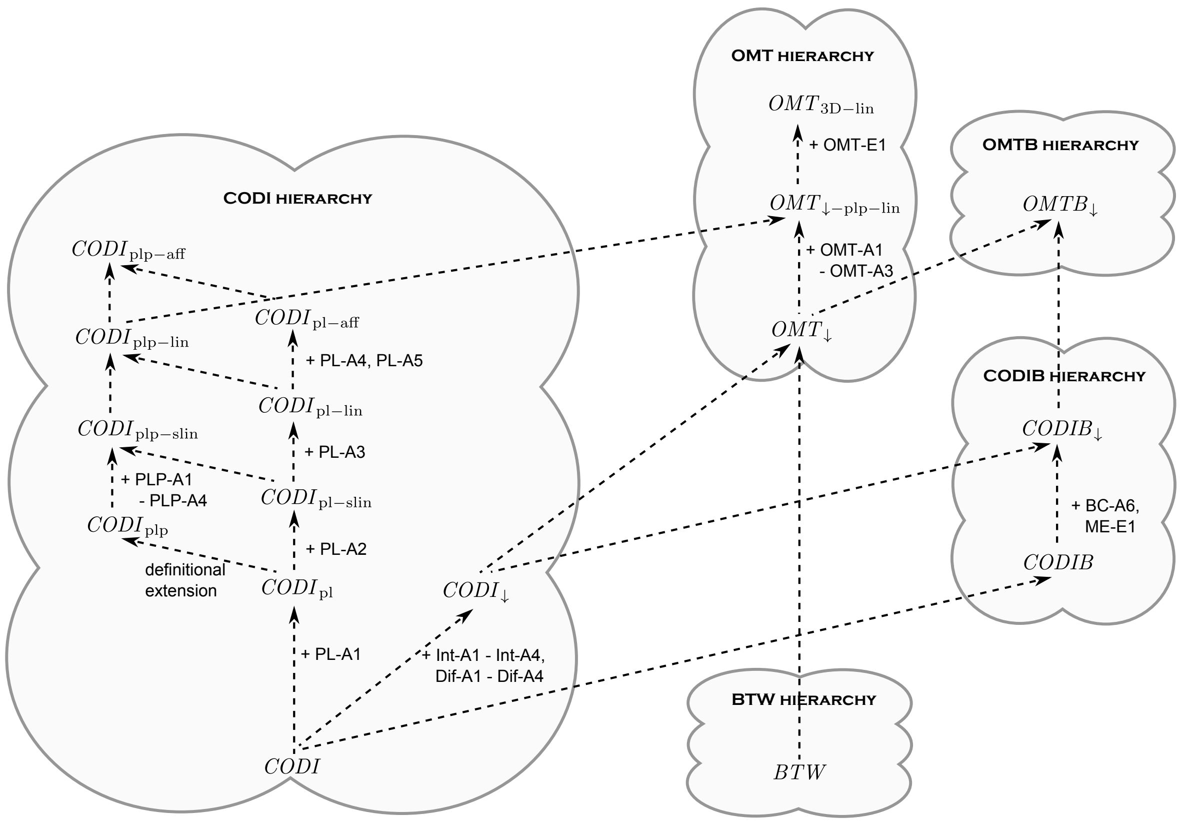

Figure 10.14: The relationship among the various theories that extend *CODI* by axioms constraining how three kinds of maximal entities—points, lines, and planes—interact. Once we add a theory of quaternary betweenness, which forms a hierarchy *BTW* by itself, we obtain the new hierarchy *OMT*, and—with an extension by the primitive relation of boundary-containment—the new hierarchy *OMTB*, which is not further explored in this thesis.

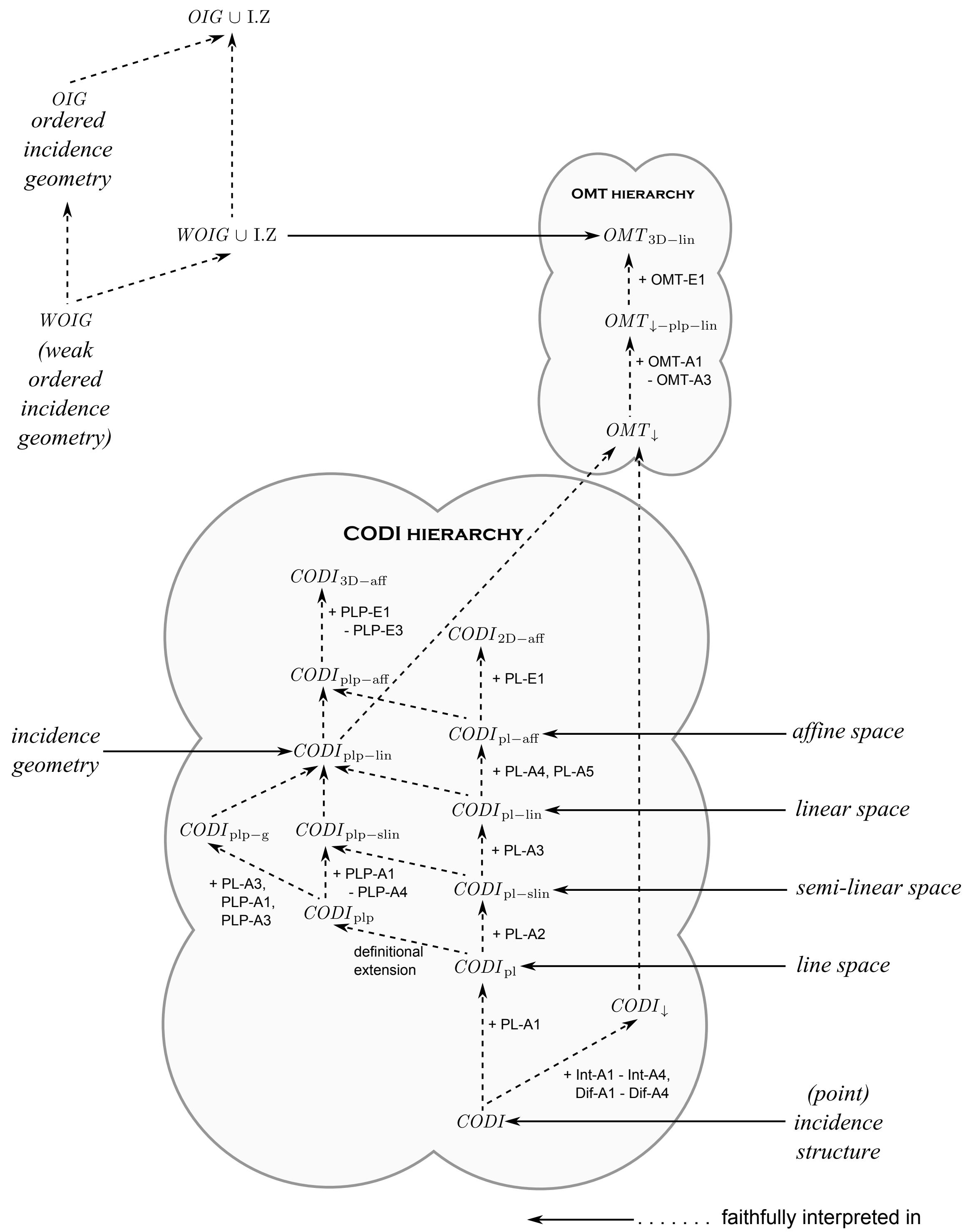

Figure 10.15: The relationships between the theories developed in Chapter 10 and (ordered) incidence structures and geometries. While extensions in *CODI* correspond to incidence structures and two- and three-dimensional incidence geometries, the theories in *OMT* generalize three-dimensional ordered incidence geometries.

Chapter 11

Modelling physical space: physical boundaries and physical voids[1]

Thus far in the thesis, we were concerned only with spatial regions from a purely mereotopological or mereogeometrical perspective. This characterizes abstract space, which we consider as a mathematical-logical construct. In that context, we introduced mereological closure operations, which help to formalize and talk about certain concepts such as intersections of space regions or differences between space regions, which are not always true to the physical reality of space. The closure operations introduce a lot of "artificial" entities which arguably do not correspond to any real physical entities, or the corresponding physical entities are not considered as physical entities per se, but are merely collections or parts of physical entities.

In this chapter, we present a way to utilize our axiomatization of abstract space to model physical space. We are especially interested in modelling two kinds of physical features. First, we discuss how we can model boundaries and in particular surfaces of physical objects and, secondly, we model physical voids that occur in rock formations and the water bodies that can be located in such physical voids. While the discussion of boundaries gives only a general idea of how thin and thick boundaries in abstract space can be used to model physical boundaries, the section on physical voids is much more detailed and fully axiomatized.

A key prerequisite for either task is to ground physical space in abstract space. We take the following approach. First, we add a new layer of enduring physical entities, which are completely disjoint from the abstract spatial regions we talked about so far. This layer intuitively contains physical entities such as rock formations, sediments, or various kinds of water bodies. Instead of devising a completely new ontology for physical space, we adopt a portion of the DOLCE ontology [Mas+03]. In particular we maintain the DOLCE categorization of so-called physical endurants, which encompasses all the physical entities we are interested in, into physical objects, matter, and features. Each physical endurant must be located in space, that is, it must occupy some nonzero region of the underlying abstract space. In the second step, we map entities located in physical space to their associated abstract spatial regions.

Subsequently, we can define physical boundaries as physical endurants whose regions are based on

[1] The work in this chapter, except for Section 11.3, is joint work with Boyan Brodaric. It has been previously published as [HB12] in the Proceedings of the 7th International Conference on Formal Ontologies in Information Systems, pp. 45-58, IOS Press. Copyright 2012, reprinted with permission from IOS Press.

the abstract boundaries that we axiomatized in Chapter 9. We will only sketch in Section 11.3 how this would work in principle when using DOLCE's categorization of physical endurants and point to the differences in comparison to other treatments of physical boundaries.

In Section 11.4 we will axiomatize the spatial character of physical voids. One piece still missing from our axiomatization of abstract space is the mereogeometric relation of convex hull. Even though we discussed how it becomes definable in a highly restricted version of ordered mereotopology *OMT* as defined in Chapter 10, we opt here to use convex hull as a primitive function, adopting the axioms from [Coh+97a; Don05] in Section 11.4.1. This mirrors the extension of equidimensional mereotopologies to equidimensional mereogeometries, compare [BM10], just in the multidimensional case. Finally, we show how we can use this new mereogeometrical axiomatization of abstract space to restrict the interpretation of the primitive relation of 'hosting a void' between physical endurants and physical voids. This generalizes holes as studied by Casati and Varzi [CV94] to voids. We also provide a classification of physical voids along three criteria: the internal connectedness of a void's host (Section 11.5.1), the connectivity of a void to other voids and to the exterior of its host (Section 11.5.2) and the scale of a void (Section 11.5.3). It should be noted that we are still not in a position to completely define voids; the identification of physical voids—like the identification of physical entities of interest to a certain domain or to humans–remains an important open issue.

The work in this chapter differs in its logical approach from the work in previous chapters. In the previous chapters, we were concerned with the expressiveness of theories as determined by their primitive languages as well as the restrictiveness of the axioms for theories in the same language. Both concerns were foundational in our analysis of the various spatial theories. Such a rigorous mathematical approach to ontology was only possible because we worked with a very small set of primitive notions, allowing us to closely examine those few relations and functions. Consequently, we have better understood the differences in expressiveness between the various theories. The development in this chapter links this mathematical-logical analysis to a more philosophical approach to ontology, which tries to capture a set of foundational categories much like the work of Chisholm [Chi96]. We reuse a portion of DOLCE [Mas+03] as upper-level ontology, which comprises a rich set of primitive notions, though only few of them are densely axiomatized or even defined in terms of others. Without a precise distinction between undefined and defined concepts in the DOLCE theory, the majority of concepts are treated as primitives, compare Figure 11.9 at the end of the chapter[2]. The expressiveness of the DOLCE ontology and its models are not well-understood—they are simply too diverse to admit a rigorous characterization. But by using the multidimensional theory of abstract space developed in the earlier chapters, we are able provide a more rigorous axiomatization for some of the spatial concepts in DOLCE.

This chapter serves multiple purposes. We offer a rigorous spatial characterization of the DOLCE categories of physical entities, we give a concrete example how our theories of abstract space can be used to model physical reality, and we show how the mathematical model-theoretical approach to ontology design complements the philosophical top-down approach. Moreover, we specifically contribute to the understanding and formalization of physical voids. Finally, we demonstrate in Section 11.6 that the distinction between different kinds of voids are indeed relevant in the domains of hydrogeology and hydro-ontology: we can give more precise characterizations of the difference between a surface water body and a ground water body, we can formalize necessary properties of so-called hydro-rock bodies,

[2]For this reason, all axioms pertaining to physical endurants are labelled as PED-Ax. Only in Sections 11.4 and 11.5 we use definitions to introduce new categories of voids.

rock bodies that store water, and we can define concepts such as a reservoir. Thereby, this chapter also gives a glimpse of the applicability of our abstract theories of space to real-world domains. The chapter, except for the discussion in Section 11.3, is joint work with Boyan Brodaric, previously published in [HB12]. This chapter here makes minor corrections and the distinction between internal and external voids is clarified. We also offer more a more in-depth discussion of some of the modelling choices, in particular concerning a suitable multidimensional axiomatization of convex hulls, and of the limitations of our approach.

11.1 Physical endurants

In addition to abstract spatial regions, we now also consider physical entities located in what we call physical space. Instead of developing our own ontology of physical entities, we reuse a portion of the upper-ontology DOLCE [Mas+03]. In particular, we reuse the DOLCE taxonomy of so-called *physical endurants* shown in Figure 11.1 and we reuse relevant relations as much as possible, though those are sparsely, if at all, axiomatized in DOLCE. Throughout, we are only concerned with endurants[3] that have some physical location. Examples of physical endurants from hydrogeology are rock formations, sediments, and various kinds of water bodies such as rivers, lakes, groundwater, aquifers, and wells. Other DOLCE categories, in particular perdurants such as processes, plus nonphysical entities, are out of scope because they are not only of spatial nature but intrinsically of spatio-temporal nature.

We maintain the symbol PED from DOLCE to denote physical endurants. DOLCE distinguishes three disjoint categories of physical endurants (PED-A1, PED-A2): physical objects POB (e.g., a body of water), amounts of matter M (e.g., the water that constitutes a body of water), and features F (e.g., the water surface), compare the top-level specialization in Figure 11.1. Moreover, the physical objects considered here, especially the hydrogeological entities, fall all into the DOLCE category of nonagentive physical objects $NAPO$, i.e., physical objects that do not act by themselves or pursue goals (PED-A3), that refines the category POB.

11.1.1 Physical features: dependent places and relevant parts

Physical features, such as material surfaces, must not be confused with abstract spatial features, such as abstract boundaries, which we treat as nonphysical entities in our theory. For brevity we use the term *feature* to denote a *physical feature*. Physical features, F, depend on physical endurants as their *host*, a term used by [CV94] and in the OWL version of DOLCE, but not axiomatized in the first-order version of DOLCE. Because features are inextricably linked to their host, they are best captured by a binary *hosts* relation. Only features are hosted (PED-A4) and all features are hosted (PED-A5). Moreover, the *hosts* relation is asymmetric (PED-A6). In DOLCE, physical features are specialized as relevant parts RPF such as bumps, edges, surfaces, boundaries, or dependent places DPF such as shadows and holes (PED-A7, PED-A8).

[3] An essential criteria for something to be an endurant is that its parts are wholly present at any point in time. This requires any proposition about an endurant to be relative to a timepoint or an interval of time. However, for simplicity we omit the time reference here completely; our axiomatization can be thought of as capturing a static view of the domain at a fixed timepoint.

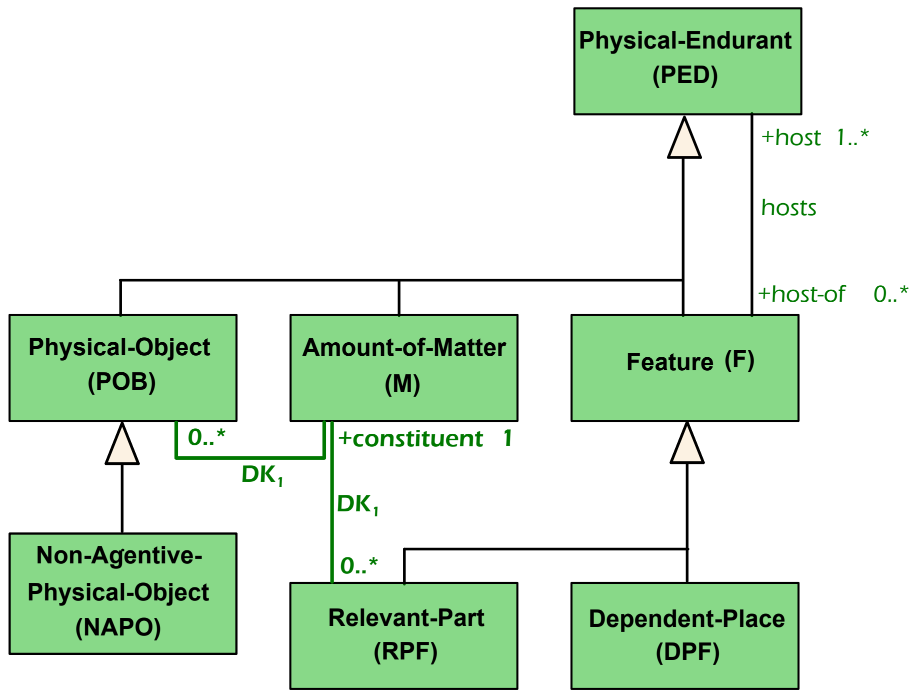

Figure 11.1: UML diagram of the DOLCE category physical endurant, its specializations and relations.

11.1.2 Matter

DOLCE further distinguishes a physical object from its constituent matter, for example, a body of rock or a "piece of rock" from its constituent rock matter, or a water body from its constituent water. The specific amount of matter that constitutes a physical object can change over time, for example, the water in a surface water body changes due to evaporation, precipitation, and water flow, but the object itself endures wholly at every timepoint of its existence: the water body that forms Lake Ontario persists even when its water matter is completely exchanged. To capture the constituency of a physical object by matter over time, DOLCE uses a general constitution relation $K(x, y, t)$ meaning 'the amount of matter x constitutes y at time t'. This constitution relation can be used for multiple levels of granularity such as a rock body being constituted by some amount of matter, the mineral grains, but also being constituted by the molecules and atoms of that amount of matter. Here, we will only deal with a single level of constituency and therefore use DOLCE's more specialized relation $DK(x, y, t)$ denoting 'x directly constitutes y at time t'. Because we only deal with a static view here, we can drop the time reference, obtaining a binary constituency relation. We further limit the binary version of direct constituency to the first step in physical scale, i.e., the direct constituency of a physical object to its matter, such as a rock to granite, as opposed to, e.g., the atomic or molecular constituency of an amount of matter, such as some granite to its chemical composition. We call this relation *primary constituency* $DK_1(x, y)$ with the intended interpretation of 'x identifies the entire matter that the physical object or relevant part feature y is constituted of' (PED-A9). Thus the matter of any physical object or relevant part must be unique (PED-A10). Moreover, every physical object or relevant part feature is constituted by some matter (PED-A11), whereas dependent places can be immaterial (holes) or material (my back yard).

(PED-A1) $PED(x) \leftrightarrow POB(x) \lor M(x) \lor F(x)$ (physical objects, amount-of-matters, and features are exhaustive categories of physical endurants)

(PED-A2) $\neg[POB(x) \land M(x)] \land \neg[POB(x) \land F(x)] \land \neg[M(x) \land F(x)]$ (physical objects, amount-of-matters, and features are disjoint categories)

(PED-A3) $NAPO(x) \rightarrow POB(x)$ (nonagentive physical objects specialize physical objects)

(PED-A4) $hosts(x, y) \rightarrow PED(x) \land F(y)$ (only features are hosted; the host being a *PED*)

(PED-A5) $F(x) \leftrightarrow \exists y[hosts(y, x)]$ (features must be hosted)

(PED-A6) $hosts(x, y) \rightarrow \neg hosts(y, x)$ (*hosts* relation is asymmetric)

(PED-A7) $F(x) \leftrightarrow RPF(x) \lor DPF(x)$ (features are either relevant parts or dependent places)

(PED-A8) $\neg RPF(x) \lor \neg DPF(x)$ (relevant parts and dependent places are disjoint categories)

(PED-A9) $DK_1(x, y) \leftrightarrow M(x) \land [POB(y) \lor F(y)]$ (primary constitution: direct constitution of an object or relevant-part feature by matter)

(PED-A10) $DK_1(x, y) \land DK_1(z, y) \rightarrow x = z$ (an object's primary constituent matter is unique)

(PED-A11) $POB(y) \lor RPF(y) \rightarrow \exists x[DK_1(x, y)]$ (all physical objects and relevant parts are constituted by some matter, i.e., are material)

Axiom Set 11.1: Axioms PED-A1 – PED-A11 of the DOLCE theory of physical endurants *PED*.

We define the theory of physical endurants as

$$PED = \{\text{PED-A1} - \text{PED-A11}\}.$$

In *PED*, the primary constitution relation DK_1 is irreflexive and asymmetric (PED-T1, PED-T2).

(PED-T1) $\neg DK_1(x, x)$ (DK_1 irreflexive)

(PED-T2) $DK_1(x, y) \rightarrow \neg DK_1(y, x)$ (DK_1 asymmetric)

Lemma 11.1. *PED* $\models$ *{PED-T1, PED-T2}*

Now, we have reviewed all the necessary categories of physical entities and the relationships among them. Most importantly, we have distinguished physical objects and their relevant parts, which are constituted of matter, from the matter they are constituted of and from dependent places that are not constituted of matter.

11.2 Physical endurants' location in space

If we extend one of our theories of abstract space from the *CODI* or *CODIB* hierarchies with the theory of physical endurants *PED*, we can assign all physical entities a location in space. For this purpose we reuse the region function $\mathrm{r}(x)$ from layered mereotopology [Don03; Don05; DS03][4]. The range of the region function defines the category of *spatial region* S (S-A2, S-A3), maintaining the DOLCE terminology. We refer to entities in the category S henceforth simply as *regions*. Note that the inverse of r may not be

[4]In the context of DOLCE, the function $\mathrm{r}(x)$ can be see as a function returning a *spatial quale* that is related to the entity x at the predetermined static time point by the *spatial location* quality.

a total function: not every region is occupied by an identifiable physical endurant. Consequently, there may be many more regions than physical endurants. But regions and physical endurants are strictly disjoint (S-A1).

In addition, we assert that all relations defined in previous chapters, in particular the primitive relations *Cont*, $<_{\text{dim}}$, *ZEX*, and *BCont* from the hierarchies *CODI* and *CODIB*, refer only to spatial regions (S-A4 – A-A7). This could equally be expressed for our quaternary betweenness relation *Btw* introduced in Chapter 10, but is unnecessary for this chapter.

(S-A1)	$\neg PED(x) \lor \neg S(x)$	(physical endurants and regions are disjoint)
(S-A2)	$S(\mathrm{r}(x))$	(the range of the region function are spatial regions)
(S-A3)	$S(x) \leftrightarrow x = \mathrm{r}(x)$	(spatial regions are their own region)
(S-A4)	$Cont(x,y) \rightarrow S(x) \land S(y)$	(*Cont* is a relation between spatial regions)
(S-A5)	$x <_{\text{dim}} y \rightarrow S(x) \land S(y)$	($<_{\text{dim}}$ is a relation between spatial regions)
(S-A6)	$ZEX(x) \rightarrow S(x)$	(the zero region is a spatial region)
(S-A7)	$BCont(x,y) \rightarrow S(x) \land S(y)$	(*BCont* is a relation between spatial regions)

Axiom Set 11.2: Axioms S-A1 – S-A7 of the theory *SPACE* of abstract and physical space.

While we introduced S-A4 – S-A7 as a way to clarify that the primitive relations only apply to regions, more is needed to combine the theory *PED* with, for example, a theory from the hierarchies *CODI* or *CODIB*. Assume we want to reuse the axiomatization of $CODIB_{\updownarrow}$ from codib/codib_updown.clif as theory of abstract space. Then, we must restrict the scope of all quantifiers in the axioms and definitions of $CODIB_{\updownarrow}$ to the type S to express that the sentences only apply to regions but not to physical endurants. Common Logic [Int07], the logical language in which we implemented all presented theories, provides a mechanism to import a set of axioms as a *module* whose quantifiers are restricted to the entities of some named category[5]. For example, we can define $CODIB_{\updownarrow}$ as a module S and then import that module into a theory consisting of the axioms S-A1 – S-A7 using the cl-imports statement. The semantic would then restrict all quantifiers in the axioms from $CODIB_{\updownarrow}$ to entities in S, that is, any universally quantified formulas $\forall v[\alpha(v)]$ would be relativized to $\forall v[S(v) \rightarrow \alpha(v)]$ and all existentially quantified formulas $\exists v[\alpha(v)]$ would be relativized to $\exists v[S(v) \land \alpha(v)]$, compare [NH12].

Because no available tool supports the relativization of Common Logic modules, we manually relativized all axioms from $CODIB_{\updownarrow}$ to the category S for the purposes of this chapter. The relativized axioms are included as theory files in the folder named space and form the basis for the theory *SPACE* that we will define in a moment. We still include S-A4 – S-A7 to assert that the relations *Cont*, $<_{\text{dim}}$, *BCont*, and *ZEX only* apply to regions. All relations and functions defined in $CODIB_{\updownarrow}$ through explicit definitions (in the sense of Definition 2.4) are then automatically restricted to regions. Other relations and functions, especially the closure functions $\cdot$, $-$, and $+$, still apply to non-regions as well, but the values for non-regions are of no interest.

For convenience, we often want to apply the abstract spatial relations directly to physical endurants to talk about the relationships of their occupied spatial regions. For example, we want to be able to express that the region of a physical endurant x is contained in the region of another physical endurant y. While can express this as $Cont(\mathrm{r}(x), \mathrm{r}(y))$, our original presentation in [HB12] introduced a general

[5] We rely on the corrected semantic for the cl-module statement from [NH12].

containment relation that equally applies to regions and physical endurants, denoted as $x \subseteq y$ meaning that 'the region of x is contained in the region of y' (S-D1). Equally, we can define $x \subset y$ (S-D2), $x \subseteq_{\mathrm{P}} y$ (parthood for non-regions, S-D3), $x \subset_{\mathrm{P}} y$ (proper parthood for non-regions, S-D4), $x <_{\mathrm{S-dim}} y$ (S-D5), $x \leq_{\mathrm{S-dim}} y$ (S-D6), $x >_{\mathrm{S-dim}} y$ (S-D7), $x \geq_{\mathrm{S-dim}} y$ (S-D8), $x =_{\mathrm{S-dim}} y$ (S-D9), and $x \prec_{\mathrm{S-dim}} y$ (S-D10). In a similar way, we could give physical equivalents of all the relations we used in the previous chapters. However, for clarity of presentation, we will not use any of those definitions subsequently and instead use the region function explicitly if we want to apply spatial relations to physical endurants.

(S-D1) $x \subseteq y \leftrightarrow \mathit{Cont}(\mathrm{r}(x), \mathrm{r}(y))$ (spatial inclusion for regions and non-regions)

(S-D2) $x \subset y \leftrightarrow x \subseteq y \wedge y \nsubseteq x$ (proper spatial inclusion)

(S-D3) $x \subseteq_{\mathrm{P}} y \leftrightarrow P(\mathrm{r}(x), \mathrm{r}(y))$ (spatial parthood)

(S-D4) $x \subset_{\mathrm{P}} y \leftrightarrow x \subseteq_{\mathrm{P}} y \wedge y \nsubseteq_{\mathrm{P}} x$ (proper spatial parthood)

(S-D5) $x <_{\mathrm{S-dim}} y \leftrightarrow \mathrm{r}(x) <_{\mathrm{dim}} \mathrm{r}(y)$ (x is of lower spatial dimension than y)

(S-D6) $x \leq_{\mathrm{S-dim}} y \leftrightarrow \mathrm{r}(x) \leq_{\mathrm{dim}} \mathrm{r}(y)$ (x is of lower or equal spatial dimension than y)

(S-D7) $x >_{\mathrm{S-dim}} y \leftrightarrow \mathrm{r}(x) >_{\mathrm{dim}} \mathrm{r}(y)$ (x is of greater spatial dimension than y)

(S-D8) $x \geq_{\mathrm{S-dim}} y \leftrightarrow \mathrm{r}(x) \geq_{\mathrm{dim}} \mathrm{r}(y)$ (x is of greater or equal spatial dimension than y)

(S-D9) $x =_{\mathrm{S-dim}} y \leftrightarrow \mathrm{r}(x) =_{\mathrm{dim}} \mathrm{r}(y)$ (x and y are of equal spatial dimension)

(S-D10) $x \prec_{\mathrm{S-dim}} y \leftrightarrow \mathrm{r}(x) \prec_{\mathrm{dim}} \mathrm{r}(y)$ (x is of the next-lower spatial dimension than y)

Axiom Set 11.3: Definitions S-D1 – S-D10 of *SPACE*.

DOLCE assumes physical endurants to be "real" in the sense that they are bodily, which means that physical endurants occupy a spatial region of maximal dimension (S-A8) and are constituted by matter (as already expressed in PED-A11), with the constituting matter occupying a subregion of the endurant's region (S-A9). For example, in 3D space, every physical entity must be 3D. While we can talk about lower-dimensional abstractions, such as lines, in abstract geometrical space (the category *S*), those abstractions have no physical equivalent in *PED*. With the help of the region function, we can now precisify the difference between relevant part features and dependent place features. The main difference is that *RPF*s are constituted by their host's matter and are therefore a spatial part thereof (S-A10), while *DPF*s cannot overlap their host or their host's matter (S-A12). The matter of relevant parts must further occupy a subregion of the region occupied by the relevant part's host (S-A11).

We define the theory of abstract and physical space as

$$\mathit{SPACE} = \mathit{CODIB}_{\updownarrow} \cup \mathit{PED} \cup \{\text{S-A1} - \text{S-A12}\}.$$

By replacing $\mathit{CODIB}_{\updownarrow}$ with a weaker or stronger axiomatization of abstract space in the definition of *SPACE*, we can adjust the necessary and allowable restrictiveness and expressiveness of the included theory of abstract space.

In *SPACE* we can prove that the region function r is idempotent (S-T1) and that every physical endurant must occupy an nonzero region (S-T2). That is consistent with how *PED* is used in DOLCE [Mas+03].

(S-T1) $\mathrm{r}(\mathrm{r}(x)) = \mathrm{r}(x)$ (region function idempotent)

(S-A8) $PED(x) \rightarrow MaxDim(\mathrm{r}(x))$ (physical endurants occupy regions of codimension 0)

(S-A9) $DK_1(x, y) \rightarrow P(\mathrm{r}(x), \mathrm{r}(y))$

(a physical endurant's constituting matter occupies a subregion of the physical endurant's region)

(S-A10) $hosts(x, y) \rightarrow [RPF(y) \leftrightarrow P(\mathrm{r}(y), \mathrm{r}(x))]$

(the region occupied by a relevant part feature is part of its host's region)

(S-A11) $hosts(x, y) \wedge RPF(y) \wedge DK_1(m, x) \wedge DK_1(n, y) \rightarrow P(\mathrm{r}(n), \mathrm{r}(m))$

(the region of a relevant part's matter is part of the region of its host's matter)

(S-A12) $hosts(x, y) \rightarrow [DPF(y) \leftrightarrow \neg PO(\mathrm{r}(y), \mathrm{r}(x))]$

(the region of a dependant place feature does not partially overlap its host's region)

Axiom Set 11.4: Axioms S-A8 – S-A12 of *SPACE*.

(S-T2) $PED(x) \rightarrow \neg ZEX(\mathrm{r}(x))$ (no zero physical endurant exists)

Lemma 11.2. *SPACE* $\vDash$ *{S-T1, S-T2}*

Proof. **(S-T1)** $\mathrm{r}(\mathrm{r}(x)) = \mathrm{r}(x)$.

$S(\mathrm{r}(x))$ by S-A1 and thus $\mathrm{r}(\mathrm{r}(x)) = \mathrm{r}(x)$ by S-A2.

(S-T2) $PED(x) \rightarrow \neg ZEX(\mathrm{r}(x))$.

We have $\exists x[MinDim(x)]$ by D-A6 and thus $\exists x[\neg ZEX(x)]$ by D-D6. Let $a \in \mathbf{M}$ be such an entity with $a \notin \mathbf{M}_{\mathcal{M}}$. Then $a >_{\mathbf{dim}} \mathrm{ze}$ for any $\mathrm{ze} \in \mathbf{ZEX}_{\mathcal{M}}$.

Now let $b \in \mathbf{M}$ be an entity such that $b \in \mathbf{PED}_{\mathcal{M}}$, then $\mathbf{r}(b) \in \mathbf{MaxDim}_{\mathcal{M}}$ by S-A8. Thus $\mathbf{r}(b) \geq_{\mathbf{dim}} a >_{\mathbf{dim}} \mathrm{ze}$ for any $\mathrm{ze} \in \mathbf{ZEX}_{\mathcal{M}}$. Thus $\mathbf{r}(b) \notin \mathbf{ZEX}_{\mathcal{M}}$ by D-A4.

□

Note that all features captured by the category F must, by PED-A1 together with S-A8, be of maximal dimension and thereby of the same dimension as their hosts. This implies that boundaries, understood as features that are either part of their host or at least dependent on their host, must be material, that is, boundaries must be of the same dimension as their hosts, and thereby bulky. Nevertheless, we can capture both bulky and bodiless boundaries in the theory *SPACE* as we discuss in the next section. Bodiless boundaries are just not features in the DOLCE sense.

11.3 Boundaries of physical endurants

As we discussed in Section 3.4, surfaces, as the most interesting boundaries, can be understood as the top-most or outer-most but still bulky, material layer of a physical endurant (Stroll's *P-surfaces*), or they can be considered as bodiless, immaterial surfaces (Stroll's *A-surfaces*) that demarcate an object from its surrounding[6]. Either kind of surfaces can be described as being dependent on a single host or being only present where two physical endurants meet (such as a physical object and the surrounding air). In Chapter 9 we defined a bodiless and a bulky abstract notion of boundaries in *CODIB* as two coexistent, nonexclusive conceptions. But how can we use them to model physical boundaries, and especially physical surfaces, of physical endurants?

[6]We use the term *physical* in this section referring to some entity present in physical space, whereas the term *material* refers to entities that are constituted by matter. Thus, entities can be physical and immaterial at once.

Abstract lower-dimensional regions readily describe bodiless, immaterial physical boundaries: the function boundary(r(x)) describes the space occupied by x's bodiless, immaterial boundary. Equally, the thick boundary thickboundary(r(x)) of the region of a physical endurant x can be used to capture the bulky region of space occupied by x's material surface. The relation of relative dimension allows us to axiomatically distinguish those two types of boundaries as physical entities that are of the same or a lower dimension than the bounded physical entity. Naturally, we want to treat either kind of boundaries as features, though the DOLCE category of features, F, only allows material features of maximal dimension, and thus no lower-dimensional, bodiless physical boundaries. We could remedy this by either retracting S-A8 to allow a special category of lower-dimensional features—maybe called *physical non-extended features*— which are different from the other physical features in RPF and DPF. This conception of bodiless physical boundaries is used, for example, in the Basic Formal Ontology (BFO [Smi+12]). If we insist that all features are extended, that is, maximal-dimensional, we can talk about abstract boundaries only on the level of abstract space—another viable option. Note that a non-extended feature of a physical endurant can host other features of its own dimension, such as a hole in an abstract surface, which is a tunnel-cavity as described in [CV94]. But it can also host features of again lower dimension, such as an edge or a singularity on a surface.

If we deny the existence of lower-dimensional, immaterial physical boundaries altogether, we only need to model material physical boundaries. If we follow DOLCE's ontological assumptions, the material physical boundary of a single object falls into the category of relevant part features (RPFs) with the following properties:

- They are bulky, i.e., of the same dimension as the object they bound;
- They are constituted by some matter (just like all relevant part features);
- They cannot be shared by two nonoverlapping physical endurants.

The third point immediately follows from the first two because if such a boundary were shared by two physical endurants, those two endurants would share a physical part and thereby overlap.

Another important kind of boundaries are what we will call *internal physical boundaries*. They are probably best described by bodiless spatial regions in the underlying abstract space [Fle96; HG09]. Internal boundaries may be potential or actual[7] (bona-fide) boundaries within the matter of a physical endurant. Actual internal boundaries exhibit an physical discontinuity: they either separate two material parts that constitute the physical endurant or they describe a physical discontinuity like an internal crack or a fissure, see [HG09] for examples. While internal boundaries that arise from parts of a physical endurant can be described the same way as boundaries of physical objects in general, internal boundaries that arise from internal, material defects such as cracks, are more difficult to deal with. They are different from physical holes (or physical voids) in that they cannot be of the same dimension as their host, rather, they exhibit the same characteristics as two touching objects (like a stack of paper): the material is the same to either side, but it is disconnected; the disconnect is not a space free of matter (at least not at the macroscopic level) but is caused only by a spatial disconnect on the atomic level, resulting in a lack of binding forces. For example, the matter of two stacked pieces of paper is only coincidentally adjacent. This is a problem, for which Fleck [Fle96] suggested using a $\mathbb{R}^n$ model of space from which all boundary points (no matter whether internal or outer boundary points) are deleted. In our theory we can easily

[7] The distinction between potential and actual boundaries is taken from Kachi's classification in [Kac09].

describe such discontinuities on the level of abstract space, but need the before-mentioned category of physical non-extended feature to capture their physical reality. Another option is to actually described the difference in on the microscopic scale, using a model similar to what we propose in Section 11.5.3 for distinguishing voids in objects from voids in its matter.

Two material objects may only share bodiless immaterial boundaries but not bulky material boundaries. In that way puzzles such as cutting a piece of paper, which pose the question of which of the newly created two pieces of paper owns the newly created boundary, are only superficially paradox—they always shared an internal, potential, bodiless boundary; by cutting along that boundary the potential boundary becomes actual but stays bodiless and therefore shareable by both newly created pieces of paper. At the same time, the cutting process creates two new bulky, material boundaries, one for each new piece of paper. Those two material boundaries are the edges of the resulting two pieces of paper, they are not identical. In other words, physical contact is nothing else but sharing a lower-dimensional boundary. Two material bodies with bulky material boundaries can share a bodiless boundary, in which case they are said to be in contact, even though their bulky, material boundaries are always distinct.

The bodiless notion of a boundary in abstract space also plays an important role in modelling fiat boundaries. Fiat boundaries are by definition artificial and thus best described in abstract space only, because they are not dependent on any physical discontinuity anyway. Because fiat boundaries may have absolutely no physical justification, it makes no sense to model them in physical space, instead, they should completely reside in abstract space. Consequently, we see no use in introducing a physical equivalent of a fiat boundary.

In this section, we have sketched how our work on boundaries in abstract space relates to the various notions of physical boundaries that have been discussed in previous work. The sole purpose of this section was to outline how physical, and in particular material, boundaries can be modelled consistently in our approach to space that clearly delineates the mathematical construct of abstract space from the space of physical objects humans experience. Of course, our discussion will seem unsatisfactory to any reader that disagrees with separating two levels of space. However, we think this separation allows us to be more precise about the different conceptions of boundaries—which all have some use in human language—and allows us to capture them in a single consistent theory. The remainder of this chapter is dedicated to physical voids—another kind of physical features—and their spatial structure.

11.4 Physical voids

A physical void is, intuitively, a physical feature whose region is not occupied by the region of its physical host. Examples are holes of maximal dimension such as caves or canyons, which stand in contrast to cracks that we choose to model as lower-dimensional non-extended features [HG09]. According to Casati & Varzi [CV94] a hole can only exist if a physical endurant's region is strictly smaller than its convex hull. The same applies to a *void*. We will capture this necessary condition by first axiomatizing the *convex hull* as a unary function that assign any spatial region a region that represents its convex hull. For a non-region, i.e., a physical endurant, it assigns the convex hull of its occupied region. Subsequently, we can define an abstract spatial notion of a *void region*—a spatial region not occupied by a specific physical entity (its *host region*) but inside the host's convex hull. All physical voids hosted by a particular physical endurant must then occupy a subregion of the sum of all its void regions, that is, a void must be located within its host's convex hull bot not overlap its host's region.

To define the convex hull function, we could build on a sufficiently restricted theory from the hierarchy *OMT*, which would be able to define convexity and, if it is a theory of dense space, also the convex hull function. But because we have not fully investigated which theory in the *OMT* hierarchy would be suitable to define the intended interpretation of the convex hull function (see the discussion in Section 10.3.5), we use a primitive function here, similar to how it has been used in extensions to equidimensional mereotopologies [Coh+97a; Don05]. This also allows us to consider convex hulls only for regions that arise from physical endurants, avoiding the tricky issues of convexity of regions of nonmaximal dimension. We still need the expressivity of *CODIB* to restrict the convex hull primitive as much as possible. In particular, we rely on the definitions of tangential containment and internal self-connectedness. How the language extension defined in this section fits into *OMT* remains to be investigated in the future (**Challenge 2**).

11.4.1 Convex hull

A convex hull function has been used successfully to discriminate several types of spatial "containment" or "inside" relations, such as "geometrical inside" from "topological inside" [Coh+97a; Don05]. While this earlier work on axiomatizations of convex hulls has been based on equidimensional mereotopology, we have to account for the multidimensionality of our underlying spatial theory. But we only axiomatize it to the extent necessary for capturing void spaces and, ultimately, voids in physical endurants. Because all physical endurants are of maximal dimension, we need to only concern ourselves with convex hulls of regions of maximal dimension. Since a region is always contained in its convex hull, the convex hull has no smaller dimension than the region itself. Therefore, we do not have to worry about the dimension of convex hulls for regions of nonmaximal dimension and thus do not have to treat the two differing notions of convex hulls discussed in Section 10.3.5 separately. The axioms we present here apply to both notions, except for CH-A9 and CH-A11, which explicitly make assertions only about regions of maximal dimensions.

The convex hull operation is a purely abstract spatial concept, applicable to space regions and resulting in another space region. For that reason, we introduce the axioms CH-A1 and CH-A2 asserting that the range of the convex hull function is always a region (CH-A1) and that the convex hull is always defined in terms of occupied regions (CH-A2). We further include CH-A3 that requires any convex hull to be internally self-connectedness. The axioms CH-A4 to CH-A13 are adopted from [Coh+97a], the original axiom numbering is included in parentheses as reference. Donnelly [Don05] also included CH-A4 to CH-A7, but did not mention CH-A8 – CH-A13. The convex hull function is idempotent (CH-A4), a property that is not provable from CH-A9, contrary to what is claimed in [Coh+97a]. Any nonzero region is contained in its convex hull (CH-A5) and any non-closed regions is tangentially contained in its convex hull (CH-A6). If a region x is contained in region y, the convex hull of x must also be contained in the convex hull of y (CH-A7). Regions that have the same bounded convex hull must be connected (CH-A8); the key to this axiom is that the convex hulls must be identical, not just one contained in the other. Note that CH-A8 differs from Cohn's version [Coh+97a] in that it requires the convex hulls to be bounded manifolds, expressed by the condition $\neg \mathit{Closed}(x)$[8]. We call a region *convex* if and only if it is its own convex hull.

A well-known geometric property requires convex regions to be closed under intersections, but it does only work as long as the convex hull of a region and the region itself are of the same dimension—as in our

[8]Thanks to Ernie Davis for pointing out that the conditions of the original axiom (32) from [Coh+97a] were insufficient.

first understanding of convex hulls discussed in Section 10.3.5. See Figure 11.2 for counterexamples when this is not the case. However, the intersection of two convex regions is convex if both convex regions are of maximal dimension (CH-A9). Moreover, the convex hull of a region y removed of a nontangentially contained region x must be nonconvex (CH-A10), when both x and y are of maximal dimension, the removed region must also be in the convex hull (CH-A11). The convex hull operation is also monotone under sums (CH-A12). Finally, CH-A13 states that if a region y is superficially connected to regions x and z without x being connected to z, and the sums $x + y$ and $y + z$ are both convex, then y is convex as well; this axiom is discussed in more detail in [Coh+97a].

(CH-A1) $S(\mathrm{ch}(x))$ (convex hull ch is a spatial region)

(CH-A2) $\mathrm{ch}(x) = \mathrm{ch}(\mathrm{r}(x))$ (ch defined with respect to occupied regions)

(CH-A3) $\neg ZEX(x) \wedge \rightarrow ICon(\mathrm{ch}(x))$ (convex hull is internally self-connected)

(CH-A4) $MaxDim(\mathrm{r}(x)) \rightarrow \mathrm{ch}(ch(x)) = \mathrm{ch}(x)$
(28: convex hull is idempotent for entities of maximal dimension)

(CH-A5) $\neg ZEX(\mathrm{r}(x)) \rightarrow Cont(\mathrm{r}(x), \mathrm{ch}(x))$ (29: any region is contained in its convex hull)

(CH-A6) $\neg ZEX(\mathrm{r}(x)) \wedge \neg Closed(\mathrm{r}(x)) \rightarrow TCont(\mathrm{r}(x), \mathrm{ch}(x))$
(29: any non-closed nonzero region is tangentially contained in its convex hull)

(CH-A7) $Cont(\mathrm{r}(x), \mathrm{r}(y)) \rightarrow Cont(\mathrm{ch}(x), \mathrm{ch}(y))$ (30: containment monotone under convex hull)

(CH-A8) $\neg ZEX(\mathrm{r}(x)) \wedge \mathrm{ch}(x) = \mathrm{ch}(y) \wedge \neg Closed(x) \rightarrow C(\mathrm{r}(x), \mathrm{r}(y))$
(32: regions with identical convex hull are connected)

(CH-A9) $MaxDim(\mathrm{r}(x)) \wedge MaxDim(\mathrm{r}(y)) \wedge \mathrm{r}(x) = \mathrm{ch}(x) \wedge \mathrm{r}(y) = \mathrm{ch}(y) \rightarrow \mathrm{ch}(x) \cdot ch(y) = \mathrm{ch}(\mathrm{ch}(x) \cdot \mathrm{ch}(y))$
(33: the intersection of convex regions of maximal dimension is convex)

(CH-A10) $ICont(\mathrm{r}(x), \mathrm{r}(y)) \wedge \neg Closed(\mathrm{r}(y) - \mathrm{r}(x)) \rightarrow \mathrm{r}(y) - \mathrm{r}(x) \neq \mathrm{ch}(\mathrm{r}(y) - \mathrm{r}(x)))$
(35: the difference $y - x$ between y and interior-contained region x is nonconvex)

(CH-A11) $ICont(\mathrm{r}(x), \mathrm{r}(y)) \wedge MaxDim(\mathrm{r}(x) \wedge MaxDim(\mathrm{r}(y)) \rightarrow Cont(\mathrm{r}(x), \mathrm{ch}(\mathrm{r}(y) - \mathrm{r}(x)))$
(special case of 35: for regions of maximal dimension, the convex hull of the difference $y - x$ between y and interior-contained region x must contain x)

(CH-A12) $[\neg ZEX(\mathrm{r}(x)) \vee \neg ZEX(\mathrm{r}(y))] \rightarrow Cont(\mathrm{ch}(x) + \mathrm{ch}(y), \mathrm{ch}(\mathrm{r}(x) + \mathrm{r}(y))$
(31: the sum of the convex hulls of x and y is contained in the convex hull of their sum)

(CH-A13) $\mathrm{r}(x) =_{\mathrm{dim}} \mathrm{r}(y) =_{\mathrm{dim}} \mathrm{r}(z) \wedge SC(\mathrm{r}(x), \mathrm{r}(y)) \wedge SC(\mathrm{r}(y), \mathrm{r}(z)) \wedge \neg C(\mathrm{r}(x), \mathrm{r}(z)) \wedge \mathrm{r}(x) + \mathrm{r}(y) = \mathrm{ch}(\mathrm{r}(x) + \mathrm{r}(y)) \wedge \mathrm{r}(y) + \mathrm{r}(z) = \mathrm{ch}(\mathrm{r}(y) + \mathrm{r}(z)) \rightarrow \mathrm{r}(y) = \mathrm{ch}(y)$
(36: if x, y, and z are of equal dimension, x and y, and y and z are superficially connected, but x and z are disconnected, and $x + y$ and $y + z$ are convex, the shared region y is convex)

Axiom Set 11.5: Axioms CH-A1 – CH-A13 of the theory *SPCH*.

We define the extension of *SPACE* by the axioms CH-A1 – CH-A13 as the theory

$$SPCH = SPACE \cup \{\text{CH-A1} - \text{CH-A13}\}.$$

In the theory *SPCH*, we can prove that an entity that is not internally self-connected and not of minimal dimension cannot be convex (CH-T1), and the universal region must be convex, i.e., it is its own convex hull (CH-T2). Finally, it trivially follows that the convex hull of entities of maximal dimension is also

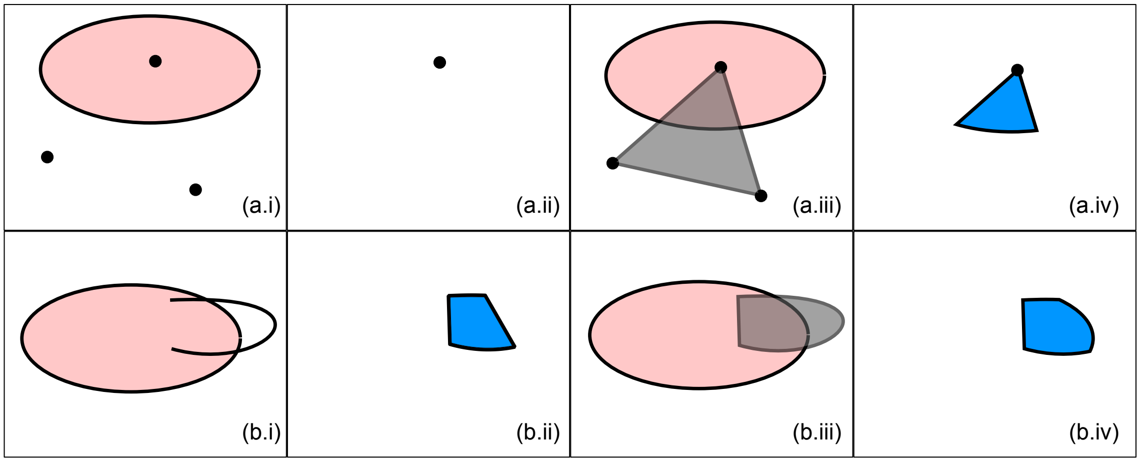

Figure 11.2: Configuration in which the convex hull of the intersection of two regions of nonmaximal dimension is not equivalent to the intersection of their convex hulls. In other words, the entities do not satisfy the consequent $\mathrm{ch}(x) \cdot \mathrm{ch}(y) = \mathrm{ch}(\mathrm{ch}(x) \cdot \mathrm{ch}(y))$ of CH-A9.
In (a), $\mathrm{ch}(x) \cdot \mathrm{ch}(y) = \mathrm{ch}(\mathrm{ch}(x) \cdot \mathrm{ch}(y))$ is only not satisfied for the interpretation of convex hull in space, i.e., for our second interpretation from Section 10.3.5. Consider the two entities in (a.i): an elliptic area (2D) and a set of points (0D). Their intersection is a single point whose convex hull is again the single point (a.ii), while the convex hull of the point set is a 2D entity (a.iii), whose intersection with the elliptic area is a convex 2D area (a.iv), not the point we obtained in (a).
In the example (b), $\mathrm{ch}(x) \cdot \mathrm{ch}(y) = \mathrm{ch}(\mathrm{ch}(x) \cdot \mathrm{ch}(y))$ fails for either interpretation of convex hulls. The horseshoe-shaped linear feature in (b.i) is convex in its own dimension. But the intersection with the ellipse resulting in two scattered linear features (not shown), which is not convex. The linear feature is not convex in space. The convex hull of its intersection with the ellipse, which is convex, results in the area shown in (b.ii). It is smaller than the area resulting from the intersection of the convex hull of the linear feature (see (b.iii)) with the ellipse, which results in the area shown in (b.iv).

of maximal dimension (CH-T3), which further implies that the convex hull of all physical endurants is of maximal dimension.

(CH-T1) $\neg ICon(\mathrm{r}(x)) \rightarrow \mathrm{r}(x) \neq \mathrm{ch}(x)$ (34: a not internally connected region is nonconvex)

(CH-T2) $\mathrm{ch}(\mathrm{u}) = \mathrm{u}$ (the universal region is its own convex hull)

(CH-T3) $MaxDim(\mathrm{r}(x)) \rightarrow \mathrm{r}(x) =_{\mathrm{dim}} \mathrm{ch}(x)$

(the convex hull of regions of maximal dimensions is also of maximal dimension)

Lemma 11.3. *SPCH* ⊨ *{CH-T1 – CH-T3}*

Proof. **(CH-T1)** $\neg ICon(\mathrm{r}(x)) \rightarrow \mathrm{r}(x) \neq \mathrm{ch}(x)$.

The following logical derivation immediately proves CH-T1:

$$\begin{aligned} \neg ICon(\mathrm{r}(x)) &\rightarrow \neg Con(\mathrm{r}(x)) \vee \exists y[PP(y,x)] && \text{(ICon-D)} \\ &\rightarrow \neg ZEX(\mathrm{r}(x)) && \text{(Con-T2)} \\ &\rightarrow ICon(\mathrm{ch}(x)) && \text{(CH-A3)} \\ &\rightarrow \mathrm{r}(x) \neq \mathrm{ch}(x) \end{aligned}$$

(CH-T2) $\mathrm{ch}(\mathrm{u}) = \mathrm{u}$.

By CH-A4, we have $Cont(\mathrm{u}, \mathrm{ch}(\mathrm{u}))$ from which by the definition of u (U-A1) and by C-A2 we immediately obtain $\mathrm{u} = \mathrm{ch}(\mathrm{u})$.

(CH-T3) $MaxDim(\mathrm{r}(x)) \to \mathrm{r}(x) =_{\mathrm{dim}} \mathrm{ch}(x)$.

Consider the following logical derivation:

$$\begin{aligned}
MaxDim(\mathrm{r}(x)) &\to MaxDim(\mathrm{r}(x)) \wedge \exists y[MinDim(\mathrm{r}(y))] && \text{(D-A6)}\\
&\to MaxDim(\mathrm{r}(x)) \wedge \exists y[\neg ZEX(\mathrm{r}(y)) \wedge \mathrm{r}(x) \geq_{\mathrm{dim}} \mathrm{r}(y)] && \text{(D-D6)}\\
&\to MaxDim(\mathrm{r}(x)) \wedge \neg ZEX(x) && \text{(D-A5)}\\
&\to MaxDim(\mathrm{r}(x)) \wedge Cont(\mathrm{r}(x), \mathrm{ch}(x)) && \text{(CH-A4)}\\
&\to MaxDim(\mathrm{r}(x)) \wedge \mathrm{r}(x) \leq_{\mathrm{dim}} \mathrm{ch}(x) && \text{(CD-A1)}\\
&\to \mathrm{r}(x) =_{\mathrm{dim}} \mathrm{ch}(x) && \text{(D-D5)}
\end{aligned}$$

□

11.4.2 The nature of voids

In order to define physical voids, we first define the abstract spatial notion of a *void region*, a spatial region not occupied by a specific physical entity (the host) but inside the host's convex hull (VS-D). A void region relies on a specific physical endurant x, its host, and must be disjoint from the host's occupied region $\mathrm{r}(x)$. Intuitively, the void regions of a given host are the regions in which voids can possibly be spatially located. Because void regions are abstract regions, we can talk about their intersections, differences, and sums in the same way as for all other regions. But in order for the resulting region to be a void region again, we must restrict the mereological operations to void regions of the same host region.

Physical voids (*voids* in short) are real physical entities that are located in a void region of some entity (V-A1). We use the relation of *hosting a void* (*hosts-v*) as a primitive relation between a void and its host, in the spirit of Casati and Varzi [CV94], who have used the primitive relation of *hosting a hole*. Thereby we generalize holes to voids, but do not address the open question concerning which void regions have physical void counterparts. We thereby do not define or identify which void regions are actually occupied by voids, we only give some necessary conditions for voids. A more thorough discussion of the challenges involved in identifying physical holes and voids is offered in [CV94]. The relation *hosts-v* specializes the *hosts* relation between physical endurants and hosted features (V-A1). But it is reasonable to assume that only void regions that are s-connected to their host qualify as the regions for voids (V-A1). We further distinguish simple from complex voids (V-D) depending on whether the void is internally connected ($\mathrm{V_S}$-D, $\mathrm{V_C}$-D). We require a complex void to be composed of simple voids, which represent the internally connected parts of the complex void (V-A2).

As an additional restriction, a void cannot be hosted by other voids (V-A3), though all other kinds of physical endurants may host voids. In particular, non-void dependent places, such as shadows, can host voids, e.g., there can be a hole in a shadow. V-A4 asserts that every void is hosted by some non-feature, that is, if a void is hosted by some feature, the host of that feature must also host the void. For example, a void hosted by a surface is also hosted by the object or matter of which it is a surface. Conversely, a void in an object or in some amount of matter must also be a void in a surface thereof (the "void lining", compare [CV94]), which is a relevant part feature of the object or matter (V-A5).

V-A6 might be more controversial: while we allow voids' regions to be contained in one another, we do not allow them to overlap without containment, in order to keep things neat. For example, a canyon as a void in the ground surface may fully include a smaller void at its bottom in which a river flows, see Figure 11.3(d). Allowing voids to be nested in this way gives us the freedom to decompose voids into meaningful parts that are voids themselves. More generally, voids do not necessarily occupy *maximal* internally connected void regions. Consider Figure 11.3(c): the region $\mathrm{r}(u) + \mathrm{r}(v')$ may not be a void at all, though it is a maximal void region hosted by $\mathrm{r}(x) + \mathrm{r}(y)$.

The regions of voids are not necessarily preserved by parthood: the void region v in Figure 11.3(a) is not the region of a void hosted by the physical endurant occupying $\mathrm{r}(x) + \mathrm{r}(y)$ because x and y are reciprocal fillers, though v's region is still a void in the endurant's part x. V-A7 and V-A8 capture weaker conditions under which voids must exist in parts and wholes, strengthening the axiom A2.4 of [CV94]. These conditions are illustrated in Figure 11.3(a)–(c).

We define the basic theory of physical voids as

$$\mathit{VOIDS} = \mathit{SPCH} \cup \{\text{V-A1} - \text{V-A8},\ \text{V}_\text{S}\text{-D},\ \text{V}_\text{C}\text{-D},\ \text{V-D}\}.$$

(VS-D) $\mathit{VS}(x, y) \leftrightarrow \mathit{PED}(x) \wedge S(y) \wedge \mathit{Cont}(y, \mathrm{ch}(x)) \wedge \neg \mathit{PO}(y, \mathrm{r}(x))$
(y is a void region in the physical endurant x, i.e., a spatial subregion of a physical endurant's convex hull not overlapping the endurant's region)

(V-A1) $\mathit{hosts}\text{-}v(x, y) \rightarrow \mathit{hosts}(x, y) \wedge \mathit{VS}(x, \mathrm{r}(y)) \wedge C_\mathrm{S}(\mathrm{r}(x), \mathrm{r}(y))$ (hosting a void)

($\mathbf{V_S}$-D) $V_\mathrm{S}(y) \leftrightarrow \mathit{ICon}(\mathrm{r}(y)) \wedge \exists x[\mathit{hosts}\text{-}v(x, y)]$ (simple void has an internally connected region)

($\mathbf{V_C}$-D) $V_\mathrm{C}(y) \leftrightarrow \neg \mathit{ICon}(\mathrm{r}(y)) \wedge \exists x[\mathit{hosts}\text{-}v(x, y)]$ (a complex void's region is not internally connected)

(V-D) $V(x) \leftrightarrow V_\mathrm{S}(x) \vee V_\mathrm{C}(x)$ (a void is a simple or complex void)

(V-A2) $\mathit{hosts}\text{-}v(x, y) \wedge V_\mathrm{C}(y) \wedge \mathit{PO}(\mathrm{r}(z), \mathrm{r}(y)) \rightarrow \exists v[\mathit{hosts}\text{-}v(x, v) \wedge V_\mathrm{S}(v) \wedge \mathit{PO}(\mathrm{r}(z), \mathrm{r}(v))]$
(any region overlapping a complex void's region overlaps a simple void's region of the same host)

(V-A3) $\mathit{hosts}(x, y) \wedge V(y) \rightarrow \neg V(x)$ (voids cannot host voids)

(V-A4) $\mathit{hosts}\text{-}v(x, y) \wedge \mathit{RPF}(x) \rightarrow \exists z[\mathit{hosts}(z, x) \wedge \neg F(z) \wedge \mathit{hosts}\text{-}v(z, y)]$
(every void hosted by a relevant part feature is also hosted by that feature's host)

(V-A5) $\mathit{hosts}\text{-}v(x, y) \wedge \neg F(x) \rightarrow \exists z[\mathit{hosts}(x, z) \wedge \mathit{RPF}(z) \wedge \mathit{hosts}\text{-}v(z, y)]$
(every void is hosted by some relevant part: the surface of its host)

(V-A6) $\mathit{hosts}\text{-}v(x, y) \wedge \mathit{hosts}\text{-}v(x, z) \wedge \mathit{PO}(\mathrm{r}(y), \mathrm{r}(z)) \rightarrow \mathit{Cont}(\mathrm{r}(y), \mathrm{r}(z)) \vee \mathit{Cont}(\mathrm{r}(z), \mathrm{r}(y))$
(one of the regions of two overlapping voids of the same host must be contained in the other)

(V-A7) $\mathit{hosts}\text{-}v(x, v) \wedge P(\mathrm{r}(x), \mathrm{r}(y)) \wedge \mathit{PED}(y) \wedge \neg \mathit{DPF}(y) \wedge \neg \mathit{Cont}(\mathrm{r}(v), \mathrm{r}(y)) \rightarrow \exists u[\mathit{Cont}(\mathrm{r}(v) - \mathrm{r}(y), \mathrm{r}(u)) \wedge \mathit{hosts}\text{-}v(y, u)]$ (if a non-dependent physical endurant y with part x, which hosts void v, does not completely fill the region of v, then $\mathrm{r}(v) - \mathrm{r}(y)$ must be in some void u of y)

(V-A8) $\mathit{hosts}\text{-}v(x, v) \wedge P(\mathrm{r}(y), \mathrm{r}(x)) \wedge \mathit{PED}(y) \wedge \neg \mathit{DPF}(y) \wedge \mathit{PO}(\mathrm{r}(v), \mathrm{ch}(y)) \rightarrow \exists u[\mathrm{r}(u) = \mathrm{r}(v) \cdot \mathrm{ch}(y) \wedge \mathit{hosts}\text{-}v(y, u)]$ (if void v in x has a region that overlaps the convex hull of part y of x, then y hosts a void u that occupies the region $\mathrm{r}(v) \cdot \mathrm{ch}(y)$)

Axiom Set 11.6: Axioms V-A1 – V-A8 and definitions VS-D, V_S-D, V_C-D, V-D of the theory *VOIDS*.

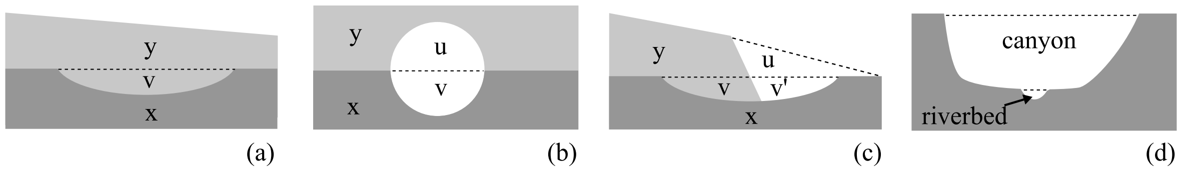

Figure 11.3: Examples of voids being contained in other voids.
(a) The void v in x is not a void occupying $\mathrm{r}(x) + \mathrm{r}(y)$: it is completely filled by y.
(b) The void with region $\mathrm{r}(v) + \mathrm{r}(u)$ in the physical endurant occupying $\mathrm{r}(x) + \mathrm{r}(y)$ has a part u with $\mathrm{r}(u) = (\mathrm{r}(v) + \mathrm{r}(u)) \cdot \mathrm{ch}(y)$ that is a void in y and a void in the entity occupying $\mathrm{r}(x) + \mathrm{r}(y)$.
(c) The void with the region $\mathrm{r}(v)+\mathrm{r}(v')$ in x is not completely filled by the endurant occupying $\mathrm{r}(x)+\mathrm{r}(y)$: $\mathrm{r}(x)+\mathrm{r}(y)$ only fills v but not v'. Thus $\mathrm{r}(v')$, a subregion of $\mathrm{r}(v)+\mathrm{r}(v')$, is part of the void region $\mathrm{r}(v')+\mathrm{r}(u)$ in the entity occupying $\mathrm{r}(x) + \mathrm{r}(y)$.
(d) The riverbed void is contained in the canyon void.

The theory *VOIDS* entails that voids are dependent places because of the earlier restriction of the *hosts* relation (S-A12).

(V-T1) $V(y) \rightarrow DPF(y)$ (voids are dependent place features)

Lemma 11.4. *VOIDS* $\models$ *V-T1*

Proof. Assume $x \in \mathbf{V}_{\mathcal{M}}$. Then for some $y \in \mathbf{M}$, $\mathbf{hosts\text{-}v}(x, y)$ by V-D, $\mathrm{V_S}$-D, and $\mathrm{V_C}$-D and hence $\mathbf{hosts}(x, y)$ and $\mathbf{VS}(x, \mathbf{r}(y))$ by V-A1. Then by VS-D, we must have $\neg\mathbf{PO}(\mathbf{r}(y), \mathbf{r}(x))$, which together with $\mathbf{hosts}(x, y)$ requires $x \in \mathbf{DPF}_{\mathcal{M}}$ by S-A12. □

11.5 Classifying physical voids

In the previous section we already distinguished simple from complex voids based on whether a void is internally self-connected or not. In this section, we will discuss three more criteria that can be used to classify voids. First, we look at the internal connectedness of a void's host to distinguish holes from gaps. Then, we study how different kinds of voids are connected to the exterior, which includes the space occupied neither by the host nor the void itself. This categorizes voids into cavities, tunnels, and hollows. The third criteria distinguishes whether a void is connected to its host's exterior, which includes the host and *all* voids of the host, discriminating internal from external voids. All three criteria are largely independent from one another, except for cavities.

A fourth and very different classification of voids arises from the interplay between objects, matter, and voids. We propose a way to separate voids in an object—*macroscopic voids*—from voids in the object's matter—*microscopic voids*. The idea behind this distinction is that microscopic voids are not recognizable on the coarser object level. Separating microscopic from macroscopic voids is of great practical concern in domains such as hydrogeology, where large amounts of water may be stored within rock bodies that appear solid, but are constituted of porous material. The distinction only becomes definable once we separate the spatial region occupied by an object from the spatial subregion occupied by its matter as we did in S-A9.

11.5.1 Internal connectedness of the host: holes vs. gaps

We already distinguish between a simple and a complex void based on the internal connectedness of the void, but we can also distinguish voids by their host's internal self-connectedness. If the host of a void is internally connected, we call the void a *hole* (V-A9, Hole-D) following [CV94] but strengthening connectedness to internal connectedness. If the host of a void is not internally connected, i.e., consists of several scattered regions or regions with only degenerate connections, we call the void a *gap* (V-A10, Gap-D). Intuitively, a gap is the space between the parts of a scattered host, such as the gap(s) between individual pebbles in a gravel pit. In hydrogeology, gaps are most prominent in rock matter, because though a rock body may appear solid, its matter consisting of individual grains or crystals is often not s-connected, i.e., not fused together in the sense that some individual grains or crystals may only be connected at edges, leaving gaps (pores) that can be filled with water.

It is easily verified that for any specific host *hosts-h* and *hosts-g* are disjoint and exhaustive subrelations of *hosts-v*. Then holes and gaps are exhaustive categories of voids (V-T2), but some voids might be gaps and holes with respect to different hosts.

(V-T2) $V(x) \leftrightarrow GAP(x) \vee HOLE(x)$ (gap and hole exhaustive classes of voids)

Lemma 11.5. *VOIDS* $\cup$ *{V-A9, V-A10, Hole-D, Gap-D}* $\models$ *V-T2*

Proof. Assume $x \in \mathbf{V}_{\mathcal{M}}$. Then for some $y \in \mathbf{M}$, $\mathbf{hosts\text{-}v}(x,y)$ by V-D, V_S-D, and V_C-D. Furthermore, we must have either $y \in \mathbf{ICon}_{\mathcal{M}}$ or $y \notin \mathbf{ICon}_{\mathcal{M}}$. Hence, either $\mathbf{hosts\text{-}h}(x,y)$ or $\mathbf{hosts\text{-}g}(x,y)$ and thus either $x \in \mathbf{Hole}_{\mathcal{M}}$ or $x \in \mathbf{Gap}_{\mathcal{M}}$. □

11.5.2 Contact to the exterior and other voids: cavities, tunnels, and hollows

We can also categorize voids by their opening(s), i.e., their connectivity to regions not occupied by the host, such as other endurants' regions and other void regions of the same host. The central concept of the *opening of a void*, a unary function op (V-A11), is defined as the lower-dimensional intersection between the void's region and the complement of the sum of the void's and host's regions. This opening is a purely abstract spatial region of nonmaximal dimension.

A *cavity* (V-A12, CAV-D) has usually no opening (an internal cavity; V-A13) or has a degenerate opening that, in the three-dimensional example, is not a surface but is a point or line (a tangential cavity; V-A14). *Hollows* are depressions[9] in an interior or exterior surface and have exactly one internally connected surface opening (V-A15, HOL-D). *Tunnels* or, more generally, tunnel systems have openings that consist of multiple not s-connected pieces (V-A16, TUN-D).

[9]We are not able to make the distinction between hollows that have a sharp edge and depressions that have a smooth edge which has been used by Casati & Varzi. This distinction is not definable in our theory.

(V-A9) $\textit{hosts-h}(x,y) \leftrightarrow \textit{hosts-v}(x,y) \wedge \textit{ICon}(\mathrm{r}(x))$ (non-scattered host of a void)

(Hole-D) $\textit{HOLE}(y) \leftrightarrow \exists x[\textit{hosts-h}(x,y)]$ (a hole has a non-scattered host)

(V-A10) $\textit{hosts-g}(x,y) \leftrightarrow \textit{hosts-v}(x,y) \wedge \neg \textit{ICon}(\mathrm{r}(x))$ (scattered host of a void)

(Gap-D) $\textit{GAP}(y) \leftrightarrow \exists x[\textit{hosts-g}(x,y)]$ (a gap has a scattered host)

Axiom Set 11.7: Axioms V-A9, V-A10 and definitions Hole-D and Gap-D of the theory $\textit{VOIDS}_{\text{extended}}$.

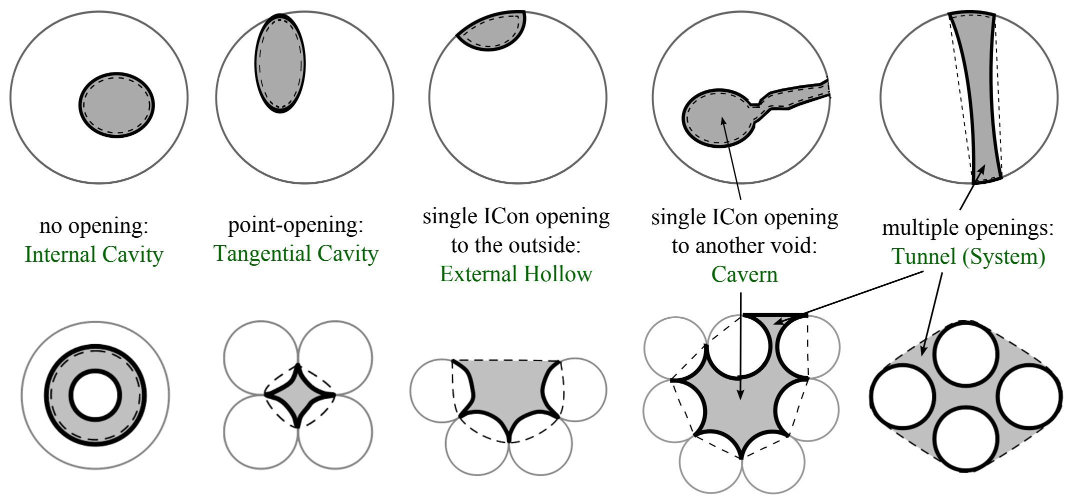

Figure 11.4: Examples of holes (top row) and gaps (bottom row), which are shown as grey areas within the convex hull (dashed lines) of their hosts (white). Thick solid lines show the surface of a host that also hosts the void. From left to right we have an internal cavity, a tangential cavity, an external hollow, an external hollow that consists of an internal hollow and an external tunnel, and an external tunnel.

We can further distinguish internal from external voids based on whether a void is connected to the outside of its host (an externally hosted void, V-A17) or merely to other voids within the host (an internally hosted void, V-A18). This distinction is not so interesting for cavities: all proper, i.e., internal cavities are connected neither to the outside nor to other voids within the same host; though tangential cavities, of rather theoretical nature, may be connected to the outside or other voids. But the distinction is useful for hollows and tunnels: *external hollows* are hollows whose (single) opening is connected to the outside, whereas *internal hollows* are only connected to other voids within the same host. In hydrogeology those may be called *caverns* as in our original axiomatization [HB12]. We can apply the same distinction to tunnels: *external tunnels* are connected to the outside, whereas *internal tunnels* are only connected to other voids within the same host. We leave it that basic distinction, though many more sophisticated notions such as an *indirectly external hollow* (or tunnel) as part of a system of connected hollows (or tunnels) are subsequently definable.

Hollows and tunnels are not required to be maximal internally connected voids: we want to maintain the flexibility to allow, for example, a hollow to consist of a tunnel leading to an internal hollow, as in the example in the second column from the right in Figure 11.4. Consider Figure 11.3(c) as another example: there are good reasons to call v' a void of the entity occupying $\mathrm{r}(x) + \mathrm{r}(y)$—even though v' is not maximal (the void region $\mathrm{r}(v') + \mathrm{r}(u)$ is maximal), $\mathrm{r}(v')$ is the greatest void region reasonably occupied by a water body such as a lake or river. In contrast, $\mathrm{r}(u)$ is a void region but u is likely not considered a void. To correctly capture v' as an *external hollow*, V-A17 treats void regions not occupied by the object's voids as parts of the exterior. Note that cavities are implicitly required to be maximal voids.

Any specific void in a host must be either a cavity, tunnel, or hollow and only one of those (V-T3, V-T4). Trivially, any void is either internally or externally hosted, but V-T5 also confirms that internal

(V-A11) $\mathit{hosts\text{-}v}(x,v) \rightarrow \mathrm{op}(x,v) = \mathrm{r}(v) \cdot (\mathrm{r}(x) + \mathrm{r}(v))'$
(the opening of a void region v as the boundary region that is not shared with its host's region $\mathrm{r}(x)$)

(V-A12) $\mathit{hosts\text{-}cavity}(x,y) \leftrightarrow \mathit{hosts\text{-}v}(x,y) \wedge \mathrm{op}(x,y) \not\prec_{\mathrm{dim}} \mathrm{r}(x)$
(cavity-hosting: hosting a void with no proper opening)

(CAV-D) $\mathit{CAVITY}(y) \leftrightarrow \exists x[\mathit{hosts\text{-}cavity}(x,y)]$ (cavity)

(V-A13) $\mathit{hosts\text{-}cavity}_{\mathrm{i}}(x,y) \leftrightarrow \mathit{hosts\text{-}cavity}(x,y) \wedge \mathit{ZEX}(\mathrm{op}(x,y))$ (an internal cavity has no opening)

(V-A14) $\mathit{hosts\text{-}cavity}_{\mathrm{t}}(x,y) \leftrightarrow \mathit{hosts\text{-}cavity}(x,y) \wedge \neg \mathit{ZEX}(\mathrm{op}(x,y))$
(a tangential cavity has a degenerate opening)

(V-A15) $\mathit{hosts\text{-}hollow}(x,y) \leftrightarrow \mathit{hosts\text{-}v}(x,y) \wedge \mathrm{op}(x,y) \prec_{\mathrm{dim}} \mathrm{r}(x) \wedge \mathit{ICon}(\mathrm{op}(x,y))$
(hollow-hosting: hosting a void with a single internally connected, proper opening)

(HOL-D) $\mathit{HOLLOW}(y) \leftrightarrow \exists x[\mathit{hosts\text{-}hollow}(x,y)]$ (hollow)

(V-A16) $\mathit{hosts\text{-}tunnel}(x,y) \leftrightarrow \mathit{hosts\text{-}v}(x,y) \wedge \mathrm{op}(x,y) \prec_{\mathrm{dim}} x \wedge \neg \mathit{ICon}(\mathrm{op}(x,y))$
(tunnel-hosting: hosting a void that has multiple, not internally connected regions as its opening)

(TUN-D) $\mathit{TUNNEL}(y) \leftrightarrow \exists x[\mathit{hosts\text{-}tunnel}(x,y)]$ (tunnel system)

(V-A17) $\mathit{hosts\text{-}v}_{\mathrm{e}}(x,y) \leftrightarrow \mathit{hosts\text{-}v}(x,y) \wedge \exists z\big[P(z,\mathrm{op}(x,y)) \wedge \forall u[\mathit{hosts\text{-}v}(x,u) \wedge z \cdot \mathrm{r}(u) =_{\mathrm{dim}} z \rightarrow \mathit{PO}(\mathrm{r}(y),\mathrm{r}(u)) \wedge \mathit{Cont}(z \cdot \mathrm{r}(u), \mathrm{op}(x,u))]\big]$ (an externally hosted void y in x is a void with a part z of its opening such that any other void u of the same host x that includes a part of that opening partially overlaps y and has $z \cdot \mathrm{r}(u)$ also in its opening)

(V-A18) $\mathit{hosts\text{-}v}_{\mathrm{i}}(x,y) \leftrightarrow \mathit{hosts\text{-}v}(x,y) \wedge \neg \mathit{hosts\text{-}v}_{\mathrm{e}}(x,y)$ (an internally hosted void)

Axiom Set 11.8: Axioms V-A11 – V-A18 and definitions of the theory $\mathit{VOIDS}_{\text{extended}}$.

cavities are indeed internally hosted.

(V-T3) $\neg[\mathit{hosts\text{-}cavity}(x,y) \wedge \mathit{hosts\text{-}hollow}(x,y)] \wedge \neg[\mathit{hosts\text{-}cavity}(x,y) \wedge \mathit{hosts\text{-}tunnel}(x,y)] \wedge \neg[\mathit{hosts\text{-}hollow}(x,y) \wedge \mathit{hosts\text{-}tunnel}(x,y)]$ (no void y is hosted in two different ways by one host x)

(V-T4) $\mathit{hosts\text{-}v}(x,y) \leftrightarrow \mathit{hosts\text{-}cavity}(x,y) \vee \mathit{hosts\text{-}tunnel}(x,y) \vee \mathit{hosts\text{-}hollow}(x,y)$

(cavity-, tunnel-, and hollow-hosting are exhaustive subrelations of hosting a void)

(V-T5) $\mathit{hosts\text{-}cavity}_{\mathrm{i}}(x,y) \rightarrow \mathit{hosts\text{-}v}_{\mathrm{i}}(x,y)$ (internal cavities are always internally hosted voids)

Lemma 11.6. *VOIDS* $\cup$ *{V-A11 – V-A18}* $\models$ *{V-T3 – V-T5}*

Proof. V-T3 and V-T4 follow immediately from the definitions of the relations *hosts-cavity*, *hosts-hollow*, and *hosts-tunnel* in V-A12, V-A15, and V-A16 for any two $x, y \in \mathbf{M}$: either $\mathbf{op}(x,y) \prec_{\mathbf{dim}} \mathbf{r}(x)$ or not and either $\mathbf{ICon}(\mathbf{op}(x,y))$ or not; and one of each of those conditions must hold.

To prove V-T5, assume $\mathbf{hosts\text{-}cavity_i}(x,y)$ for arbitrary $x, y \in \mathbf{M}$. Then $\mathbf{op}(x,y) \in \mathbf{ZEX}_{\mathcal{M}}$ by V-A13. Then by EP-D and C-A4 no $z \in \mathbf{M}$ can exists such that $\mathbf{P}(z, \mathbf{op}(x,y))$, hence $\neg\mathbf{hosts\text{-}v_e}(x,y)$ by V-A17 and thereby $\mathbf{hosts\text{-}v_i}(x,y)$ by V-A18. □

Note that the distinction between holes and gaps is independent of the distinction between cavities, tunnels, and hollows as Figure 11.4 demonstrates: a gap can form a cavity (bottom row, first and second from the left), a tunnel (bottom row, first and second from the right), or a hollow (bottom row, second and third from the right), while a hole can also be any of those (in the same order, top row).

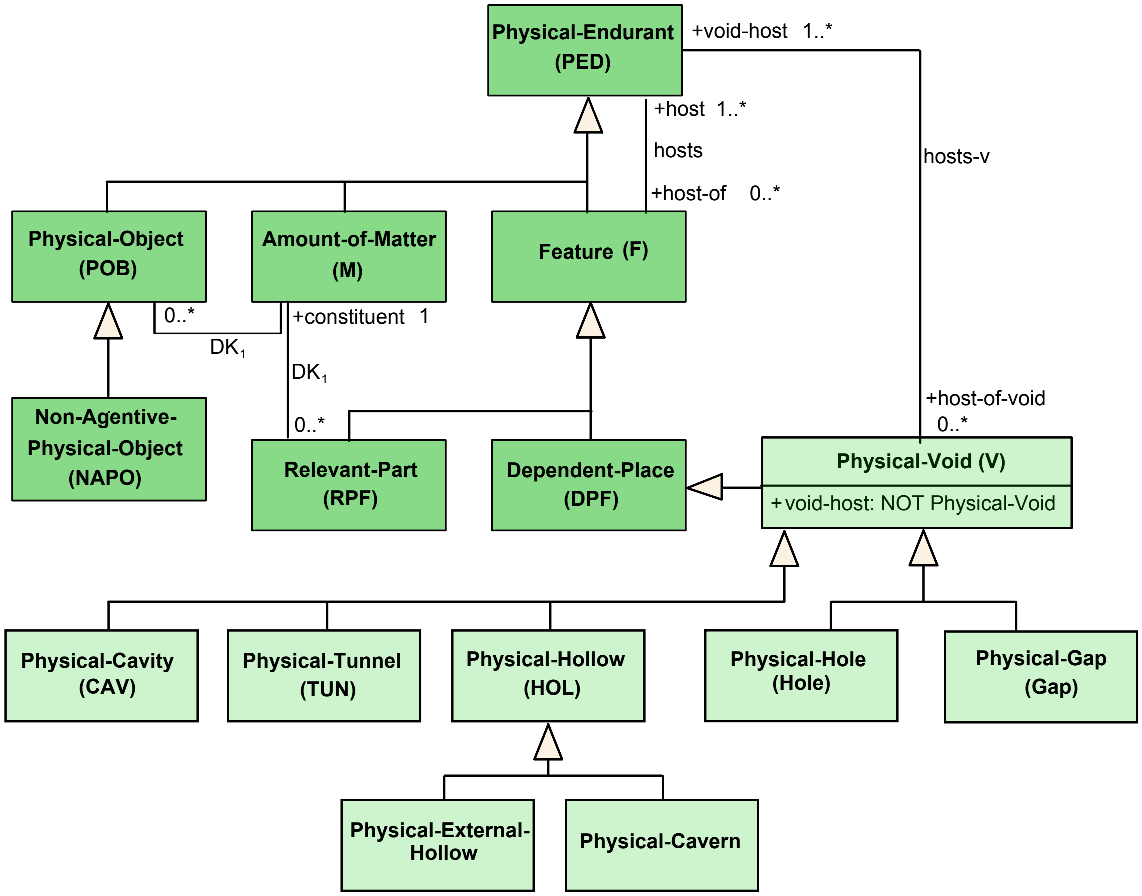

Figure 11.5: The category of physical voids within the taxonomy of physical entities. The bottom of the diagram shows four of the categorizations of voids discussed in this chapter: into simple and complex voids; into cavities, tunnels, and hollows; into holes and gaps; and into internal and external voids. All those categories refer to a specific void-host pair. Some voids may be categorized differently with respect to different hosts.

11.5.3 Voids in objects vs. voids in matter

In many real-world domains such as hydrogeology, it is not only important to distinguish holes from gaps and between voids with different kinds of openings, but also to distinguish macroscopic voids in an object from microscopic voids in its constituting matter. This clearly separates two notions of voids that are often confused in natural language: "a hole in the limestone", for a example, typically refers to a hole, such as a cave, that is in a rock body constituted by some limestone; it usually does not refer to the microscopic spaces between the individual grains of the limestone. Such a distinction shares many properties with the hybrid representation of matter discussed by Davis [Dav10] from a chemical perspective.

Thus far, voids in an object can also be voids in its matter or co-located with voids in its matter, for example, a cave in a rock body can also be (or be co-located with) a cave in its limestone. But the converse is impossible: microscopic voids in the matter of an object are never voids in the object itself because they are within the object's region (compare S-A9). To formally capture the idea of the microscopic voids in an object's matter, we define *pore space* of an object as the sum of all regions occupied by voids in its matter that do not overlap voids in the object itself (V-A19). The pore space then only contains the microscopic voids in an object's matter. For example, a cave in a rock body is not considered to be part of the rock body's pore space. The *void space* of an object is its pore space together with the regions occupied by the object's voids (V-A20). Because *hosts-v* is a primitive relation, the definitions of pore and void space presuppose an identification of all voids, because the sum of the regions occupied by an object's voids (determined by *hosts-v*) may be smaller than the sum of the objects void regions (defined by VS-D). To properly appreciate V-A19 and V-A20, recall that strong supplementation (EP-E2) ensures that the extension of *PO* uniquely identifies a region; thereby V-A19 and V-A20 effectively capture sums. While the pore and void space of a physical endurant are spatial regions, in fact void regions, as opposed to voids, they manifest themselves in (simple or complex) voids hosted by the endurant's matter (V-A21, V-A22). The portion of a physical object's void or pore space that has direct or indirect external openings is called its *connected void* or *pore space* (V-A23, V-A24).

(V-A19) $PO(\mathrm{r}(v), \mathrm{porespace}(o)) \leftrightarrow \exists m\big[DK_1(m,o) \land \forall u[\mathit{hosts\text{-}v}(o,u) \rightarrow \neg PO(\mathrm{r}(v),\mathrm{r}(u))] \land \exists u[\mathit{hosts\text{-}v}(m,u) \land PO(\mathrm{r}(v),\mathrm{r}(u))]\big]$ (pore space of an object overlaps any region that overlaps some void's region in the matter and does not overlap a region occupied by a void in the object)

(V-A20) $PO(\mathrm{r}(v), \mathrm{voidspace}(o)) \leftrightarrow PO(\mathrm{r}(v), \mathrm{porespace}(o)) \lor \exists u[\mathit{hosts\text{-}v}(o,u) \land PO(\mathrm{r}(v),\mathrm{r}(u))]]$
(void space of an object comprises its pore space and all its voids' regions)

(V-A21) $\neg ZEX(\mathrm{porespace}(o)) \rightarrow \exists v,m[\mathrm{r}(v) = \mathrm{porespace}(o) \land \mathit{hosts\text{-}v}(m,v) \land DK_1(m,o)]$
(nonempty pore space is the region of a void in the object's matter)

(V-A22) $\neg ZEX(\mathrm{voidspace}(o)) \rightarrow \exists m,v[\mathrm{r}(v) = \mathrm{voidspace}(o) \land \mathit{hosts\text{-}v}(m,v) \land DK_1(m,o)]$
(nonempty void space is the region of a void in the object's matter)

(V-A23) $PO(\mathrm{r}(v), \text{con-voidspace}(o)) \leftrightarrow \exists u[PO(\mathrm{r}(v),u) \land ICon(u) \land Cont(u,\mathrm{voidspace}(o)) \land C_{\mathrm{S}}(u,(\mathrm{r}(o)+\mathrm{voidspace}(o))')]$ (connected void space is the sum of the pieces of maximal internally connected void space with some external opening)

(V-A24) $PO(v, \text{con-porespace}(o)) \leftrightarrow \exists u[PO(\mathrm{r}(v),u) \land ICon(u) \land Cont(u,\mathrm{porespace}(o)) \land C_{\mathrm{S}}(u,(\mathrm{r}(o)+\mathrm{porespace}(o))')]$ (connected pore space is the sum of the pieces of maximal internally connected pore space with some external opening)

Axiom Set 11.9: Axioms V-A19 – V-A24 of the theory $VOIDS_{\mathrm{extended}}$.

It is entailed that matter never has pore space (V-T6), which is a property of an object. But all physical endurants that are matter or constituted by matter can have void space. However, if we identify the entire matter of an object as a separate object, this new object may have void space whose extent is equivalent to the former object's pore space, in that it covers exactly the voids in the matter.

(V-T6) $M(x) \rightarrow ZEX(\mathrm{porespace}(x))$ (matter and dependent places have no pore space)

Lemma 11.7. *VOIDS* $\cup$ *{V-A19 – V-A24}* $\models$ *V-T6*

Proof. First, assume $x \in \mathbf{M}_{\mathcal{M}}$. Suppose for some $v \in \mathbf{M}$, $\mathbf{PO}(v, \mathbf{porespace}(x))$. Then for some $m \in \mathbf{M}$, $\mathbf{DK_1}(m,x)$. However, by PED-A9 we must have $x \in \mathbf{POB}_{\mathcal{M}} \cup \mathbf{RPF}_{\mathcal{M}}$ and as such $x \notin \mathbf{M}_{\mathcal{M}}$ by PED-A2, which contradicts our initial assumption $x \in \mathbf{M}_{\mathcal{M}}$. Hence the supposition was false and no v can exist such that $\mathbf{PO}(v, \mathbf{porespace}(x))$. In particular, not $\mathbf{PO}(\mathbf{porespace}(x), \mathbf{porespace}(x))$ and thereby $\mathbf{porespace}(x) \in \mathbf{ZEX}_{\mathcal{M}}$ by PO-T1. □

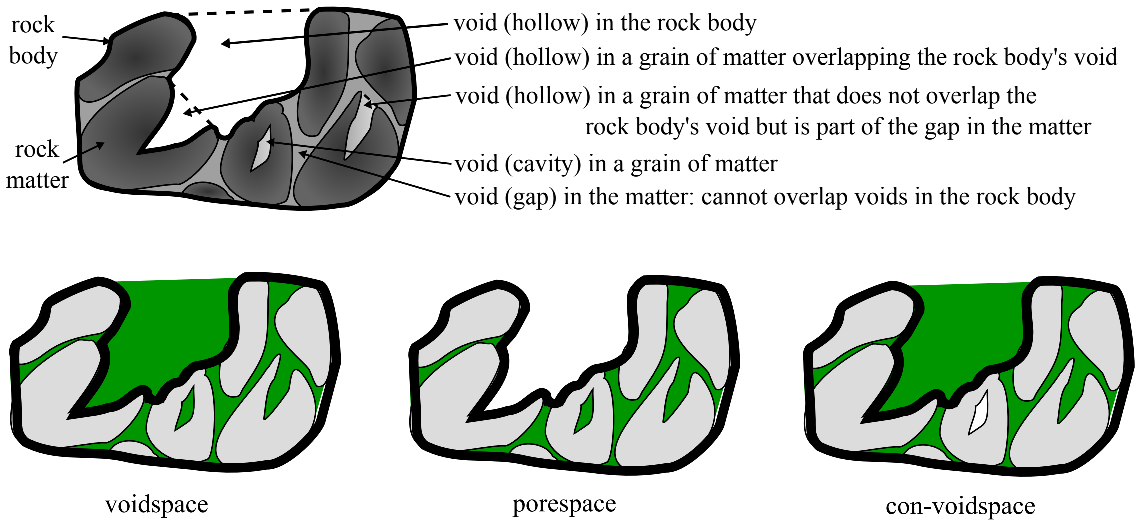

Figure 11.6: Examples of voids in an rock body, its matter (dark grey), and individual grains of matter (top). Light grey areas are gaps in the rock body's matter, and dashed lines are some void openings. The pore space of the object contains the void space in the rock matter that does not overlap voids in the object. For example, the hollow in a grain of matter that overlaps the hollow in the rock body (both shown in white with dashed lines marking their respective openings) may be a void in the matter, but cannot be part of the rock body's pore space. The other gaps and holes in the rock matter are part of the rock body's pore space, shown in light gray. The void space of the rock body is the sum of the voids in the rock body (here the large hollow) and its pore space. The connected pore space and void space exclude the cavity in the pore space, since it has no direct or indirect opening to the outside. The bottom row illustrates the void space, pore space, and connected void space for the example in the top.

We define the refined theory of physical voids as

$$\mathit{VOIDS}_{\text{extended}} = \mathit{VOIDS} \cup \{\text{V-A9} - \text{V-A24, Hole-D, Gap-D, CAV-D, TUN-D, HOL-D}\},$$

which contains all categorizations of voids that we discussed in this section.

11.6 Physical voids in hydrogeology

In this section we will give an excerpt of the domain theory of hydrogeology and outline which conceptual distinctions that are relevant to hydrogeology can be expressed in the theory $\mathit{VOIDS}_{\text{extended}}$. In particular, we want to distinguish ground water bodies, such as wells and aquifers, from surface water bodies, such as rivers and lakes. Those water bodies are illustrated in Figure 11.7).

In the domain theory, we talk about different kinds of matter: rock matter, water, soil, and organic matter. Rock matter can vary in its degree of consolidation from unconsolidated material, such as sand or gravel, to consolidated material composed of grains or crystals, such as sandstone or granite. Soil is a mixture of rock, water, organic matter, and gases, and water is primarily H_2O with other suspended or dissolved materials, most notably rock or soil matter. Soils are minimally composed of rock matter or organic matter, but might have other stuff as well [Soi12].

$$\mathit{RockMatter}(x) \rightarrow M(x)$$

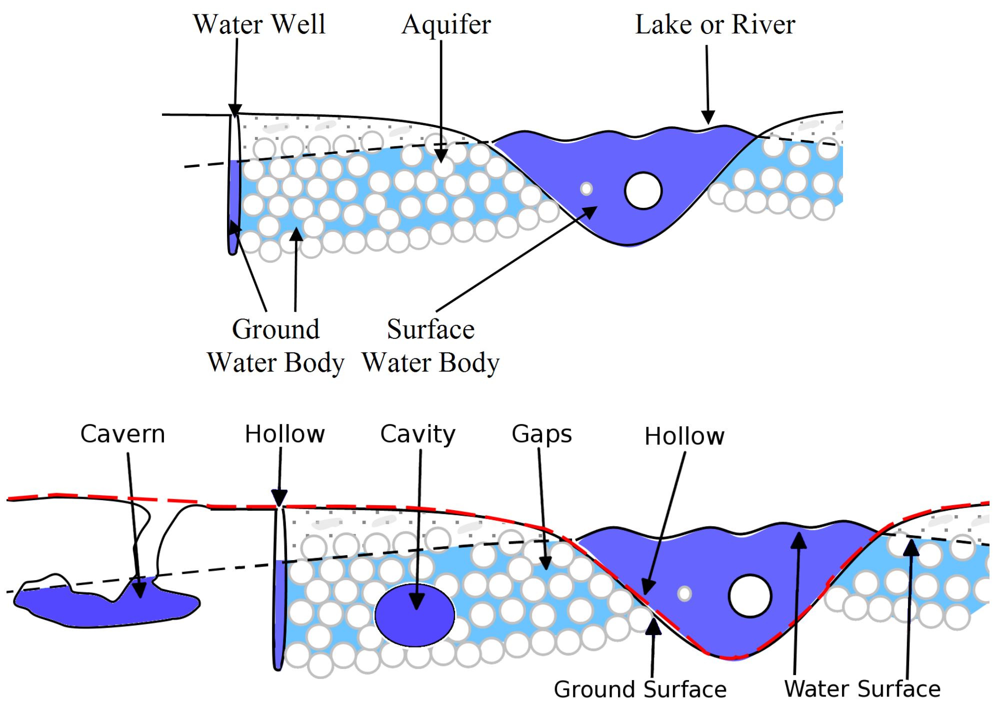

Figure 11.7: Different kinds of ground and surface water bodies relevant to hydrogeology (top) and the categories of physical voids that serve as containers for those water bodies. The dashed lines denote the water table and the ground surface (only in the bottom figure), respectively.

$$\begin{aligned} \mathit{OrganicMatter}(x) &\rightarrow M(x) \\ \mathit{Water}(x) &\rightarrow M(x) \\ \mathit{Soil}(x) &\rightarrow M(x) \wedge \exists y[P(\mathrm{r}(y), \mathrm{r}(x)) \wedge (\mathit{RockMatter}(y) \vee \mathit{OrganicMatter}(y)] \end{aligned}$$

We can now define two principal categories of hydrogeological physical objects, namely rock bodies and water bodies, by their constituting matter. Rock bodies are constituted by some amount of rock matter and only by rock matter, whereas water bodies are only constituted by water, if they are constituted by some matter at all. This allows for extreme cases of water bodies such as a lake that has dried out (permanently or temporarily). Either category is a subcategory of the category of nonagentive physical objects *NAPO*.

$$\begin{aligned} \mathit{WB}(x) &\rightarrow \mathit{NAPO}(x) \wedge \forall y[DK_1(y,x) \rightarrow \mathit{Water}(y)] \\ \mathit{RB}(x) &\leftrightarrow \mathit{NAPO}(x) \wedge \exists y[DK_1(y,x)] \wedge \forall y[DK_1(y,x) \rightarrow \mathit{RockMatter}(y)] \end{aligned}$$

To define more interesting categories of hydrogeological physical objects, we need the notion of a *ground surface*. The ground surface is not definable, we treat it as a primitive, unary relation that must be specified explicitly for any particular hydrogeological scenario. We only know that it is a feature, more precisely a relevant part feature of some nonagentive object, which may, for example be a rock body.

$$\mathit{GS}(gs) \rightarrow \mathit{RPF}(gs) \wedge \exists o[\mathit{NAPO}(o) \wedge \mathit{hosts}(o, gs)]$$

Then we can distinguish between *surface* and *ground*, also called *subsurface*, water bodies. This essential distinction in hydrogeology has not been offered by other hydro-ontologies such as the INSPIRE schema [INS11] or the Groundwater Markup Language (GWML) [BB12].

$$\begin{aligned}
&\mathit{SurfaceWB}(wb) \rightarrow \mathit{WB}(wb) \wedge \exists gs[\mathit{hosts\text{-}hollow}_{\mathrm{e}}(wb, gs) \wedge \mathit{GS}(gs)] \\
&\mathit{GroundWB}(wb) \rightarrow \mathit{WB}(wb) \wedge \exists rb, gs\big[\mathit{RB}(rb) \wedge \mathit{hosts}(rb, gs) \wedge \mathit{GS}(gs) \wedge P(\mathrm{r}(wb), \mathrm{voidspace}(rb)) \wedge \\
&\qquad\qquad \forall v[\mathit{hosts\text{-}hollow}_{\mathrm{e}}(rb, v) \rightarrow \neg \mathit{PO}(\mathrm{r}(wb), \mathrm{r}(v))]\big]
\end{aligned}$$

The most interesting kind of physical objects are neither water bodies nor rock bodies but a mixture of both, we call them *hydro-rock bodies*. They are partially constituted by rock matter and partially constituted by water, the latter being the matter of some ground water body. The three main categories of hydro-rock bodies in hydrogeology are aquifers, aquitards, and aquicludes. They differ in their permeability of water: aquifers are generally permeable, aquitards have a low permeability, and aquicludes are impermeable. We cannot define those differences in permeability in $\mathit{VOIDS}_{\mathrm{extended}}$. The differences in permeability depend on factors such as the size of the void spaces and the size of the connection between void spaces, not just on the general presence or connectivity of void space within a rock body.

$$\begin{aligned}
\mathit{HydroRockBody}(aq) \rightarrow & \mathit{NAPO}(aq) \wedge \exists rb, wb\big[\mathrm{r}(aq) = \mathit{RB}(rb) \wedge \mathit{GroundWB}(wb) \wedge \\
& \qquad \mathrm{r}(rb) + \mathrm{r}(wb) \wedge P(\mathrm{r}(wb), \text{con-voidspace}(rb))\big] \\
\mathit{Aquifer}(aq) \rightarrow & \mathit{HydroRockBody}(aq) \\
\mathit{Aquitard}(aq) \rightarrow & \mathit{HydroRockBody}(aq) \\
\mathit{Aquiclude}(aq) \rightarrow & \mathit{HydroRockBody}(aq)
\end{aligned}$$

The difference between a lake and a river is not yet definable in $\mathit{VOIDS}_{\mathrm{extended}}$—both have a container that is an external hollow hosted by the ground surface. Rivers usually have a different width-length ratio than lakes, but even that is no crisp distinction [BMT08]. The only clear distinction may be the connectivity to other water bodies: two lakes are never directly connected, only certain kinds of non-lake water bodies (any kind of natural or artificial linking water body such as a river, a canal, or a strait) can connect two lakes. For example, even though we talk about Lake Michigan and Lake Huron, hydrologically they are a single lake called Lake Michigan-Huron or Huron-Michigan. If we consider them as separate lakes, we must consider their connecting water body, the Straits of Mackinac, as a separate water body as well. Maybe the only reliable distinction between a lake and a river concerns their water flow: lakes are relatively still, whereas rivers have a stronger and directed flow of water. Neither of those properties can be captured in our framework.

But we are able to tell a water well from a river or a lake. While a water well is below the ground surface, i.e., is a ground water body, a lake or river is a surface water body. But the distinction is only possible if a suitable definition of ground surface is assumed. More precisely, the ground surface would need to include the bed of the lake or river and the opening of a well, but exclude the well liner [compare CV99a]. Altogether, we have to admit that—except for the fairly crisp definition of a hydro-rock body—we are still far from providing formal definitions for many foundational hydrological entities. However, we have made significant advances compared to earlier work [BB12; INS11] in the following ways: (1) we offer a unified way in which we can talk about the commonalities and the differences between surface and subsurface water bodies; (2) we define hydro-rock bodies as physical objects constituted partly by

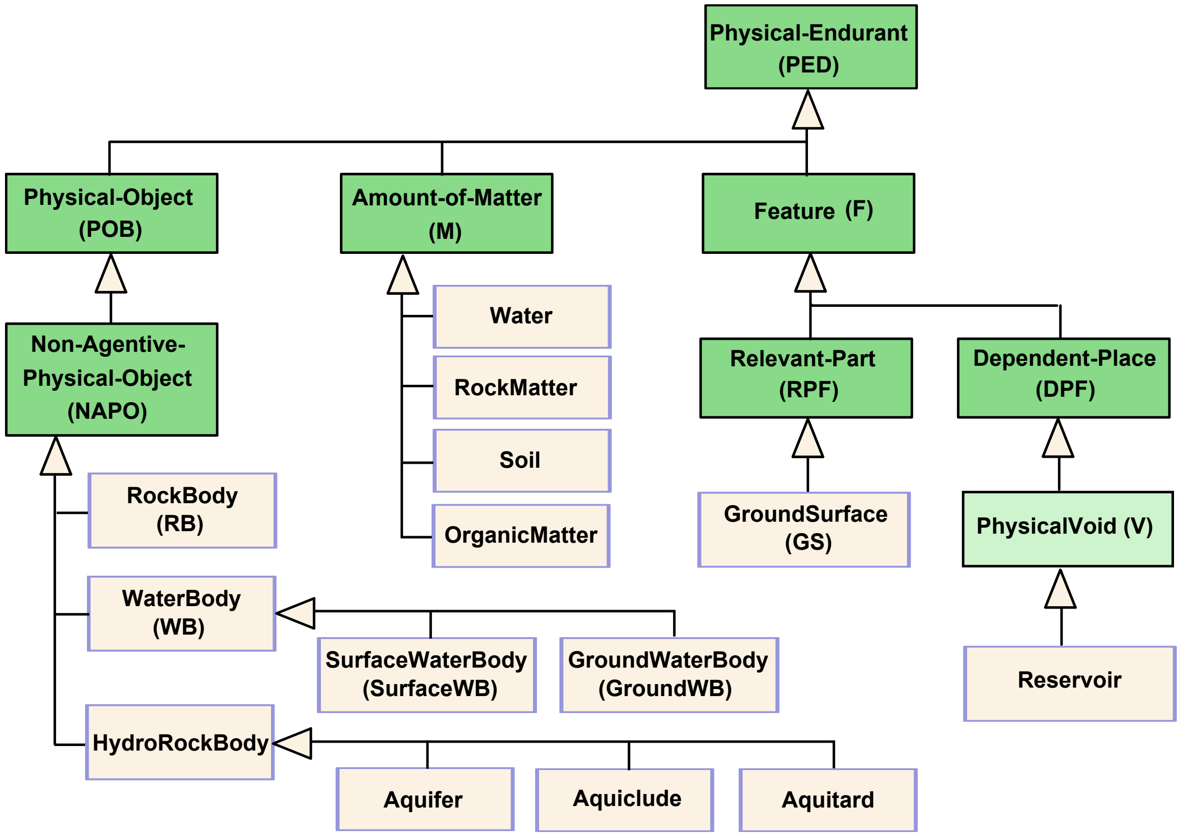

Figure 11.8: The extension of the DOLCE categories of physical endurants from Figure 11.5 by the categories defined as part of the hydrogeology domain theory.

the rock matter of a rock body and by water; and (3) we formalize some necessary characteristics of key hydrological entities. Now we are also able to prove more precise definitions of concepts such as a reservoir, a concept that has not been clearly defined in GWML or INSPIRE, as a void in a rock body.

$$Reservoir(wr) \leftrightarrow V(wr) \land \exists rb[RB(rb) \land \mathrm{r}(wr) = \mathrm{voidspace}(rb)]$$

We could further refine the concept of a reservoir into ground water reservoirs such as aquifers and surface water reservoirs such as a lake or an impounded (damned) lake. We could also formally distinguish a well that possibly yields water from a dry well. We leave those definitions as future work. Figure 11.8 summarizes how the discussed hydrological/hydrogeological concepts relate to the general DOLCE concepts and our proposed category of voids.

11.7 Summary

In this chapter, we explored how two kinds of physical features—boundaries and voids—can be modelled using our axiomatization of abstract space from the previous chapters as underlying spatial theory and an axiomatization of physical entities. The theory of physical entities is based on DOLCE's taxonomy of physical endurants and the relations between them; we adopted DOLCE's axiomatization to our needs in Section 11.1, resulting in the theory *PED*. We have then used the layering technique from [Don03;

Don05; DS03] to relate physical endurants to their location in space, captured by abstract spatial regions (Section 11.2). Thereby we obtained the theory *SPACE* as the combination of *PED* with the theory $CODIB_{\updownarrow}$ from Chapter 9.

In Section 11.3 we discussed how we can use the bodiless and bulky boundaries axiomatized in abstract space to model different physical boundaries. The key conclusion of this section is that bodiless, immaterial boundaries as well as bulky, material boundaries are definable in principle, though our interpretation of DOLCE's category of features does not allow lower-dimensional, non-extended physical features. However, we could fix this by introducing an appropriate category of non-extended features.

As main focus of the chapter, we formalized the spatial nature of different kinds of physical voids in Section 11.4. For this purpose, we first adopted an axiomatization of a primitive convex hull function from [Coh+97a; Don05] to the multidimensional case. Because a full investigation of the definability of convex hulls, which we started in Section 10.3.5, is still outstanding, we relied on the convex hull as a primitive function instead of defining it in a theory obtained as the extension of $CODIB_{\updownarrow}$ with a theory from the *OMT* hierarchy. Extending *SPACE* with the axiomatization of the convex hull function resulted in the theory *SPCH*.

The theory *SPCH* subsequently allowed us to define so-called void spaces—space regions in which voids can be located. While the identification of voids has not been tackled and therefore remains as an open issue, we proposed axioms restricting which void regions qualify as the regions of some physical voids. We also distinguished simple from complex voids based on the internal connectedness of a void in the resulting theory *VOIDS* that extends *SPCH*.

In Section 11.5, we studied the classification of voids using three additional criteria: (1) the internal connectedness of a void's host (holes from gaps), (2) the connection of the void to the exterior of the host and to other voids in the same host (cavities, hollows, and tunnels as well as internal vs. external voids), and (3) the level of granularity on which a void is present (microscopic voids that are only present on the level of matter vs. macroscopic voids that are present at the object level). Based on the formalization of voids, we have proposed to extend the taxonomy of physical endurants by a category of physical voids as subcategory of dependent place features. The four classifications of voids allow, in principle, a further refinement of the category of voids, though in orthogonal ways. It remains to investigate whether there is some natural precedence over those four classification criteria that leads to a hierarchy of subcategories of voids (**Question 8**). The theory *VOIDS* extended by all four classification of voids defines the theory $VOIDS_{\text{extended}}$. The relationships between the theories in this chapter is illustrated in Figure 11.9.

In the final section of the chapter, Section 11.6, we showed how the theory of physical voids can be applied to the domain of hydrogeology to formalize key hydrological concepts, in particular the difference between surface and ground water bodies and the notion of a hydro-rock body as a rock body that can store water and is of particular importance to modelling ground water. In the future, we intend to extend this application to distinguish different kinds of containment relations, thereby generalizing the analysis from [Don05] to all kinds of containment relations that may involve physical voids.

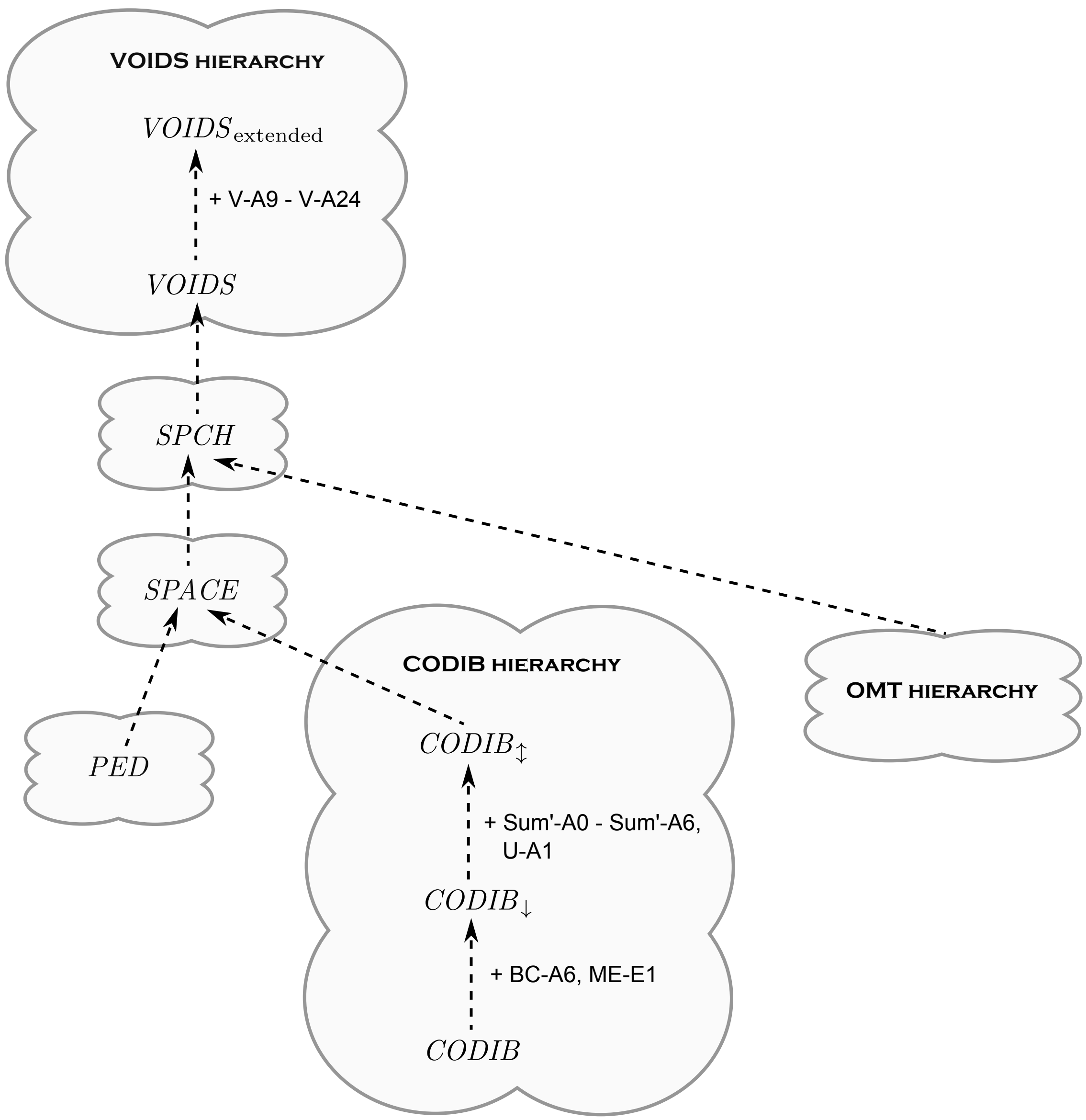

Figure 11.9: The theories and hierarchies constructed in Chapter 11. Each of *PED*, *SPACE*, and *SPCH* reside in their own hierarchy. *SPACE* is combination of $CODIB_{\updownarrow}$ and *PED*, while *SPCH* introduces the convex hull function ch as new primitive. The *VOIDS* hierarchy introduces *hosts-v* as new primitive relation, which is not present in the other hierarchies. *SPCH* is also an extension of a theory in the *OMT* hierarchy, though it remains to be investigated of which theory in the *OMT* hierarchy.

Chapter 12

Summary and conclusions

In this thesis we explored a range of spatial ontologies that involve some kind of mereotopological relations and we designed a family of ontologies to partially fill the expressivity gap between existing qualitative theories and geometric theories of space.

We started by studying equidimensional mereotopologies (Chapter 4). We introduced the notion of spatial representability to analyze which equidimensional mereotopologies are capable of representing space in a mereotopological way, in which all models are closed under mereological or topological definitions of the closure operations intersection, sum, and complementation. We thereby established necessary conditions for spatially representable equidimensional mereotopologies: their algebraic counterparts are either generalized Boolean contact algebras (GBCAs) or Stonian p-ortholattices equipped with a contact relation defined as $x\mathbf{C}y \leftrightarrow x \not\leq y^{\perp}$. Both classes of contact algebras define intersections and sums mereotopologically, but complementation is defined only mereologically in the former and only topologically in the latter. The algebraic counterparts of equidimensional mereotopologies allowed us to partially order equidimensional mereotopologies with respect to the restrictiveness of their lattice (the parthood relation) and of their contact relation within a single hierarchy. But to find more expressive ontologies with mereotopological relations, we have to move beyond the hierarchy of equidimensional mereotopologies.

Therefore, we subsequently focused on multidimensional mereotopological theories of space (from Chapter 5 on). We developed a family of multidimensional spatial theories that are grouped into hierarchies of theories of equal expressivity. The hierarchies are related to one another by their sets of primitives, resulting in the hierarchy of hierarchies depicted in Figure 12.1, which is partially ordered by the expressivity of the hierarchies' nonlogical languages. Within each hierarchy, theories only vary in how restrictive their axioms are; we partially ordered them by nonconservative extensions. We also considered definitional extensions within each hierarchy, which extend the language with defined nonlogical symbols, but which maintain the primitive language of undefined nonlogical symbols.

At the core of our work, we devised the new general multidimensional mereotopological theory *CODI* (Chapter 6), which we successively extended throughout the thesis. Firstly, we definably extended the theory—staying within the same hierarchy—by mereological closure functions that work for all pairs of entities, independent of their dimension (Theorems 7.1, 7.2, 7.5, and 7.7 in Chapter 7). Secondly, we extended the primitive language: we introduced $BCont(x, y)$ as new primitive relation capturing the notion of 'x is contained in the boundary of y' of the intended structures (Chapter 9) and we

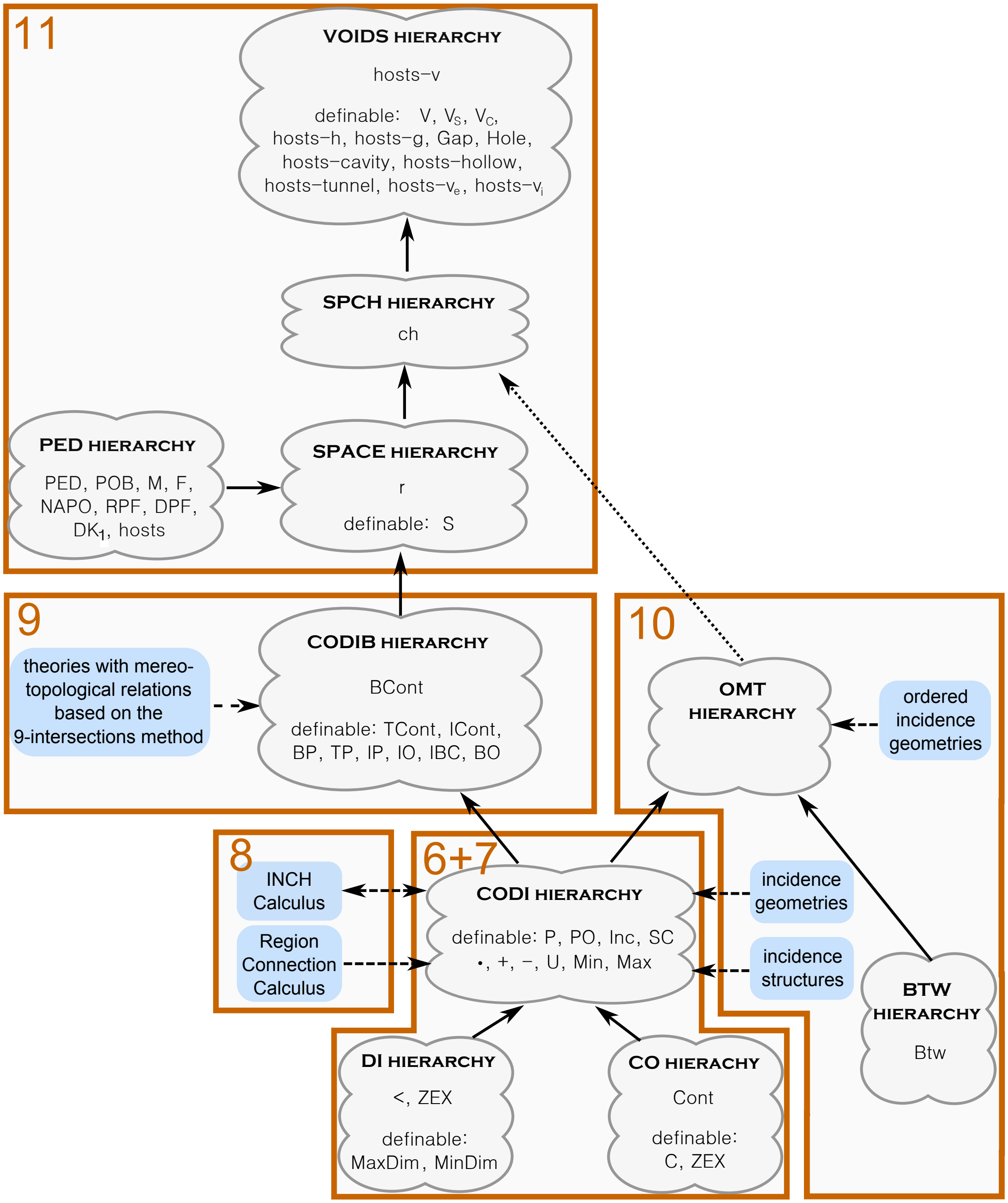

Figure 12.1: An overview of the hierarchies constructed in this thesis. For each hierarchy, we indicate the new primitive relations and functions it introduces to increase its expressivity and which relations and functions then become definable (only the most important ones are listed). The hierarchies are related by extensions of the primitive language indicated by solid arrows. The dotted arrow also denotes a language extension, but one which requires a more thorough investigation. The rounded boxes are external theories (or sets of theories) to which we formally proved a relationship. The dashed arrow shows a interpretation relation: the target theory interprets the source theory. The numbered boxes indicate in which chapter the work was completed.

introduced $Btw(r, x, y, z)$ as new primitive relation capturing the notion of 'x separates y from z within r' (Chapter 10).

Up to Chapter 10, we concentrated on abstract theories of multidimensional space to avoid having to deal with physical restrictions. In a final step we demonstrated how to model physical space by supplementing the theories of abstract space from the previous chapters with a categorization of physical entities based on the DOLCE upper ontology (Chapter 11). We further demonstrated how to capture different kinds of physical voids, which in turn allowed us to model physically meaningful differences between different kinds of water bodies in hydrogeology.

The next two sections summarize the specific results that we obtained towards the thesis' two overarching contributions discussed in the introduction (Chapter 1). We first show how we partially filled the expressivity gap between equidimensional mereotopological theories and geometric theories of space and then give an overview of the results that help semantically integrate a range of qualitative and geometric ontologies of space that involve some kinds of mereotopological relations.

12.1 Qualitative theories of space with expressivity in between mereotopology and geometry

We extended the expressivity of equidimensional mereotopologies in three ways: (1) we broadened the domain of the intended structures from sets of manifolds of equal dimension to sets of manifolds of varying dimensions, (2) we distinguished whether an entity is in contact with another entity's interior or boundary regardless of either entities' dimension, and (3) we captured a notion of order between entities of arbitrary dimension within a common entity. While we expressed the first two conditions in the definition of the class of intended structures $\mathbb{M}$ and its subclass $\mathbb{M}_{\text{dense}}$ (Chapter 5 and Section 9.1), we did not directly express the third condition in the intended structures, but treated it as a relation defined on top of every intended structure. To logically capture the three extensions we developed three hierarchies of ontologies of qualitative space: *CODI*, *CODIB*, and *OMT*. We will now summarize the results pertaining to those three hierarchies.

In the *CODI* hierarchy, $CODI_{\downarrow}$ is the key theory, for which we showed satisfiability with respect to the class $\mathbb{M}$ of intended structures (Theorem 7.4). $CODI_{\downarrow}$ is an extension of the theory *CODI*, our most general multidimensional mereotopology (presented in Chapter 6) that is already more expressive than equidimensional mereotopologies: it can express relations such as incidence and superficial contact between entities of different dimensions. Lower-dimensional entities are first-class objects in *CODI*: they can be in relation to one another just like entities of maximal dimension. In *CODI* the relations PO, Inc, SC, and $\neg C$ form a small intuitive set of JEPD mereotopological relations for multidimensional space (Theorem 6.2). Closing *CODI* under intersection and differences defines the theory $CODI_{\downarrow}$. We showed satisfiability of $CODI_{\downarrow}$ with respect to $\mathbb{M}$, but we also demonstrated that any particular model of $CODI_{\downarrow}$ can represent multiple distinct models of $\mathbb{M}$. In that sense the primitive language of the *CODI* hierarchy is not expressive enough to distinguish between some structures in $\mathbb{M}$: it cannot distinguish between structures in which entities may be in contact to the interior instead of the boundary of another entity (or vice versa). Further closing the models of $CODI_{\downarrow}$ under sums defines the theory $CODI_{\updownarrow}$, which is no longer satisfiable with respect to $\mathbb{M}$ because it introduces additional entities which may violate some conditions of a complex manifold. All three mereological operations are definable in *CODI* (Theorems 7.1, 7.2, 7.5, and 7.7 in Chapter 7). While their axiomatizations are fairly complicated, we were able to verify key properties of the operations.

To verify $CODI_{\updownarrow}$, we also provided a model-theoretic characterization of $CODI_{\updownarrow}$: all models are sets of Boolean algebras related by parthood, each set capturing the entities of a single dimension (Theorem 7.6). Thus, the extension of containment *Cont* in any model of *CODI* is partitioned into the extensions of parthood P and of lower-dimensionality $<_{\text{dim}}$ (Theorem 6.1). As a consequence, the set of entities of each dimension of a model of *CODI* form a structure that is a model of the RCC, because RCC models also form Boolean algebras. We showed this specifically for the entities of maximal dimension (Theorem 8.2 in Section 8.1). We also cross-verified $CODI_{\updownarrow}$ against the INCH Calculus—another multidimensional mereotopology that had not been thoroughly studied beforehand. We showed that, apart from minor differences in their ontological assumptions, $CODI_{\updownarrow}$ and the INCH Calculus are definably equivalent (Theorem 8.3). Through our verification, we now better understand the models of the theories in the *CODI* hierarchy. Thereby, we provided the theoretical foundation for the reuse of those multidimensional mereotopologies of space.

To capture the distinction between the interior or the boundary of an entity being in contact to another entity, we supplemented the primitive language of *CODI* by a relation of boundary-containment, *BCont*, resulting in the hierarchy *CODIB*. The relation of boundary-containment is not definable in the language of *CODI*. Theories in the *CODIB* hierarchy (Chapter 9) can again be closed mereologically—similarly to the theories in the *CODI* hierarchy. Of particular interest is the theory $CODIB_{\downarrow}$, whose models are completely closed under intersections and differences. $CODIB_{\downarrow}$ is satisfiable with respect to the intended structures that have densely ordered dimensions (Theorem 9.2), i.e., with respect to the class $\mathbb{M}_{\text{dense}} \subset \mathbb{M}$. We further closed $CODIB_{\downarrow}$ under sums of entities that do not meet in their interior, resulting in the theory $CODIB_{\updownarrow}$. Then satisfiability extends to $CODIB_{\updownarrow}$: all models of $CODIB_{\updownarrow}$ still represent structures in $\mathbb{M}_{\text{dense}}$, which was not the case for the models of the theory $CODI_{\updownarrow}$.

We further examined the expressive power of the new hierarchy *CODIB*. It lets us define both a bodiless and a bulky notion of interior containment (*ICont* and *IP*) and tangential containment (*TCont* and *TP*) as well as bodiless and bulky boundaries (Sections 9.3 and 9.4). In any model of $CODIB_{\downarrow}$ the extensions of *ICont* and *TCont* and of *IP* and *TP* partition the extensions of *Cont* and P, respectively (Theorems 9.1 and 9.3). Furthermore, we refined contact exhaustively into four subrelations in $CODIB_{\downarrow}$: interior overlap *IO*, boundary overlap *BO*, and interior-boundary contact *IBC* and its inverse, IBC^{-1} (Theorem 9.4). This classification is largely independent from the classification of contact into *PO*, *Inc*, and *SC* though some dependencies exist, such as $PO(x, y) \rightarrow IO(x, y)$, which need to be fully analyzed in the future. Other definable relations in *CODIB* concern the intersection of one entity's interior, boundary or exterior with the exterior of a second entity (Section 9.5). Together with the earlier relations, we effectively generalized Egenhofer's 9-intersection relations [EF91; EH91] to the general finite-dimensional case, in which all relations equally apply to manifolds with different absolute dimensions.

To deal with the third expressivity extension, we proposed a multidimensional relation of betweenness, *Btw*, in Chapter 10. In combination with *CODI*, it can be used to preserve order properties within structures in the class $\mathbb{M}$. The resulting hierarchy of ordered multidimensional mereotopologies, *OMT*, allows us to capture simple orderings over spatial entities that are preserved neither in models of *CODI* nor in models of *CODIB*. This is especially useful to model maps that involve many nonintersecting entities, such as a grid network of roads or the floors in a three-dimensional representation of a building. Ternary geometric betweenness is definable in terms of this multidimensional betweenness relation.

12.2 Semantic integration results

This thesis also contributes to the integration of spatial ontologies. We used three methods to integrate ontologies: (1) model expansions including definable expansions to show that the models of one ontology can always be expanded to models of another ontology, (2) interpretations including faithful interpretations to show that one theory entails the translation of another theory, and (3) definable relations to show that one theory can define all the relations of another theory. Thereby we either directly integrated external spatial theories with theories within our hierarchies or we indirectly related their classes of models, which in turn implies a relationship between the theories themselves. Note that most of our integration results, with the exception of the integration of the INCH Calculus, are not between theories of equally expressive languages; therefore, we cannot find definably equivalent theories.

We first related the algebraic counterparts of different equidimensional mereotopologies to one another (Chapter 4), including those of the RCC [Coh+97b; RCC92] (as Boolean contact algebras), of the GRCC [LY04] (as generalized Boolean contact algebras), and of Asher and Vieu's mereotopology *RT* [AV95] (as contact algebras with Stonian p-ortholattices). However, Chapter 4's contribution is not the integration of those theories, but a thorough exploration of the ensuing hierarchy of equidimensional mereotopologies for theories that are suitable to represent space topologically.

We further integrated the multidimensional theories from the *CODI* hierarchy with the equidimensional RCC (Section 8.1): the entities of maximal dimension of any model of $CODI_{\downarrow}$ define an RCC model (Theorem 8.2). In the converse direction, an RCC model cannot always be uniquely expanded to a model of *CODI* because the language of *CODI* is strictly more expressive than the language of the RCC. Moreover, we related the *CODI* hierarchy to the INCH Calculus (Section 8.2): every model of $CODI_{\updownarrow}$ that satisfies C-E4 is definably equivalent to a model of $INCH_{\text{calculus}}$ (Theorem 8.3). In the converse direction, every model of $INCH_{\text{calculus}}$ that satisfies I-E1 (a lowest dimension apart from that of the zero region exists), I-E2 (any two connected entities have a maximal shared constituent), and I-E3 (a universal entity exists of which every other entity is a constituent) is a model of $CODI_{\updownarrow}$ (also Theorem 8.3).

We integrated the *CODIB* theories with the spatial theories constructed using Egenhofer's 9-intersection method [EF91; EH91] (Chapter 9). We did so by showing the intended interpretations of all nine intersection relations are definable in any finite-dimensional complex manifold (Theorem 9.5). Consequently, each spatial ontology constructed on the basis of those nine intersection relations, including the theories presented in [CDF98; CDFO93; Ege91; EH91; EM95; ME94; McK+05], can be interpreted in some theory that extends $CODIB_{\downarrow}$ without having to extend $CODIB_{\downarrow}$'s primitive language.

Afterwards, we related the *CODI* and the *OMT* hierarchies to incidence structures, incidence geometries, and ordered incidence geometries (Chapter 10). First, we related finite-dimensional incidence structures to models of *CODI*: every model $\mathcal{M}$ of *CODI* defines an incidence structure (Theorem 10.1), and as special case thereof, every model of *CODI* defines a point incidence structure (Corollary 10.1). In the converse direction, every point incidence can be (definably) expanded to a model of *CODI* (Theorem 10.2). Any model of the theory $CODI_{\text{pl}}$, which introduces definitions of points and lines and postulates that every line contains at least two distinct points, defines a line space (Theorem 10.3), the most basic kind of two-dimensional incidence geometry. For the converse, any line space can be (definably) expanded to a model of $CODI_{\text{pl}}$ (Theorem 10.4), though not all models of $CODI_{\text{pl}}$ can be constructed in that way. Analogue relationships are obtained between the models of $CODI_{\text{pl-slin}}$, $CODI_{\text{pl-lin}}$, and $CODI_{\text{pl-aff}}$—all extensions of $CODI_{\text{pl}}$—and semi-linear, linear, and affine two-dimensional incidence geometries (Theorems 10.5 and 10.6).

We further extended the results about planar geometries to relationships between $CODI_{\text{plp}-\text{lin}}$, an extension of $CODI_{\text{pl}-\text{lin}}$ by a definition of planes, and incidence geometries, which are more precisely linear incidence geometries with three-dimensional point incidence structures and are axiomatized by the theory *IG*. Every model of $CODI_{\text{plp}-\text{lin}}$ defines a linear incidence geometry (Theorem 10.7) and every model of *IG* can be (definably) expanded to a model of $CODI_{\text{plp}-\text{lin}}$ (Theorem 10.8). As final step in Chapter 10, we related the theory $OMT_{\text{3d}-\text{lin}}$, an ordered multidimensional mereotopology from the *OMT* hierarchy, to weak ordered incidence geometries, axiomatized by the theory *WOIG*. Then any model of $OMT_{\text{3d}-\text{lin}}$ defines a weak ordered incidence geometry (Theorem 10.9) and any weak ordered incidence geometry defines a model of $OMT_{\text{3d}-\text{lin}}$ in a natural way (Theorem 10.10). All the relationships between the theories developed in this thesis and external spatial ontologies are illustrated in Figure 12.1.

As a by-product, Chapter 10 identified qualitative analogues of incidence geometries and ordered incidence geometries that omit two central geometric requirements: (1) that lines are straight and planes flat and (2) that lines have no endpoints and planes have no borders.

Summarily, ontology integration is an arduous task, though we showed how it can be expedited through automated theorem proving and model finding. We demonstrated that—even for large and complex ontologies—proving consistency, theorems, and interpretations between theories can be automated to a large extent. An often repeated criticism of using expressive logical languages, such as first-order logic, to specify ontologies has been that due to the intractability of first-order logic, reasoning with first-order theories is often impractical or requires lots of tuning of the theorem prover as in [HV06]. We demonstrated that despite its theoretical intractability, many reasoning tasks verifying first-order ontologies can still be successfully accomplished in reasonable time in practice. Importantly, this is not restricted to ontologies with only a few nonlogical symbols and a few axioms. For example, we can automatically construct interesting models and prove theorems of the theory *VOIDS* that consists of 117 axioms (translated to over 250 clauses) in a nonlogical language that includes 57 nonlogical symbols (clausification adds another 40 skolem functions). Intractability did not stop us from effectively utilizing first-order theorem proving for many of our ontology verification tasks as Appendix D shows without partitioning the ontology (as in [AM05]) or manually tuning the theorem provers or model finders. More generally, meaningful semantic integration that goes beyond mapping of concepts seems only possible with expressive logical languages in which subtle semantic differences can actually be captured. Note, however, that we only reasoned about ontologies that contained few or no named individuals. If we borrow the terminology of description logics, we focused on reasoning with first-order ontologies with large *TBoxes* (terminological knowledge) but empty ABoxes (instance knowledge). This can also be thought of as reasoning with large and complex database schemata of an essentially empty database. While reasoning about large sets of individual domain entities (with large *ABoxes*) using a first-order ontologies is generally thought of as impractical, this is not necessarily true for reasoning about first-order ontologies with large sets of axioms themselves. Note that the term "large" used with respect to the set of domain entities implies a size several order of magnitudes greater than with respect to axioms. Consequently, it is realistic to automate verification of first-order ontologies with small or empty ABoxes to a large extent using off-the-shelf theorem provers in the future.

12.3 Open questions

Some questions posed in this thesis remain open, we collect them in the following list.

Question 1. Provide topological embedding theorems for WBCAs and SPOCAs with C5′ (Chapter 4).

Question 2. Prove or disprove axiomatizability of the finite models of $CODI_{\updownarrow}$ with respect to $\mathbb{M}$.

Question 3. Find an intuitive way (if there is one) to extend RCC models to models of $CODI_{\updownarrow}$ that does not suffer from the problems discussed in Section 8.1.4.

Question 4. Verify the relationships between the classes of models of (ordered) incidence geometries and *CODI* theories in Chapter 10 through automated theory interpretation proofs.

Question 5. Identify the theories in the *CODI* and *OMT* hierarchies that are definably equivalent to incidence geometries and ordered incidence geometries.

Question 6. Verify the definitions of (line) segments in *OMT* from Section 10.3.5 and determine the least restrictive theory in *OMT* in which segments corresponds to their intended geometric interpretation.

Question 7. Fully axiomatize both notions of convexity discussed in Section 10.3.5 and determine the least restrictive theory in *OMT* that properly captures the intended geometric interpretations of convexity.

Question 8. Construct the lattice of all possible voids that can arise from the four different categorizations we presented in Chapter 11.

Question 9. Determine whether every CW-complex is in $\mathbb{M}_{\text{dense}}$ and determine whether every structure in the class $\mathbb{M}_{\text{dense}}$ is a CW-complex. Ideally, we could characterize the intended models in terms of special classes of CW-complexes or as generalization of CW-complexes.

Question 10. Determine whether the class $\mathbb{M}_{\text{dense}}$ is a subset or superset of the class of structures $\overline{P}_r^n$ with finite n (the closed polytopes $\overline{P}_n^n$ of dimension n recursively closed under the polytopes $\overline{P}_i^n$ with $0 \leq i \leq n-1$ that represent their boundaries) defined by Galton [Gal04]. Intuitively, we would think that every series $\overline{P}_r^n$ with finite n is in $\mathbb{M}_{\text{dense}}$. If it turns out that $\mathbb{M}_{\text{dense}}$ is neither a subset nor a superset of the class of structures $\overline{P}_r^n$ with finite n, a precise characterization of the set of series $\overline{P}_r^n$ that are not in $\mathbb{M}_{\text{dense}}$ would provide valuable insight.

Next, we list some opportunities for future work that directly arise from this thesis. This includes axiomatizations of spatial concepts that we have not fully investigated, additional verification tasks, and the establishment of new or the strengthening of existing relationships between theories of our hierarchies.

Challenge 1. Axiomatize the interaction between *BCont* and *Btw* in a theory that extends both *CODIB* and *OMT*. This would formally relate the two hierarchies.

Challenge 2. Investigate the relationship between the *SPCH* hierarchy that includes a convex hull primitive and the *OMT* hierarchy in which notions of convexity can be defined.

Challenge 3. Axiomatize the distinction between partial and full contact in addition to the other classifications of contact presented in Chapter 6 and 9.

Challenge 4. Explore and formalize the interdependencies among the various JEPD classifications of contact and containment.

Challenge 5. Extend the specification of the class of intended structures $\mathbb{M}$ or $\mathbb{M}_{\text{dense}}$ by including a notion of betweenness to provide a formal class of structures on which we can base a satisfiability theorem for the *OMT* theories.

Challenge 6. Explore the suitability of $OMT_\downarrow$ for modelling order between higher-dimensional entities. In particular, are axioms missing from $OMT_\downarrow$ that only apply to orders between higher-dimensional entities?

Challenge 7. Formalize the various weak notions of order discussed in Section 10.3.4.

Challenge 8. Devise an efficient implementation of the closure operations intersection, difference, and sum for geometric models. We expect an implementation—similar to the procedure for checking whether a set of manifolds forms a complex manifold—that leverages existing geometric calculations for vector-based representations to be feasible.

We will finish with a set of challenges regarding the integration of other spatial ontologies. Various other spatial ontologies are still waiting to be integrated into the hierarchies discussed here. Of particular interest are other spatial theories that involve mereotopological relations in a multidimensional setting. The following integration tasks are of primary interest.

Challenge 9. Fully integrate the theories from [CDF98; CDFO93; Ege91; EH91; EM95; ME94; McK+05] that are based on Egenhofer's 9-intersection method into the *CODIB* hierarchy by finding the extension of *CODIB* that faithfully interpret or are definably equivalent to a particular theory that is based on the 9-intersection method.

Challenge 10. Integrate the General Formal Ontology's (GFO) spatial ontology [BH11], which distinguishes spatial relations between entities of different dimensions and offers basic mereotopological relations.

Challenge 11. Integrate Galton's multidimensional mereotopology [Gal96] that is based on boundaries with theories within our hierarchies.

Challenge 12. Study whether Smith's mereotopology [Smi96] can be integrated into our hierarchies. If a formal integration is possible at all, it will pose a difficult task due to the rather unusual treatment of boundaries as special kinds of regions in [Smi96].

Challenge 13. Investigate whether we can use our hierarchies to precisify the formal semantics for many of the relations within the recently proposed GeoSPARQL [Ope12] and supplement the GeoSPARQL schema with knowledge about how its spatial relations are related to one another, using in particular the RDF notion of a subProperty. It would help to integrate spatial knowledge using different relations. For example, we could express that the RCC and Egenhofer relations of disconnection, rcc8dc and ehDisjoint, are identical. By addressing this challenge, the user community of GeoSPARQL would immediately benefit from this thesis' integration efforts.

Appendix A

List of logical functions and relations

$EC(x, y)$	x and y are externally connected, i.e., they are connected but do not share a part (relation of the Region Connection Calculus)	164
$ED(x, y)$	x and y are of the same dimension (relation of the INCH Calculus)	171
$EL(x, y)$	x is an element, i.e., a lower-dimensional constituent CS of y (relation of the INCH Calculus)	171
$EPt(x, y)$	y is an endpoint of the linear part y	278
$F(x)$	physical feature (DOLCE relation)	287
$GAP(x)$	physical gap	299
$GED(x, y)$	x is of the same or a greater dimension than y (relation of the INCH Calculus)	171
$GD(x, y)$	x is of a greater dimension than y (relation of the INCH Calculus)	171
$HOLE(x)$	physical hole	299
$HOLLOW(x)$	physical hollow	301
$hosts(x, y)$	physical endurant x hosts features y (DOLCE relation)	287
$hosts\text{-}g(x, y)$	physical endurant x hosts physical gap y	299
$hosts\text{-}cavity(x, y)$	physical endurant x hosts physical cavity y	301
$hosts\text{-}h(x, y)$	physical endurant x hosts physical hole y	297
$hosts\text{-}hollow(x, y)$	physical endurant x hosts physical hollow y	301
$hosts\text{-}tunnel(x, y)$	physical endurant x hosts physical tunnel y	301
$hosts\text{-}v(x, y)$	physical endurant x hosts physical void y	297
$hosts\text{-}v_{\mathrm{e}}(x, y)$	physical endurant x externally hosts physical void y	301
$hosts\text{-}v_{\mathrm{i}}(x, y)$	physical endurant x internally hosts physical void y	301
$ICon(x)$	strong (internally) self-connected entity (equidimensional mereotopology)	38
$ICon(x)$	internally self-connected entity	207
$ICont(x, y)$	x is contained in the interior of y (not touching the boundary of y)	213

$O(x,y)$	x and y overlap, i.e., they share a part (relation of the Region Connection Calculus)	164
$OV(x,y)$	x and y overlap, i.e., share a chunk CH (relation of the INCH Calculus)	171
$P(x,y)$	x is a part of y (equidimensional mereotopology)	36
$P(x,y)$	x is a spatial part of y, i.e., x is contained in y and of the same dimension as y	103
$P(x,y)$	x is a part of y (relation of the Region Connection Calculus)	164
$PED(x)$	physical endurant (DOLCE relation)	287
$Pl(x)$	x is a plane, i.e., a maximal entity in its dimension and of two dimensions greater than the minimal dimension	258
$PO(x,y)$	x and y partially overlap, i.e., they share a part (an entity of the same dimension as x and y)	105
$POB(x)$	physical object (DOLCE relation)	287
$PP(x,y)$	x is a part of y (equidimensional mereotopology)	36
$PP(x,y)$	x is a spatial proper part of y, i.e., x is properly contained in y and of the same dimension as y	103
$PP(x,y)$	x is a proper part of y (relation of the Region Connection Calculus)	164
$\mathrm{prod}(x,y)$	the product (intersection) of x and y (**function** of the Region Connection Calculus)	164
$Pt(x)$	x is a point, i.e., a maximal entity in its dimension and of minimal dimension	253
$\mathrm{r}(x)$	**spatial region of an physical endurant or a region (primitive function)**	288
$RPF(x)$	(physical) relevant part feature (such as physical surfaces) (DOLCE relation)	287
$S(x)$	abstract spatial region (DOLCE relation)	288
$SBP(x,y)$	x is a strong boundary part of y (tangentially contained in y, a part of y, contains a part of the boundary of y, and contains no interior part of y)	226
$SC(x,y)$	x and y are in superficial contact, i.e., they are in contact but only share entities that are of a lower-dimension than both x and y	107

$SPH(x)$	x is a simple sphere (equidimensional mereogeometry)	41
$SR(x)$	x is a simple, i.e., strongly self-connected region (equidimensional mereogeometry)	41
$STP(x, y)$	x is a strong tangential part of y (x is tangentially contained in y, a part of y, and contains a part of the boundary of y)	225
$\mathrm{sum}(x, y)$	the sum of x and y (**function** of the Region Connection Calculus)	164
$\mathrm{strongthickboundary}(x)$	strong thick (equidimensional) boundary (**function**)	227
$Sum(x, y, z)$	$x + y = z$ as ternary relation that does not force the existence of sums	161
$TCont(x, y)$	x is tangentially contained in y (x touches the boundary of y)	214
$\mathrm{thickboundary}(x)$	thick (equidimensional) boundary (**function**)	227
$TP(x, y)$	x is a tangential part of y (equidimensional mereotopology)	37
$TP(x, y)$	x is a tangential part of y (x is tangentially contained in y and a part of y)	223
$TPP(x, y)$	x is a tangential proper part of y (equidimensional mereotopology)	37
$TPP(x, y)$	x is a tangential proper part of y (relation of the Region Connection Calculus)	164
$TUNNEL(x)$	physical tunnel	301
$U(x, y)$	x and y underlap, i.e., they are part of a common entity (equidimensional mereotopology)	36
$UCon(x)$	uniform self-connected entity	207
u	the entity denoting the entire space or universe of highest dimension	157
$ZEX(x)$	zero region of no location that neither contains nor is contained in any other entity (**initially a primitive relation in the theory of dimension, but not primitive in the theory of containment and any extensions thereof**)	98
$ZEX_I(x)$	zero region that does not $INCH$ any other region (relation of the INCH Calculus)	171

Appendix B

List of logical axioms, definitions, theorems, and mappings

Appendix C

List of logical theories

Unless explicitly ruled out, all theories listed here come in the variants T^0 and $T^{\neg 0}$ with the additional axiom Z-A1 or NZ-A1, respectively, to force or prevent a zero region.

Dimension

Containment

Containment and dimension

$CODI_{\text{linear}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D}

(basic theory of containment and linear dimension)

codi/codi_linear.clif 102

$CODI$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2}

(basic theory of containment and linear dimension with all definitions; definitional extension of $CODI_{\text{linear}}$)

codi/codi.clif 108

$CODI_{\text{unbounded}}$ = {D-A1 – D-A5, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2}

(basic theory of containment and unbounded linear dimension with all definitions)

codi/codi_unbounded.clif 108

$CODI_{\text{pl}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, PL-A1, PL-E1, CD-E1}

(*CODI* extended by definitions of points and lines—lines containing at least two distinct points)

codi/codi_pl.clif 253

$CODI_{\text{pl}-\text{slin}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, PL-A1, PL-A2, PL-E1, CD-E1}

(*CODI* extended by definitions of points and lines forming a semi-linear space)

codi/codi_pl_slin.clif 255

$CODI_{\text{pl}-\text{lin}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, PL-A1 – PL-A3, PL-E1, CD-E1}

(*CODI* extended by definitions of points and lines forming a linear space)

codi/codi_pl_lin.clif 255

$CODI_{\text{pl}-\text{aff}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, PL-A1 – PL-A5, PL-E1, CD-E1}

(*CODI* extended by definitions of points and lines forming an affine space)

codi/codi_pl_aff.clif 255

$\mathit{CODI}_{\text{plp}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, Pl-D, PL-A1, PL-E1, CD-E1}

(*CODI* extended by definitions of points and lines—lines containing at least two distinct points)

codi/codi_plp.clif 258

$\mathit{CODI}_{\text{plp-g}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, PL-A1, PL-A3, PL-E1, CD-E1, Pl-D, PLP-A1, PLP-A3}

(the mereotopological generalization of three-dimensional point incidence geometry)

codi/codi_plp_g.clif 263

$\mathit{CODI}_{\text{plp-slin}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, PL-A1, PL-A2, PL-E1, CD-E1, Pl-D, PLP-A1 – PLP-A4}

(*CODI* extended by definitions of points and lines forming a semi-linear three-dimensional incidence geometry)

codi/codi_plp_slin.clif 259

$\mathit{CODI}_{\text{plp-lin}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, PL-A1 – PL-A3, PL-E1, CD-E1, Pl-D, PLP-A1 – PLP-A4}

(*CODI* extended by definitions of points and lines forming a linear three-dimensional incidence geometry)

codi/codi_plp_lin.clif 259

$\mathit{CODI}_{\text{plp-aff}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Pt-D, L-D, PL-A1 – PL-A5, PL-E1, CD-E1, Pl-D, PLP-A1 – PLP-A4}

(*CODI* extended by definitions of points and lines forming an affine three-dimensional incidence geometry)

codi/codi_plp_aff.clif 259

Containment and dimension with mereological closure operations

Containment, dimension, boundary-containment

$CODIB_{\updownarrow}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, BC-A1 – BC-A6, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1 – Dif-A4, Con-D, ICon-D, UCon-D, CL-D, Z-A1, ME-E1, Sum′-A0 – Sum′-A6, U-A1}

(theory of containment, linear dimension, and boundary-containment closed under intersections and differences, closed under sums of equidimensional entities when they do not intersect in the interior of their minimal parts, and restricted to atomic models)

Betweenness

BTW = {B-A1 – B-A6}

(relativized strict strong betweenness)

Containment, dimension, betweenness (ordered mereotopology)

$OMT_{\downarrow}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1 – Dif-A4, Con-D, Z-A1, B-A1 – B-A6, OMT-A1 – OMT-A3}

($CODI_{\downarrow}$ extended by relativized strict strong betweenness)

$OMT_{\downarrow-\mathrm{plp}-\mathrm{lin}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1 – Dif-A4, Con-D, Z-A1, Pt-D, L-D, PL-A1 – PL-A3, PL-E1, CD-E1, Pl-D, PLP-A1 – PLP-A4, B-A1 – B-A6, OMT-A1 – OMT-A3}

($OMT_{\downarrow}$ extended to contain an linear incidence geometry)

$OMT_{3\mathrm{D}-\mathrm{lin}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1 – Dif-A4, Con-D, Z-A1, Pt-D, L-D, PL-A1 – PL-A3, PL-E1, CD-E1, Pl-D, PLP-A1 – PLP-A4, B-A1 – B-A6, OMT-A1 – OMT-A3, OMT-E1}

($OMT_{\downarrow}$ extended to contain a three-dimensional linear incidence geometry)

Containment, dimension, boundary, betweenness

OMTB = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, BC-A1 – BC-A6, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1, Dif-A2, Dif-A3a, Dif-A3b, Dif-A3c, Dif-A4, Con-D, ICon-D, UCon-D, CL-D, Z-A1, ME-E1, B-A1 – B-A6, OMT-A1 – OMT-A4}

(*CODIB* extended by relativized strict strong betweenness)

omtb/omtb_down.clif 268

Abstract and physical space

PED = {PED-A1 – PED-A11}

(Categorization of physical entities and their relations)

space/ped.clif 287

SPACE = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, BC-A1 – BC-A6, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1 – Dif-A4, Con-D, ICon-D, UCon-D, CL-D, Z-A1, ME-E1, Sum′-A0 – Sum′-A6, U-A1, PED-A1 – PED-A11, S-A1 – S-A12, S-D1 – S-D10}

(basic theory of abstract and physical space, combining $CODIB_{\downarrow}$ with *PED* and axiomatizing the relationship between regions and physical endurants)

space/space.clif 289

SPCH = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, BC-A1 – BC-A6, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1 – Dif-A4, Con-D, ICon-D, UCon-D, CL-D, Z-A1, ME-E1, Sum′-A0 – Sum′-A6, U-A1, PED-A1 – PED-A11, S-A1 – S-A12, S-D1 – S-D10, CH-A1 – CH-A13}

(*SPACE* extended by an axiomatization of convex hulls)

space/spch.clif 294

VOIDS = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, BC-A1 – BC-A6, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1 – Dif-A4, Con-D, ICon-D, UCon-D, CL-D, Z-A1, ME-E1, Sum′-A0 – Sum′-A6, U-A1, PED-A1 – PED-A11, S-A1 – S-A12, S-D1 – S-D10, CH-A1 – CH-A13, V-A1 – V-A8, V_S-D, V_C-D, V-D}

(*SPCH* extended by the axioms characterizing voids)

space/voids.clif 297

$VOIDS_{\text{extended}}$ = {D-A1 – D-A6, D-D1 – D-D7, C-A1 – C-A4, CD-A1, BC-A1 – BC-A6, C-D, C-T1, EP-D, EPP-D, PO-D, Inc-D, SC-D, ME-D1, ME-D2, Int-A1 – Int-A4, Dif-A1 – Dif-A4, Con-D, ICon-D, UCon-D, CL-D, Z-A1, ME-E1, Sum′-A0 – Sum′-A6, U-A1, PED-A1 – PED-A11, S-A1 – S-A12, S-D1 – S-D10, CH-A1 – CH-A13, V-A1 – V-A24, V_S-D, V_C-D, V-D, Hole-D, Gap-D, CAV-D, TUN-D, HOL-D}

(*VOIDS* extended by the axioms classifying physical voids according to three criteria)

space/voids-refined.clif 303

External spatial theories

$INCH_{\text{original}}$ = {I-A1 – I-A10, I-D1 – I-D9}

(INCH Calculus)

inch/inch_original.clif 170

$INCH_{\text{calculus}}$ = {I-A1 – I-A10, I-D1 – I-D9, I-A7’}

(extended INCH Calculus)

inch/inch_calculus.clif 174

$INCH_{\text{weak}}$ = {I-A1 – I-A5, I-D1 – I-D6}

(weak version of the INCH Calculus)

inch/inch_weak.clif 175

$INCH_{\text{weak-closed}}$ = {I-A1 – I-A5, I-A9, I-A10, I-D1 – I-D6}

(weak version of the INCH Calculus with closure under equidimensional sums and complements)

inch/inch_weak_closed.clif 175

RCC = {RCC1 – RCC7, RCC-Ext, RCC-P, RCC-PP, RCC-O, RCC-EC, RCC-NTPP}

(strict Region Connection Calculus that allows atomic models)

mt/rcc_basic_strict.clif 164

IG_{2D} = {I.0, I.1}

(line space as a two-dimensional incidence geometry)

ig/ig_2d.clif 244

IG = {I.0 – I.5}

(incidence geometry over three disjoint sets of points, lines, and planes)

ig/ig.clif 247

WOIG = {I.0 – I.5, O.1 – O.4}

(weak ordered incidence geometry with a three-dimensional point incidence structure)

oig/woig.clif 249

Contact algebras

Appendix D

List of automated and manual proofs

Theory	Sentences	Page	Manual Proof	Output of automated theorem prover (if automated proof is available) and necessary lemmas (if applicable)
SPOCA	M-I	66	no	ca/theorems/spoca_M-I.clif
SPOCA	M-S	66	no	ca/theorems/spoca_M-S.clif
SPOCA ∪ M-C	Uni	66	no	ca/theorems/spoca_M-C_Uni.clif
SPOCA ∪ Uni	M-C	66	no	ca/theorems/spoca_Uni_M-C.clif
SPOCA ∪ M-C	O-Ext	66	no	ca/theorems/spoca_M-C_O-Ext.clif
SPOCA	T-I	67	yes	no
SPOCA	T-S$^{\leftarrow}$	67	yes	no
SPOCA ∪ C4	T-S	67	yes	no
SPOCA ∪ C5	C4	68	no	ca/theorems/spoca_C5_C4.clif
SPOCA ∪ C5	C-Ext	68	no	ca/theorems/spoca_C5_C-Ext.clif
SPOCA ∪ C5	¬Con	68	yes	ca/theorems/spoca_C5_notCon.clif
SPOCA ∪ C5	Int	68	yes	ca/theorems/spoca_C5_Int.clif
SPOCA ∪ C5′	Con	69	no	ca/theorems/spoca_C5prime_Con.clif
SPOCA ∪ C-Ext	Dis	72	no	ca/theorems/spoca_C-Ext_Dis.clif

Theory	Sentences	Page	Manual Proof	Output of automated theorem prover (if automated proof is available) and necessary lemmas (if applicable)
$SPOCA \cup \{\text{Uni, Dis}, \neg\text{C4}, \neg\text{Triv}, \neg\text{Con}\}$	⊥ (unsatisfiable)	72	Proof partially manual, partially automatic: spoca_Uni_Dis_notC4_notTriv_notCon.clif	
$SPOCA \cup \{\text{Dis, Atom}, \neg\text{Triv}, \text{Con}\}$	⊥ (unsatisfiable)	72	yes	no
$EWBCA \cup \{\text{Atom, Con}\}$	⊥ (unsatisfiable)	72	yes	no
$DI_{\text{linear}-\text{unbounded}}$	D-T1 – D-T5	98		dim/theorems/dim_prime_linear_unbounded_theorems.clif
CO_{basic}	C-T1	101		cont/theorems/cont_basic_ext.clif
$CO_{\text{basic}} \cup$ C-D	C-T2 – C-T5	101		cont/theorems/cont_c_theorems.clif
$CODI_{\text{linear}} \cup \{\text{EP-D, EPP-D}\}$	EP-T1 – EP-T9	103		codi/theorems/ep_theorems.clif
$CODI_{\text{linear}} \cup \{\text{EP-D, PO-D}\}$	PO-T1 – PO-T3	105		codi/theorems/po_theorems.clif
$CODI_{\text{linear}} \cup \{\text{EP-D, Inc-D}\}$	Inc-T1 – Inc-T6	106		codi/theorems/inc_theorems.clif
$CODI_{\text{linear}} \cup \{\text{EP-D, SC-D}\}$	SC-T1 – SC-T4	107		codi/theorems/sc_theorems.clif
CODI	CD-T1 – CD-T9	108		codi/theorems/codi_theorems.clif
CODI	Int-T1 – Int-T4	114	no	codi/theorems/int_theorems.clif
$CODI \cup \{\text{Int-A1 – Int-A4}\}$	Int-T5 – Int-T9, Int-E1	115	no	codi/theorems/codi_int_theorems.clif
$CODI \cup \{\text{Int-A1 – Int-A4}\}$	Int-T10 – Int-T12	116	no	codi/theorems/codi_int_theorems.clif, and codi/theorems/codi_int_theoremsT11prime.clif
$CODI \cup \{\text{Dif-A1 – Dif-A4}\}$	Z-A1	119	no	codi/theorems/codi_down_theoremsZ-A1.clif
$CODI_{\downarrow}$	Dif-T1, Dif-T2	119	yes	codi/theorems/codi_down_theoremsT1-T2.clif
$CODI_{\downarrow}$	Dif-T3	119	yes	codi/theorems/codi_down_theoremsT3-T5.clif
$CODI_{\downarrow}$	Dif-T4	120	yes	no
$CODI_{\downarrow}$	Dif-T5	120	yes	no
$CODI_{\downarrow}$	EP-E1	121	yes	codi/theorems/codi_down_theoremsEP-E1.clif (necessary lemmas: Dif-T1 – Dif-T3)
$CODI_{\downarrow}$	EP-E2	121	yes	codi/theorems/codi_down_theoremsEP-E2.clif (necessary lemma: EP-E1)
$CODI_{\downarrow}$	EP-E3	121	yes	codi/theorems/codi_down_theoremsEP-E3.clif (necessary lemma: Dif-T1)
$CODI_{\downarrow}$	PO-E1	123	yes	codi/theorems/codi_down_theoremsPO-E1.clif (necessary lemmas: EP-E2 (manual); Dif-T1, Dif-T3 (automatic))

Theory	Sentences	Page	Manual Proof	Output of automated theorem prover (if automated proof is available) and necessary lemmas (if applicable)
$CODI_\downarrow$	Dif-T6	124	yes	no
$CODI_\downarrow$	Dif-T7	124	yes	codi/theorems/codi_down_theoremsT6-T10.clif
$CODI_\downarrow$	Dif-T8	125	yes	no
$CODI_\downarrow$	Dif-T9	125	yes	codi/theorems/codi_down_theoremsT6-T10.clif
$CODI_\downarrow$	Dif-T10	126	yes	codi/theorems/codi_down_theoremsT6-T10.clif
$CODI_\downarrow \cup$ Con-D	Con-T1 – Con-T5	127	Con-T3 – Con-T5	codi/theorems/con_theorems.clif (only Con-T1 – Con-T3)
$CODI_\downarrow$	Dif-T11	129	yes	codi/theorems/codi_down_theoremsT11.clif
$CODI_\downarrow \cup$ Dif-T11	Dif-T11′	129	yes	codi/theorems/codi_down_theoremsT11prime.clif
$CODI_\downarrow \cup$ ME-E1	Dif-T12	130	yes	no
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T1	136	yes	codi/theorems/codi_down_sum_theorems.clif (necessary lemma: EP-E3)
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T2	137	yes	codi/theorems/codi_down_sum_theorems.clif
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T3	137	yes	no
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T4	137	yes	codi/theorems/codi_down_sum_theorems.clif
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T5	137	yes	codi/theorems/codi_down_sum_theorems.clif
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T6	138	yes	no
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T7	138	yes	no (necessary lemmas: PO-E1, Sum-T3, Sum-T6 (manual))
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T8	140	yes	no
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T8′	141	yes	no
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T9	141	yes	codi/theorems/codi_down_sum_theorems.clif (the six separate cases only; necessary lemmas: Sum-T8′)
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T10	143	yes	no (necessary lemmas: Sum-T5, Sum-T9 (manual))
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T11	143	yes	no (necessary lemma: Sum-T10 (manual))
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T12	146	yes	codi/theorems/codi_down_sum_theoremsT11-T14.clif
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T13	146	yes	codi/theorems/codi_down_sum_theoremsT11-T14.clif
$CODI_\downarrow \cup$ {Sum-A1 – Sum-A4}	Sum-T14	146	yes	no (necessary lemmas: Sum-T8′, Sum-T10 (manual))

Theory	Sentences	Page	Manual Proof	Output of automated theorem prover (if automated proof is available) and necessary lemmas (if applicable)
$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}	Sum-T15	147	yes	no
$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}	Sum-T16	148	yes	no
$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}	Sum-T17	149	yes	codi/theorems/codi_down_sum_theoremsT16-T20.clif
$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}	Sum-T18	149	yes	codi/theorems/codi_down_sum_theoremsT16-T20.clif
$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}	Sum-T19	149	yes	no
$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}	Sum-T20	151	yes	no (necessary lemma: Sum-T19 (manual))
$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}	Sum-T21	152	yes	no
$CODI_{\downarrow} \cup$ {Sum-A1 – Sum-A4}	Sum-T22	153	yes	no
$CODI_{\updownarrow}$	U-T1, U-T2	157	yes	codi/theorems/codi_updown_theorems.clif
$CODI_{\updownarrow}$	U-T3 – U-T5	157	yes	codi/theorems/codi_updown_theorems.clif
$CODI_{\updownarrow}$	U-T6, U-T7	158	yes	no (necessary lemmas: Sum-T21, Sum-T22 (manual))
$INCH_{\text{original}}$	I-T1 – I-T13	171	no	inch/theorems/inch_original_theorems.clif
$INCH_{\text{calculus}}$	I-T14	171	no	inch/theorems/inch_calculus_theorems.clif
$INCH_{\text{weak}} \cup$ {I-M1′ – I-M3′}	CO_{basic}	175	no	inch/theorems/inch_cont_basic.clif
$INCH_{\text{weak}} \cup$ {D-D1 – D-D6, I-M1′ – I-M3′}	D-A1 – D-A5	176	no	inch/theorems/inch_dim_prime_linear.clif
$INCH_{\text{weak}} \cup$ {D-D1 – D-D6, I-M1′ – I-M3′}	CD-A1	176	no	inch/theorems/inch_CD-A1.clif
$INCH_{\text{weak-closed}} \cup$ {I-M1′ – I-M3′}	Z-A1	176	no	inch/theorems/inch_weak_closed_Z-A1.clif
$INCH_{\text{weak}} \cup$ I-E1 $\cup$ {D-D1 – D-D6, I-M1′ – I-M3′}	D-A6	176	no	inch_weak_I-E1_D-A6.clif
$CODI^0 \cup$ {I-D1 – I-D9, I-M1}	I-A3	177	no	inch/theorems/codi_linear_I-PA3.clif
$CODI^0 \cup$ {I-D1 – I-D9, I-M1}	I-A6	177	no	inch/theorems/codi_linear_I-PA6.clif
$CODI^0 \cup$ {I-D1 – I-D9, I-M1}	I-A7	177	no	inch/theorems/codi_linear_I-PA7.clif
$CODI^0 \cup$ {I-D1 – I-D9, I-M1}	I-M2	177	no	inch/theorems/codi_linear_I-M2.clif
$CODI^0 \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1, I-M2}	I-M3	178	no	inch/theorems/codi_linear_C-E4_I-M3.clif
$CODI^0 \cup$ {I-D1 – I-D9, I-M1}	I-E1	178	no	inch/theorems/codi_linear_C-E4_I-E1.clif
$CODI^0 \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M3}	I-A4	178	no	inch/theorems/codi_linear_C-E4_I-PA4.clif
$CODI^0 \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M3}	I-A5	178	no	inch/theorems/codi_linear_C-E4_I-PA5.clif

Theory	Sentences	Page	Manual Proof	Output of automated theorem prover (if automated proof is available) and necessary lemmas (if applicable)
$CODI^0 \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M3}	I-M4	178	no	inch/theorems/codi_linear_C-E4_I-M4.clif
$CODI^0 \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M3}	I-M5	178	no	inch/theorems/codi_linear_C-E4_I-M5.clif
$CODI^0 \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M3}	I-M6	178	no	inch/theorems/codi_linear_C-E4_I-M6.clif
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M6}	I-M7	179		Proof partially manual, partially automatic: inch/theorems/codi_down_C-E4_I-M7.clif
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M6}	I-M8	179		Proof partially manual, partially automatic: inch/theorems/codi_down_C-E4_I-M8.clif
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8}	I-A8	180	yes	no
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8}	I-M9	180	no	inch/theorems/codi_down_C-E4_I-M9.clif
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8}	I-M10	180	no	inch/theorems/codi_down_C-E4_I-M10.clif
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8}	I-A1	181	yes	no
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8}	I-A2	181	yes	no
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8}	I-A10	182	yes	no
$CODI_\downarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M8}	I-A9	183	yes	no
$INCH_{\text{calculus}} \cup$ {I-E1, I-E2} $\cup$ {I-M1′ – I-M6′} $\cup$ {EP-D, EPP-D, PO-D}	Int-A1 – Int-A4	189	yes	no
$INCH_{\text{calculus}} \cup$ {I-E1, I-E2} $\cup$ {I-M1′ – I-M6′} $\cup$ {EP-D, EPP-D, PO-D}	Dif-A1 – Dif-A4	190	yes	no
$INCH_{\text{calculus}} \cup$ {I-E1, I-E2} $\cup$ {I-M1′ – I-M6′} $\cup$ {EP-D, EPP-D, PO-D}	Sum-A1 – Sum-A4	196	yes	no
$INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} $\cup$ {I-M1′ – I-M6′}	U-A1	199	yes	no
$INCH_{\text{calculus}} \cup$ {I-E1 – I-E3} $\cup$ {I-M1′ – I-M6′} $\cup$ {EP-D, EPP-D, PO-D}	C-E4	200	yes	no
$CODI_\updownarrow \cup$ C-E4 $\cup$ {I-D1 – I-D9, I-M1 – I-M10}	I-E1 – I-E3	200	yes	no
$CODI_\downarrow \cup$ {Con-D, ICon-D, UCon-D}	ICon-T1	207	no	codi/theorems/icon_theorems.clif
$CODI_\downarrow \cup$ {Con-D, ICon-D, UCon-D}	UCon-T1	207	no	codi/theorems/icon_theorems.clif
$CODI \cup$ {BC-A1 – BC-A5}	BC-T1 – BC-T3	211	yes	codib/theorems/codi_bcont_theorems.clif
$CODI \cup$ {BC-A1 – BC-A5} $\cup$ U-A1	BC-T4	212	yes	codib/theorems/codi_bcont_U-A1_theorems.clif
CODIB	BC-T5	212	yes	codib/theorems/codib_theorems.clif

Theory	Sentences	Page	Manual Proof	Output of automated theorem prover (if automated proof is available) and necessary lemmas (if applicable)
$CODIB \cup$ IC-D	IC-T1 – IC-T6	213	no	codib/theorems/codib_icont_theorems.clif
$CODIB \cup$ TC-D	TC-T1 – TC-T5	213	no	codib/theorems/codib_tcont_theorems.clif
$CODIB \cup$ {IC-D, TC-D}	BC-T6 – BC-T7	214	no	codib/theorems/codib_icont_tcont_theorems.clif
$CODIB \cup$ BC-E1	BC-T8 – BC-T9	220	yes	codib/theorems/codib_boundary_theorems.clif
$CODIB_{\updownarrow}$	Sum′-T1	221	yes	no
$CODIB_{\downarrow} \cup$ {TP-D, IP-D}	TP-T1 – TP-T5	223	no	codib/theorems/codib_tp_ip_theorems.clif
$CODIB_{\downarrow} \cup$ {TP-D, IP-D, STP-D, BP-D, SBP-D}	BP-T1 – BP-T5	226	yes	codib/theorems/codib_bp_theorems.clif
$CODIB_{\downarrow} \cup$ {IO-D, IBC-D, BO-D}	BC-T10 – BC-T13	228	no	codib/theorems/codib_down_9intersection_theorems.clif
$CODIB_{\downarrow} \cup$ {IO-D, IBC-D, BO-D}	BC-T14	228	yes	no
$WOIG$	O.7′, O.8′	249	no	oig/theorems/woig_3d_theorems.clif
$CODI_{\mathrm{pl}}$	PL-T1 – PL-T3	253	no	codi/theorems/codi_pl_theorems.clif
$CODI_{\mathrm{plp}}$	PLP-T1, PLP-T2	259	no	codi/theorems/codi_plp_theorems.clif
BTW	B-T1	265	no	btw/theorems/btw_basic_theorems.clif
$OMT_{\downarrow}$	OMT-T1	268	no	omt/theorems/omt_down_theorems.clif
PED	PED-T1, PED-T2	287	no	space/theorems/ped_theorems.clif
$SPACE$	S-T1, S-T2	289	yes	space/theorems/space_theorems.clif
$SPCH$	CH-T1 – CH-T3	295	yes	space/theorems/spch_theorems.clif
$VOIDS$	V-T1	298	yes	space/theorems/voids_theorems.clif
$VOIDS \cup$ {V-A9, V-A10, Hole-D, Gap-D}	V-T2	299	yes	space/theorems/voids_extended_theorems.clif
$VOIDS \cup$ {V-A11 – V-A18}	V-T3 – V-T5	301	yes	space/theorems/voids_extended_theorems.clif
$VOIDS \cup$ {V-A19 – V-A24}	V-T6	303	yes	space/theorems/voids_extended_theorems.clif

Appendix E

Summary of results from automated theorem provers and model finders

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----
lat
----
lat\output\bounded_lattice_meet_join tested with: Mace4, Prover9
    FAILED: lat\output\bounded_lattice_meet_join.p9.out IN 1.93 ON 2012-10-30 19:08:20
  SUCCESS: lat\output\bounded_lattice_meet_join.m4.out IN 0.01 ON 2012-10-30 19:08:18, MODEL SIZE=2
lat\output\lattice_meet_join tested with: Mace4, Prover9
    FAILED: lat\output\lattice_meet_join.p9.out IN 1.90 ON 2012-10-30 19:08:12
  SUCCESS: lat\output\lattice_meet_join.m4.out IN 0.01 ON 2012-10-30 19:08:10, MODEL SIZE=2
lat\output\ortho_complemented_lattice_meet_join tested with: Mace4, Prover9
    FAILED: lat\output\ortho_complemented_lattice_meet_join.p9.out IN 1.94 ON 2012-10-30 19:08:28
  SUCCESS: lat\output\ortho_complemented_lattice_meet_join.m4.out IN 0.01 ON 2012-10-30 19:08:26, MODEL SIZE=2

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ca
----
ca\consistency\output\oca_nontrivial tested with: Mace4, Prover9
    FAILED: ca\consistency\output\oca_nontrivial.p9.out IN 1.94 ON 2012-11-5 16:42:37
  SUCCESS: ca\consistency\output\oca_nontrivial.m4.out IN 0.01 ON 2012-11-5 16:42:35, MODEL SIZE=4
ca\consistency\output\spoca_c5prime_nontrivial tested with: Mace4, Prover9
    FAILED: ca\consistency\output\spoca_c5prime_nontrivial.p9.out IN 1.91 ON 2012-11-5 16:40:16
  SUCCESS: ca\consistency\output\spoca_c5prime_nontrivial.m4.out IN 0.01 ON 2012-11-5 16:40:14, MODEL SIZE=4
ca\consistency\output\spoca_c5_nontrivial tested with: Mace4, Prover9
    FAILED: ca\consistency\output\spoca_c5_nontrivial.p9.out IN 1.91 ON 2012-11-5 16:40:47
  SUCCESS: ca\consistency\output\spoca_c5_nontrivial.m4.out IN 0.01 ON 2012-11-5 16:40:45, MODEL SIZE=6
ca\consistency\output\spoca_nontrivial tested with: Mace4, Prover9
    FAILED: ca\consistency\output\spoca_nontrivial.p9.out IN 1.91 ON 2012-11-5 16:42:24
  SUCCESS: ca\consistency\output\spoca_nontrivial.m4.out IN 0.01 ON 2012-11-5 16:42:22, MODEL SIZE=6
ca\consistency\output\wbca_c5prime_c4_nontrivial tested with: Mace4, Prover9
    FAILED: ca\consistency\output\wbca_c5prime_c4_nontrivial.p9.out IN 1.94 ON 2012-11-5 16:43:52
  SUCCESS: ca\consistency\output\wbca_c5prime_c4_nontrivial.m4.out IN 0.01 ON 2012-11-5 16:43:50, MODEL SIZE=4
ca\consistency\output\wbca_c5prime_dis_nontrivial tested with: Mace4, Prover9
  UNKNOWN: ca\consistency\output\wbca_c5prime_dis_nontrivial
    FAILED: ca\consistency\output\wbca_c5prime_dis_nontrivial.p9.out
    FAILED: ca\consistency\output\wbca_c5prime_dis_nontrivial.m4.out IN 0.88 ON 2012-11-5 12:33:55
ca\consistency\output\wbca_c5prime_nontrivial tested with: Mace4, Prover9
    FAILED: ca\consistency\output\wbca_c5prime_nontrivial.p9.out IN 1.96 ON 2012-11-5 16:43:10
  SUCCESS: ca\consistency\output\wbca_c5prime_nontrivial.m4.out IN 0.01 ON 2012-11-5 16:43:08, MODEL SIZE=4
ca\consistency\output\wbca_notc4_atom_con_nontrivial tested with: Mace4, Prover9
    FAILED: ca\consistency\output\wbca_notc4_atom_con_nontrivial.p9.out IN 1.90 ON 2012-11-5 16:03:07
  SUCCESS: ca\consistency\output\wbca_notc4_atom_con_nontrivial.m4.out IN 0.01 ON 2012-11-5 16:03:05, MODEL SIZE=4
ca\theorems\output\spoca_c-ext_dis tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c-ext_dis.p9.out IN 1.92 ON 2012-11-5 17:45:17
  SUCCESS: ca\theorems\output\spoca_c-ext_dis.m4.out IN 0.01 ON 2012-11-5 17:45:15, MODEL SIZE=2
ca\theorems\output\spoca_c-ext_dis_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_c-ext_dis_1.p9.out IN 0.05 ON 2012-11-5 17:45:22, PROOF LENGTH=41
    FAILED: ca\theorems\output\spoca_c-ext_dis_1.m4.out IN 1.98 ON 2012-11-5 17:45:24
ca\theorems\output\spoca_c4_c5prime_c-ext tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c4_c5prime_c-ext.p9.out IN 1.93 ON 2012-11-5 17:48:03
  SUCCESS: ca\theorems\output\spoca_c4_c5prime_c-ext.m4.out IN 0.01 ON 2012-11-5 17:48:01, MODEL SIZE=2
ca\theorems\output\spoca_c4_c5prime_c-ext_1 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c4_c5prime_c-ext_1.p9.out IN 1.93 ON 2012-11-5 17:48:10
  SUCCESS: ca\theorems\output\spoca_c4_c5prime_c-ext_1.m4.out IN 0.01 ON 2012-11-5 17:48:08, MODEL SIZE=4
ca\theorems\output\spoca_c4_c5prime_c-ext_2 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_c4_c5prime_c-ext_2.p9.out IN 0.01 ON 2012-11-5 17:48:14, PROOF LENGTH=6
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    FAILED: ca\theorems\output\spoca_c4_c5prime_c-ext_2.m4.out IN 1.02 ON 2012-11-5 17:48:15
ca\theorems\output\spoca_c4_c5prime_int tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c4_c5prime_int.p9.out IN 1.94 ON 2012-11-5 17:47:50
  SUCCESS: ca\theorems\output\spoca_c4_c5prime_int.m4.out IN 0.01 ON 2012-11-5 17:47:48, MODEL SIZE=2
ca\theorems\output\spoca_c4_c5prime_int_1 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c4_c5prime_int_1.p9.out IN 1.98 ON 2012-11-5 17:47:57
  SUCCESS: ca\theorems\output\spoca_c4_c5prime_int_1.m4.out IN 0.02 ON 2012-11-5 17:47:56, MODEL SIZE=6
ca\theorems\output\spoca_c4_c5prime_uni tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c4_c5prime_uni.p9.out IN 1.94 ON 2012-11-5 17:47:25
  SUCCESS: ca\theorems\output\spoca_c4_c5prime_uni.m4.out IN 0.01 ON 2012-11-5 17:47:23, MODEL SIZE=2
ca\theorems\output\spoca_c4_c5prime_uni_1 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c4_c5prime_uni_1.p9.out IN 1.94 ON 2012-11-5 17:47:32
  SUCCESS: ca\theorems\output\spoca_c4_c5prime_uni_1.m4.out IN 0.02 ON 2012-11-5 17:47:30, MODEL SIZE=6
ca\theorems\output\spoca_c5prime_c4 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5prime_c4.p9.out IN 1.88 ON 2012-11-5 17:45:53
  SUCCESS: ca\theorems\output\spoca_c5prime_c4.m4.out IN 0.01 ON 2012-11-5 17:45:51, MODEL SIZE=2
ca\theorems\output\spoca_c5prime_c4_1 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5prime_c4_1.p9.out IN 1.96 ON 2012-11-5 17:46:00
  SUCCESS: ca\theorems\output\spoca_c5prime_c4_1.m4.out IN 0.03 ON 2012-11-5 17:45:58, MODEL SIZE=8
ca\theorems\output\spoca_c5prime_con tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5prime_con.p9.out IN 1.93 ON 2012-11-5 17:45:34
  SUCCESS: ca\theorems\output\spoca_c5prime_con.m4.out IN 0.01 ON 2012-11-5 17:45:32, MODEL SIZE=2
ca\theorems\output\spoca_c5prime_con_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_c5prime_con_1.p9.out IN 0.02 ON 2012-11-5 17:45:39, PROOF LENGTH=12
    FAILED: ca\theorems\output\spoca_c5prime_con_1.m4.out IN 0.45 ON 2012-11-5 17:45:39
ca\theorems\output\spoca_c5_c-ext tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5_c-ext.p9.out IN 1.93 ON 2012-11-5 17:46:52
  SUCCESS: ca\theorems\output\spoca_c5_c-ext.m4.out IN 0.01 ON 2012-11-5 17:46:50, MODEL SIZE=2
ca\theorems\output\spoca_c5_c-ext_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_c5_c-ext_1.p9.out IN 1.06 ON 2012-11-5 17:46:58, PROOF LENGTH=25
    FAILED: ca\theorems\output\spoca_c5_c-ext_1.m4.out IN 0.32 ON 2012-11-5 17:46:57
ca\theorems\output\spoca_c5_c-ext_2 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_c5_c-ext_2.p9.out IN 0.00 ON 2012-11-5 17:47:01, PROOF LENGTH=6
    FAILED: ca\theorems\output\spoca_c5_c-ext_2.m4.out IN 1.95 ON 2012-11-5 17:47:03
ca\theorems\output\spoca_c5_c4 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5_c4.p9.out IN 1.94 ON 2012-11-5 17:47:00
  SUCCESS: ca\theorems\output\spoca_c5_c4.m4.out IN 0.01 ON 2012-11-5 17:46:58, MODEL SIZE=2
ca\theorems\output\spoca_c5_c4_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_c5_c4_1.p9.out IN 0.02 ON 2012-11-5 17:47:05, PROOF LENGTH=25
    FAILED: ca\theorems\output\spoca_c5_c4_1.m4.out IN 1.97 ON 2012-11-5 17:47:07
ca\theorems\output\spoca_c5_int tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5_int.p9.out IN 1.91 ON 2012-11-5 17:46:38
  SUCCESS: ca\theorems\output\spoca_c5_int.m4.out IN 0.01 ON 2012-11-5 17:46:36, MODEL SIZE=2
ca\theorems\output\spoca_c5_int_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_c5_int_1.p9.out IN 0.02 ON 2012-11-5 17:46:43, PROOF LENGTH=19
    FAILED: ca\theorems\output\spoca_c5_int_1.m4.out IN 0.32 ON 2012-11-5 17:46:43
ca\theorems\output\spoca_c5_notcon tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5_notcon.p9.out IN 1.89 ON 2012-11-5 17:46:23
  SUCCESS: ca\theorems\output\spoca_c5_notcon.m4.out IN 0.01 ON 2012-11-5 17:46:21, MODEL SIZE=2
ca\theorems\output\spoca_c5_notcon_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_c5_notcon_1.p9.out IN 0.02 ON 2012-11-5 17:46:28, PROOF LENGTH=13
    FAILED: ca\theorems\output\spoca_c5_notcon_1.m4.out IN 0.32 ON 2012-11-5 17:46:28
ca\theorems\output\spoca_c5_uni tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5_uni.p9.out IN 1.95 ON 2012-11-5 17:46:12
  SUCCESS: ca\theorems\output\spoca_c5_uni.m4.out IN 0.01 ON 2012-11-5 17:46:10, MODEL SIZE=2
ca\theorems\output\spoca_c5_uni_1 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_c5_uni_1.p9.out IN 1.92 ON 2012-11-5 17:46:18
  SUCCESS: ca\theorems\output\spoca_c5_uni_1.m4.out IN 0.01 ON 2012-11-5 17:46:16, MODEL SIZE=6
ca\theorems\output\spoca_dis_atom_nottriv_con tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_dis_atom_nottriv_con.p9.out IN 1.91 ON 2012-11-5 18:27:53
  SUCCESS: ca\theorems\output\spoca_dis_atom_nottriv_con.m4.out IN 0.01 ON 2012-11-5 18:27:51, MODEL SIZE=2
ca\theorems\output\spoca_dis_atom_nottriv_con_1 tested with: Mace4, Prover9
  UNKNOWN: ca\theorems\output\spoca_dis_atom_nottriv_con_1
    FAILED: ca\theorems\output\spoca_dis_atom_nottriv_con_1.p9.out IN 600.00 ON 2012-11-5 18:38:17
    FAILED: ca\theorems\output\spoca_dis_atom_nottriv_con_1.m4.out IN 41.16 ON 2012-11-5 18:28:39
ca\theorems\output\spoca_m-c_o-ext tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_m-c_o-ext.p9.out IN 1.97 ON 2012-11-5 17:44:50
  SUCCESS: ca\theorems\output\spoca_m-c_o-ext.m4.out IN 0.01 ON 2012-11-5 17:44:48, MODEL SIZE=2
ca\theorems\output\spoca_m-c_o-ext_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_m-c_o-ext_1.p9.out IN 0.03 ON 2012-11-5 17:44:55, PROOF LENGTH=34
    FAILED: ca\theorems\output\spoca_m-c_o-ext_1.m4.out IN 0.33 ON 2012-11-5 17:44:55
ca\theorems\output\spoca_m-c_o-ext_2 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_m-c_o-ext_2.p9.out IN 0.01 ON 2012-11-5 17:44:59, PROOF LENGTH=11
    FAILED: ca\theorems\output\spoca_m-c_o-ext_2.m4.out IN 0.36 ON 2012-11-5 17:44:59
ca\theorems\output\spoca_m-c_uni tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_m-c_uni.p9.out IN 1.95 ON 2012-11-5 17:44:39
  SUCCESS: ca\theorems\output\spoca_m-c_uni.m4.out IN 0.01 ON 2012-11-5 17:44:37, MODEL SIZE=2
ca\theorems\output\spoca_m-c_uni_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_m-c_uni_1.p9.out IN 0.04 ON 2012-11-5 17:44:44, PROOF LENGTH=52
    FAILED: ca\theorems\output\spoca_m-c_uni_1.m4.out IN 0.34 ON 2012-11-5 17:44:45
ca\theorems\output\spoca_m-i tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_m-i.p9.out IN 1.92 ON 2012-11-5 17:44:23
  SUCCESS: ca\theorems\output\spoca_m-i.m4.out IN 0.01 ON 2012-11-5 17:44:21, MODEL SIZE=2
ca\theorems\output\spoca_m-i_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_m-i_1.p9.out IN 0.04 ON 2012-11-5 17:44:28, PROOF LENGTH=37
    FAILED: ca\theorems\output\spoca_m-i_1.m4.out IN 1.98 ON 2012-11-5 17:44:30
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ca\theorems\output\spoca_m-s tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_m-s.p9.out IN 1.93 ON 2012-11-5 17:38:00
  SUCCESS: ca\theorems\output\spoca_m-s.m4.out IN 0.01 ON 2012-11-5 17:37:58, MODEL SIZE=2
ca\theorems\output\spoca_m-s_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_m-s_1.p9.out IN 227.89 ON 2012-11-5 17:41:59, PROOF LENGTH=95
    FAILED: ca\theorems\output\spoca_m-s_1.m4.out IN 113.19 ON 2012-11-5 17:39:58
ca\theorems\output\spoca_t-i tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_t-i.p9.out IN 1.93 ON 2012-11-5 17:37:33
  SUCCESS: ca\theorems\output\spoca_t-i.m4.out IN 0.01 ON 2012-11-5 17:37:31, MODEL SIZE=2
ca\theorems\output\spoca_t-i_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_t-i_1.p9.out IN 0.03 ON 2012-11-5 17:37:38, PROOF LENGTH=26
    FAILED: ca\theorems\output\spoca_t-i_1.m4.out IN 1.98 ON 2012-11-5 17:37:40
ca\theorems\output\spoca_uni_c5 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_uni_c5.p9.out IN 1.95 ON 2012-11-5 17:31:51
  SUCCESS: ca\theorems\output\spoca_uni_c5.m4.out IN 0.01 ON 2012-11-5 17:31:49, MODEL SIZE=2
ca\theorems\output\spoca_uni_c5_1 tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_uni_c5_1.p9.out IN 1.97 ON 2012-11-5 17:31:58
  SUCCESS: ca\theorems\output\spoca_uni_c5_1.m4.out IN 0.01 ON 2012-11-5 17:31:56, MODEL SIZE=4
ca\theorems\output\spoca_uni_c5_2 tested with: Mace4, Prover9
  UNKNOWN: ca\theorems\output\spoca_uni_c5_2
    FAILED: ca\theorems\output\spoca_uni_c5_2.p9.out IN 600.00 ON 2012-11-5 17:42:03
    FAILED: ca\theorems\output\spoca_uni_c5_2.m4.out IN 0.67 ON 2012-11-5 17:32:03
ca\theorems\output\spoca_uni_dis_notc4_nottriv_notcon tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_uni_dis_notc4_nottriv_notcon.p9.out IN 1.91 ON 2012-11-5 12:11:10
  SUCCESS: ca\theorems\output\spoca_uni_dis_notc4_nottriv_notcon.m4.out IN 0.32 ON 2012-11-5 12:11:08, MODEL SIZE=16
ca\theorems\output\spoca_uni_dis_notc4_nottriv_notcon_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_uni_dis_notc4_nottriv_notcon_1.p9.out IN 369.45 ON 2012-11-5 12:17:25, PROOF LENGTH=115
    FAILED: ca\theorems\output\spoca_uni_dis_notc4_nottriv_notcon_1.m4.out IN 368.74 ON 2012-11-5 12:17:27
ca\theorems\output\spoca_uni_m-c tested with: Mace4, Prover9
    FAILED: ca\theorems\output\spoca_uni_m-c.p9.out IN 1.96 ON 2012-11-5 17:30:22
  SUCCESS: ca\theorems\output\spoca_uni_m-c.m4.out IN 0.01 ON 2012-11-5 17:30:20, MODEL SIZE=2
ca\theorems\output\spoca_uni_m-c_1 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_uni_m-c_1.p9.out IN 0.34 ON 2012-11-5 17:30:28, PROOF LENGTH=65
    FAILED: ca\theorems\output\spoca_uni_m-c_1.m4.out IN 0.59 ON 2012-11-5 17:30:28
ca\theorems\output\spoca_uni_m-c_2 tested with: Mace4, Prover9
  SUCCESS: ca\theorems\output\spoca_uni_m-c_2.p9.out IN 0.02 ON 2012-11-5 17:30:32, PROOF LENGTH=31
    FAILED: ca\theorems\output\spoca_uni_m-c_2.m4.out IN 0.58 ON 2012-11-5 17:30:32


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dim
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dim\consistency\output\dim_prime_linear_bounded_discrete_nozex_nontrivial tested with: Mace4, Prover9
    FAILED: dim\consistency\output\dim_prime_linear_bounded_discrete_nozex_nontrivial.p9.out IN 1.98 ON 2011-4-29 17:43:00
  SUCCESS: dim\consistency\output\dim_prime_linear_bounded_discrete_nozex_nontrivial.m4.out IN 0.00 ON 2011-4-29 17:42:58, MODEL SIZE=3
dim\output\dim_basic tested with: Mace4, Prover9
    FAILED: dim\output\dim_basic.p9.out IN 0.00 ON 2012-11-28 13:46:08
  SUCCESS: dim\output\dim_basic.m4.out IN 0.00 ON 2012-11-28 13:46:08, MODEL SIZE=2
dim\output\dim_prime_linear tested with: Mace4, Prover9
    FAILED: dim\output\dim_prime_linear.p9.out IN 0.00 ON 2011-5-2 11:25:47
  SUCCESS: dim\output\dim_prime_linear.m4.out IN 0.01 ON 2011-5-2 11:25:47, MODEL SIZE=3
dim\output\dim_prime_linear_bounded_discrete tested with: Mace4, Prover9
    FAILED: dim\output\dim_prime_linear_bounded_discrete.p9.out IN 0.02 ON 2011-4-29 12:24:38
  SUCCESS: dim\output\dim_prime_linear_bounded_discrete.m4.out IN 0.01 ON 2011-4-29 12:24:38, MODEL SIZE=3
dim\theorems\output\dim_eqdimpossible_theorems_1 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_eqdimpossible_theorems_1.relevance1.p9.out IN 0.01 ON 2011-10-21 16:34:53, PROOF LENGTH=8
    FAILED: dim\theorems\output\dim_eqdimpossible_theorems_1.relevance1.m4.out IN 0.25 ON 2011-10-21 16:34:53
dim\theorems\output\dim_eqdimpossible_theorems_2 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_eqdimpossible_theorems_2.relevance1.p9.out IN 0.01 ON 2011-10-21 16:34:57, PROOF LENGTH=10
    FAILED: dim\theorems\output\dim_eqdimpossible_theorems_2.relevance1.m4.out IN 0.27 ON 2011-10-21 16:34:58
dim\theorems\output\dim_eqdimpossible_theorems_3 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_eqdimpossible_theorems_3.relevance1.p9.out IN 0.01 ON 2011-10-21 16:35:02, PROOF LENGTH=17
    FAILED: dim\theorems\output\dim_eqdimpossible_theorems_3.relevance1.m4.out IN 0.26 ON 2011-10-21 16:35:02
dim\theorems\output\dim_eqdimpossible_theorems_4 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_eqdimpossible_theorems_4.relevance1.p9.out IN 0.02 ON 2011-10-21 16:35:06, PROOF LENGTH=14
    FAILED: dim\theorems\output\dim_eqdimpossible_theorems_4.relevance1.m4.out IN 0.27 ON 2011-10-21 16:35:06
dim\theorems\output\dim_eqdimpossible_theorems_5 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_eqdimpossible_theorems_5.relevance1.p9.out IN 0.02 ON 2011-10-21 16:35:10, PROOF LENGTH=10
    FAILED: dim\theorems\output\dim_eqdimpossible_theorems_5.relevance1.m4.out IN 0.28 ON 2011-10-21 16:35:10
dim\theorems\output\dim_prime_linear_unbounded_theorems tested with: Mace4, Prover9
    FAILED: dim\theorems\output\dim_prime_linear_unbounded_theorems.p9.out IN 1.96 ON 2012-12-6 17:18:44
  SUCCESS: dim\theorems\output\dim_prime_linear_unbounded_theorems.m4.out IN 0.00 ON 2012-12-6 17:18:42, MODEL SIZE=2
dim\theorems\output\dim_prime_linear_unbounded_theorems_1 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_prime_linear_unbounded_theorems_1.p9.out IN 0.01 ON 2012-12-6 17:18:49, PROOF LENGTH=8
    FAILED: dim\theorems\output\dim_prime_linear_unbounded_theorems_1.m4.out IN 0.14 ON 2012-12-6 17:18:49
dim\theorems\output\dim_prime_linear_unbounded_theorems_2 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_prime_linear_unbounded_theorems_2.p9.out IN 0.01 ON 2012-12-6 17:18:53, PROOF LENGTH=10
    FAILED: dim\theorems\output\dim_prime_linear_unbounded_theorems_2.m4.out IN 0.13 ON 2012-12-6 17:18:53
dim\theorems\output\dim_prime_linear_unbounded_theorems_3 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_prime_linear_unbounded_theorems_3.p9.out IN 0.01 ON 2012-12-6 17:18:57, PROOF LENGTH=22
    FAILED: dim\theorems\output\dim_prime_linear_unbounded_theorems_3.m4.out IN 0.13 ON 2012-12-6 17:18:57
dim\theorems\output\dim_prime_linear_unbounded_theorems_4 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_prime_linear_unbounded_theorems_4.p9.out IN 0.01 ON 2012-12-6 17:19:01, PROOF LENGTH=10
    FAILED: dim\theorems\output\dim_prime_linear_unbounded_theorems_4.m4.out IN 0.14 ON 2012-12-6 17:19:01
dim\theorems\output\dim_prime_linear_unbounded_theorems_5 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_prime_linear_unbounded_theorems_5.p9.out IN 0.01 ON 2012-12-6 17:19:05, PROOF LENGTH=17
    FAILED: dim\theorems\output\dim_prime_linear_unbounded_theorems_5.m4.out IN 0.13 ON 2012-12-6 17:19:05
```

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dim\theorems\output\dim_prime_linear_unbounded_theorems_6 tested with: Mace4, Prover9
  SUCCESS: dim\theorems\output\dim_prime_linear_unbounded_theorems_6.p9.out IN 0.01 ON 2012-12-6 17:19:09, PROOF LENGTH=7
    FAILED: dim\theorems\output\dim_prime_linear_unbounded_theorems_6.m4.out IN 0.13 ON 2012-12-6 17:19:09

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cont
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cont\output\cont_basic tested with: Mace4, Prover9
    FAILED: cont\output\cont_basic.p9.out IN 0.01 ON 2011-4-29 12:34:46
  SUCCESS: cont\output\cont_basic.m4.out IN 0.00 ON 2011-4-29 12:34:46, MODEL SIZE=3
cont\output\cont_c tested with: Mace4, Prover9
    FAILED: cont\output\cont_c.p9.out IN 0.02 ON 2011-4-29 12:36:36
  SUCCESS: cont\output\cont_c.m4.out IN 0.01 ON 2011-4-29 12:36:36, MODEL SIZE=3
cont\output\cont_c_a5 tested with: Mace4, Prover9
    FAILED: cont\output\cont_c_a5.p9.out IN 1.98 ON 2012-5-28 20:00:09
  SUCCESS: cont\output\cont_c_a5.m4.out IN 0.01 ON 2012-5-28 20:00:07, MODEL SIZE=2
cont\output\cont_c_ext tested with: Mace4, Prover9
    FAILED: cont\output\cont_c_ext.p9.out IN 1.97 ON 2011-4-29 12:37:02
  SUCCESS: cont\output\cont_c_ext.m4.out IN 0.01 ON 2011-4-29 12:37:00, MODEL SIZE=3
cont\theorems\output\cont_basic_ext tested with: Mace4, Prover9
    FAILED: cont\theorems\output\cont_basic_ext.p9.out IN 1.96 ON 2012-5-20 16:04:07
  SUCCESS: cont\theorems\output\cont_basic_ext.m4.out IN 0.01 ON 2012-5-20 16:04:05, MODEL SIZE=2
cont\theorems\output\cont_basic_ext_1 tested with: Mace4, Prover9
  SUCCESS: cont\theorems\output\cont_basic_ext_1.p9.out IN 0.01 ON 2012-10-30 13:38:50, PROOF LENGTH=21
    FAILED: cont\theorems\output\cont_basic_ext_1.m4.out IN 0.06 ON 2012-10-30 13:38:50
cont\theorems\output\cont_c_theorems tested with: Mace4, Prover9
    FAILED: cont\theorems\output\cont_c_theorems.p9.out IN 1.94 ON 2012-10-30 14:01:04
  SUCCESS: cont\theorems\output\cont_c_theorems.m4.out IN 0.01 ON 2012-10-30 14:01:02, MODEL SIZE=2
cont\theorems\output\cont_c_theorems_1 tested with: Mace4, Prover9
  SUCCESS: cont\theorems\output\cont_c_theorems_1.p9.out IN 0.01 ON 2012-10-30 14:01:08, PROOF LENGTH=10
    FAILED: cont\theorems\output\cont_c_theorems_1.m4.out IN 0.11 ON 2012-10-30 14:01:08
cont\theorems\output\cont_c_theorems_2 tested with: Mace4, Prover9
  SUCCESS: cont\theorems\output\cont_c_theorems_2.p9.out IN 0.01 ON 2012-10-30 14:01:12, PROOF LENGTH=10
    FAILED: cont\theorems\output\cont_c_theorems_2.m4.out IN 0.11 ON 2012-10-30 14:01:12
cont\theorems\output\cont_c_theorems_3 tested with: Mace4, Prover9
  SUCCESS: cont\theorems\output\cont_c_theorems_3.p9.out IN 0.01 ON 2012-10-30 14:01:16, PROOF LENGTH=10
    FAILED: cont\theorems\output\cont_c_theorems_3.m4.out IN 0.11 ON 2012-10-30 14:01:16
cont\theorems\output\cont_c_theorems_4 tested with: Mace4, Prover9
  SUCCESS: cont\theorems\output\cont_c_theorems_4.p9.out IN 0.01 ON 2012-10-30 14:01:20, PROOF LENGTH=15
    FAILED: cont\theorems\output\cont_c_theorems_4.m4.out IN 0.11 ON 2012-10-30 14:01:20

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codi
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codi\consistency\output\codi_down_nontrivial tested with: Mace4, Paradox3
    FAILED: codi\consistency\output\codi_down_nontrivial.m4.out IN 600.07 ON 2012-9-22 11:47:43
  SUCCESS: codi\consistency\output\codi_down_nontrivial.tptp.out, MODEL SIZE=10
codi\consistency\output\codi_down_simple_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: codi\consistency\output\codi_down_simple_nontrivial.p9.out IN 1.91 ON 2012-11-4 10:15:49
  SUCCESS: codi\consistency\output\codi_down_simple_nontrivial.m4.out IN 0.02 ON 2012-11-4 10:15:47, MODEL SIZE=4
  SUCCESS: codi\consistency\output\codi_down_simple_nontrivial.tptp.out, MODEL SIZE=4
codi\consistency\output\codi_down_sum_nontrivial tested with: Mace4, Paradox3
    FAILED: codi\consistency\output\codi_down_sum_nontrivial.m4.out IN 6000.03 ON 2012-2-27 18:55:35
  SUCCESS: codi\consistency\output\codi_down_sum_nontrivial.tptp.out, MODEL SIZE=7
codi\consistency\output\codi_down_sum_prime_nontrivial tested with: Mace4
  SUCCESS: codi\consistency\output\codi_down_sum_prime_nontrivial.m4.out IN 0.03 ON 2012-7-9 12:45:57, MODEL SIZE=4
codi\consistency\output\codi_down_sum_theorems_nontrivial tested with: Paradox3
  SUCCESS: codi\consistency\output\codi_down_sum_theorems_nontrivial.tptp.out, MODEL SIZE=8
codi\consistency\output\codi_int_nontrivial tested with: Mace4, Paradox3
    FAILED: codi\consistency\output\codi_int_nontrivial.m4.out IN 6000.01 ON 2011-4-29 19:07:35
  SUCCESS: codi\consistency\output\codi_int_nontrivial.tptp.out, MODEL SIZE=7
codi\consistency\output\codi_int_simple_nontrivial tested with: Paradox3
  SUCCESS: codi\consistency\output\codi_int_simple_nontrivial.tptp.out, MODEL SIZE=4
codi\consistency\output\codi_linear_nontrivial tested with: Mace4, Paradox3
    FAILED: codi\consistency\output\codi_linear_nontrivial.m4.out IN 6000.02 ON 2011-4-29 14:42:33
  SUCCESS: codi\consistency\output\codi_linear_nontrivial.tptp.out, MODEL SIZE=4
codi\consistency\output\codi_linear_nozex_nontrivial tested with: Mace4, Paradox3
  SUCCESS: codi\consistency\output\codi_linear_nozex_nontrivial.m4.out IN 0.01 ON 2012-7-9 13:05:27, MODEL SIZE=3
  SUCCESS: codi\consistency\output\codi_linear_nozex_nontrivial.tptp.out, MODEL SIZE=5
codi\consistency\output\codi_linear_zex_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: codi\consistency\output\codi_linear_zex_nontrivial.p9.out IN 600.14 ON 2012-11-4 10:26:27
    FAILED: codi\consistency\output\codi_linear_zex_nontrivial.m4.out IN 600.02 ON 2012-11-4 10:26:31
  SUCCESS: codi\consistency\output\codi_linear_zex_nontrivial.tptp.out, MODEL SIZE=6
codi\consistency\output\codi_plp_aff_nontrivial tested with: Paradox3
  SUCCESS: codi\consistency\output\codi_plp_aff_nontrivial.tptp.out, MODEL SIZE=11
codi\consistency\output\codi_plp_lin_nontrivial tested with: Paradox3
  SUCCESS: codi\consistency\output\codi_plp_lin_nontrivial.tptp.out, MODEL SIZE=14
codi\consistency\output\codi_plp_slin_nontrivial tested with: Mace4, Paradox3
    FAILED: codi\consistency\output\codi_plp_slin_nontrivial.m4.out IN 436.14 ON 2012-8-21 13:13:18
  SUCCESS: codi\consistency\output\codi_plp_slin_nontrivial.tptp.out, MODEL SIZE=5
codi\consistency\output\codi_updown_nontrivial tested with: Mace4, Paradox3
    FAILED: codi\consistency\output\codi_updown_nontrivial.m4.out IN 6000.12 ON 2012-2-24 17:45:46
  SUCCESS: codi\consistency\output\codi_updown_nontrivial.tptp.out, MODEL SIZE=8
codi\consistency\output\connected_nontrivial tested with: Mace4
  SUCCESS: codi\consistency\output\connected_nontrivial.m4.out IN 3.03 ON 2012-4-9 16:38:36, MODEL SIZE=7
codi\consistency\output\con_icon_notucon tested with: Mace4, Paradox3
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  SUCCESS: codi\consistency\output\con_icon_notucon.m4.out IN 0.02 ON 2012-4-9 16:39:45, MODEL SIZE=2
  SUCCESS: codi\consistency\output\con_icon_notucon.tptp.out, MODEL SIZE=6
codi\consistency\output\con_noticon_notucon tested with: Mace4, Paradox3
  SUCCESS: codi\consistency\output\con_noticon_notucon.m4.out IN 0.01 ON 2012-4-9 16:39:32, MODEL SIZE=2
  SUCCESS: codi\consistency\output\con_noticon_notucon.tptp.out, MODEL SIZE=7
codi\consistency\output\con_ucon_noticon tested with: Mace4, Paradox3
  SUCCESS: codi\consistency\output\con_ucon_noticon.m4.out IN 0.02 ON 2012-4-9 16:39:10, MODEL SIZE=2
  SUCCESS: codi\consistency\output\con_ucon_noticon.tptp.out, MODEL SIZE=6
codi\consistency\output\pl_aff_nontrivial tested with: Paradox3
  SUCCESS: codi\consistency\output\pl_aff_nontrivial.tptp.out, MODEL SIZE=10
codi\consistency\output\pl_lin_nontrivial tested with: Paradox3
  SUCCESS: codi\consistency\output\pl_lin_nontrivial.tptp.out, MODEL SIZE=10
codi\consistency\output\pl_slin_nontrivial tested with: Paradox3
  SUCCESS: codi\consistency\output\pl_slin_nontrivial.tptp.out, MODEL SIZE=3
codi\defs\output\connected tested with: Mace4, Prover9
    FAILED: codi\defs\output\connected.p9.out IN 1.91 ON 2012-11-4 11:19:29
  SUCCESS: codi\defs\output\connected.m4.out IN 0.01 ON 2012-11-4 11:19:27, MODEL SIZE=2
codi\output\codi tested with: Mace4, Prover9
    FAILED: codi\output\codi.p9.out IN 1.97 ON 2012-5-28 20:05:16
  SUCCESS: codi\output\codi.m4.out IN 0.01 ON 2012-5-28 20:05:14, MODEL SIZE=2
codi\output\codi_basic tested with: Mace4, Prover9
    FAILED: codi\output\codi_basic.p9.out IN 1.98 ON 2012-5-28 20:02:56
  SUCCESS: codi\output\codi_basic.m4.out IN 0.01 ON 2012-5-28 20:02:54, MODEL SIZE=2
codi\output\codi_down tested with: Mace4, Prover9
    FAILED: codi\output\codi_down.p9.out IN 0.69 ON 2011-11-15 14:50:01
  SUCCESS: codi\output\codi_down.m4.out IN 0.01 ON 2011-11-15 14:49:59, MODEL SIZE=3
codi\output\codi_down_sum tested with: Mace4, Prover9
    FAILED: codi\output\codi_down_sum.p9.out IN 6000.00 ON 2011-11-18 19:58:46
  SUCCESS: codi\output\codi_down_sum.m4.out IN 600.02 ON 2011-11-18 18:28:15, MODEL SIZE=4
codi\output\codi_down_sum_prime tested with: Mace4, Prover9
    FAILED: codi\output\codi_down_sum_prime.p9.out IN 1.95 ON 2012-7-9 12:42:55
  SUCCESS: codi\output\codi_down_sum_prime.m4.out IN 0.01 ON 2012-7-9 12:42:53, MODEL SIZE=2
codi\output\codi_linear tested with: Mace4, Prover9
    FAILED: codi\output\codi_linear.p9.out IN 1.97 ON 2012-5-28 20:05:01
  SUCCESS: codi\output\codi_linear.m4.out IN 0.01 ON 2012-5-28 20:04:59, MODEL SIZE=2
codi\output\codi_plp_g tested with: Paradox3
  SUCCESS: codi\output\codi_plp_g.tptp.out, MODEL SIZE=1
codi\output\codi_plp_slin tested with: Mace4, Prover9
    FAILED: codi\output\codi_plp_slin.p9.out IN 1.95 ON 2012-8-21 12:52:23
  SUCCESS: codi\output\codi_plp_slin.m4.out IN 0.02 ON 2012-8-21 12:52:21, MODEL SIZE=2
codi\output\codi_pl_aff tested with: Mace4, Prover9
    FAILED: codi\output\codi_pl_aff.p9.out IN 1.93 ON 2012-8-21 11:26:03
  SUCCESS: codi\output\codi_pl_aff.m4.out IN 0.27 ON 2012-8-21 11:26:01, MODEL SIZE=10
codi\output\codi_pl_lin tested with: Mace4, Prover9
    FAILED: codi\output\codi_pl_lin.p9.out IN 1.95 ON 2012-8-21 11:23:40
  SUCCESS: codi\output\codi_pl_lin.m4.out IN 0.01 ON 2012-8-21 11:23:38, MODEL SIZE=2
codi\output\codi_pl_slin tested with: Mace4, Prover9
    FAILED: codi\output\codi_pl_slin.p9.out IN 1.86 ON 2012-8-21 11:23:30
  SUCCESS: codi\output\codi_pl_slin.m4.out IN 0.01 ON 2012-8-21 11:23:28, MODEL SIZE=2
codi\output\codi_updown tested with: Mace4, Prover9
    FAILED: codi\output\codi_updown.p9.out IN 1.97 ON 2011-5-5 21:46:25
  SUCCESS: codi\output\codi_updown.m4.out IN 0.02 ON 2011-5-5 21:46:23, MODEL SIZE=3
codi\theorems\output\codi_down_sum_commutative_theorems_1 tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_down_sum_commutative_theorems_1.relevance1.p9.out IN 1.92 ON 2011-10-20 16:45:30
  SUCCESS: codi\theorems\output\codi_down_sum_commutative_theorems_1.relevance1.m4.out IN 0.02 ON 2011-10-20 16:45:28, MODEL SIZE=3
codi\theorems\output\codi_down_sum_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_down_sum_theorems.p9.out IN 1.90 ON 2012-11-21 11:18:54
  SUCCESS: codi\theorems\output\codi_down_sum_theorems.m4.out IN 0.01 ON 2012-11-21 11:18:52, MODEL SIZE=2
codi\theorems\output\codi_down_sum_theoremst11-t14_1 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst11-t14_1
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_1.relevance1.p9.out IN 600.00 ON 2011-11-29 20:04:45
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_1.vam.out IN 119.9
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_1.relevance1.m4.out IN 600.02 ON 2011-11-29 20:04:52
codi\theorems\output\codi_down_sum_theoremst11-t14_2 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst11-t14_2
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_2.relevance1.p9.out IN 600.00 ON 2011-11-29 20:14:56
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_2.vam.out IN 119.8
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_2.relevance1.m4.out IN 600.03 ON 2011-11-29 20:15:02
codi\theorems\output\codi_down_sum_theoremst11-t14_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_3.relevance1.p9.out IN 0.48 ON 2011-11-29 20:15:06, PROOF LENGTH=52
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_3.vam.out IN 0.69
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_3.relevance1.m4.out IN 1.95 ON 2011-11-29 20:15:07
codi\theorems\output\codi_down_sum_theoremst11-t14_4 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_4.relevance1.p9.out IN 3.36 ON 2011-11-29 20:15:15, PROOF LENGTH=93
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_4.vam.out IN 0.539
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_4.relevance1.m4.out IN 3.96 ON 2011-11-29 20:15:15
codi\theorems\output\codi_down_sum_theoremst11-t14_5 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_5.relevance1.p9.out IN 0.03 ON 2011-11-29 20:15:20, PROOF LENGTH=8
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_5.vam.out IN 0.437
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_5.relevance1.m4.out IN 1.96 ON 2011-11-29 20:15:22
codi\theorems\output\codi_down_sum_theoremst11-t14_6 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_6.relevance1.p9.out IN 0.04 ON 2011-11-29 20:15:26, PROOF LENGTH=8
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_6.vam.out IN 0.406
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_6.relevance1.m4.out IN 1.96 ON 2011-11-29 20:15:28
codi\theorems\output\codi_down_sum_theoremst11-t14_7 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_7.relevance1.p9.out IN 0.03 ON 2011-11-29 20:15:32, PROOF LENGTH=8
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  SUCCESS: codi\theorems\output\codi_down_sum_theoremst11-t14_7.vam.out IN 0.446
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_7.relevance1.m4.out IN 1.93 ON 2011-11-29 20:15:34
codi\theorems\output\codi_down_sum_theoremst11-t14_8 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst11-t14_8
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_8.relevance1.p9.out IN 600.00 ON 2011-11-29 20:25:41
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_8.vam.out IN 119.9
    FAILED: codi\theorems\output\codi_down_sum_theoremst11-t14_8.relevance1.m4.out IN 600.01 ON 2011-11-29 20:25:47
codi\theorems\output\codi_down_sum_theoremst15-t20 tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20.p9.out IN 1.93 ON 2012-11-21 13:56:12
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst15-t20.m4.out IN 0.02 ON 2012-11-21 13:56:10, MODEL SIZE=2
codi\theorems\output\codi_down_sum_theoremst15-t20_1 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst15-t20_1
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_1.p9.out IN 600.01 ON 2012-11-21 12:03:28
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_1.vam.out IN 599.73
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_1.m4.out IN 600.03 ON 2012-11-21 12:03:27
codi\theorems\output\codi_down_sum_theoremst15-t20_2 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst15-t20_2
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_2.p9.out IN 600.01 ON 2012-11-21 12:13:41
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_2.vam.out IN 599.73
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_2.m4.out IN 600.02 ON 2012-11-21 12:13:42
codi\theorems\output\codi_down_sum_theoremst15-t20_3 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst15-t20_3
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_3.p9.out IN 600.01 ON 2012-11-21 12:23:59
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_3.vam.out IN 599.569
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_3.m4.out IN 600.04 ON 2012-11-21 12:24:04
codi\theorems\output\codi_down_sum_theoremst15-t20_4 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst15-t20_4
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_4.p9.out IN 600.01 ON 2012-11-21 12:34:13
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_4.vam.out IN 599.8
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_4.m4.out IN 600.06 ON 2012-11-21 12:34:11
codi\theorems\output\codi_down_sum_theoremst15-t20_5 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst15-t20_5.p9.out IN 0.04 ON 2012-11-21 12:34:15, PROOF LENGTH=23
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst15-t20_5.vam.out IN 0.444
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_5.m4.out IN 1.87 ON 2012-11-21 12:34:17
codi\theorems\output\codi_down_sum_theoremst15-t20_6 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst15-t20_6.p9.out IN 0.05 ON 2012-11-21 12:34:21, PROOF LENGTH=23
  SUCCESS: codi\theorems\output\codi_down_sum_theoremst15-t20_6.vam.out IN 0.062
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_6.m4.out IN 1.92 ON 2012-11-21 12:34:23
codi\theorems\output\codi_down_sum_theoremst15-t20_7 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst15-t20_7
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_7.p9.out IN 600.00 ON 2012-11-21 12:44:32
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_7.vam.out IN 599.8
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_7.m4.out IN 600.03 ON 2012-11-21 12:44:36
codi\theorems\output\codi_down_sum_theoremst15-t20_8 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst15-t20_8
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_8.p9.out IN 600.01 ON 2012-11-21 12:54:46
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_8.vam.out IN 599.7
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_8.m4.out IN 600.02 ON 2012-11-21 12:54:45
codi\theorems\output\codi_down_sum_theoremst15-t20_9 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theoremst15-t20_9
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_9.p9.out IN 600.00 ON 2012-11-21 13:04:55
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_9.vam.out IN 599.704
    FAILED: codi\theorems\output\codi_down_sum_theoremst15-t20_9.m4.out IN 600.02 ON 2012-11-21 13:04:53
codi\theorems\output\codi_down_sum_theorems_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_1.p9.out IN 0.39 ON 2012-11-21 11:19:01, PROOF LENGTH=24
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_1.vam.out IN 0.024
    FAILED: codi\theorems\output\codi_down_sum_theorems_1.m4.out IN 1.84 ON 2012-11-21 11:19:03
codi\theorems\output\codi_down_sum_theorems_10 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_10.p9.out IN 0.84 ON 2012-11-21 12:00:08, PROOF LENGTH=33
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_10.vam.out IN 0.611
    FAILED: codi\theorems\output\codi_down_sum_theorems_10.m4.out IN 1.68 ON 2012-11-21 12:00:09
codi\theorems\output\codi_down_sum_theorems_11 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_11.p9.out IN 0.03 ON 2012-11-21 12:00:13, PROOF LENGTH=7
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_11.vam.out IN 0.017
    FAILED: codi\theorems\output\codi_down_sum_theorems_11.m4.out IN 1.91 ON 2012-11-21 12:00:15
codi\theorems\output\codi_down_sum_theorems_12 tested with: Mace4, Prover9, Vampire
    FAILED: codi\theorems\output\codi_down_sum_theorems_12.p9.out IN 600.01 ON 2012-11-21 12:10:25
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_12.vam.out IN 18.199
    FAILED: codi\theorems\output\codi_down_sum_theorems_12.m4.out IN 600.08 ON 2012-11-21 12:10:24
codi\theorems\output\codi_down_sum_theorems_13 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_13.p9.out IN 0.05 ON 2012-11-21 12:10:28, PROOF LENGTH=13
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_13.vam.out IN 0.008
    FAILED: codi\theorems\output\codi_down_sum_theorems_13.m4.out IN 1.85 ON 2012-11-21 12:10:30
codi\theorems\output\codi_down_sum_theorems_14 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_14.p9.out IN 0.02 ON 2012-11-21 12:10:34, PROOF LENGTH=5
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_14.vam.out IN 0.006
    FAILED: codi\theorems\output\codi_down_sum_theorems_14.m4.out IN 1.88 ON 2012-11-21 12:10:36
codi\theorems\output\codi_down_sum_theorems_15 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_15
    FAILED: codi\theorems\output\codi_down_sum_theorems_15.p9.out IN 600.01 ON 2012-11-21 12:21:14
    FAILED: codi\theorems\output\codi_down_sum_theorems_15.vam.out IN 599.581
    FAILED: codi\theorems\output\codi_down_sum_theorems_15.m4.out IN 600.04 ON 2012-11-21 12:21:12
codi\theorems\output\codi_down_sum_theorems_16 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_16
    FAILED: codi\theorems\output\codi_down_sum_theorems_16.p9.out IN 600.00 ON 2012-11-21 12:31:23
    FAILED: codi\theorems\output\codi_down_sum_theorems_16.vam.out IN 599.836
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    FAILED: codi\theorems\output\codi_down_sum_theorems_16.m4.out IN 600.03 ON 2012-11-21 12:31:23
codi\theorems\output\codi_down_sum_theorems_17 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_17
    FAILED: codi\theorems\output\codi_down_sum_theorems_17.p9.out IN 600.01 ON 2012-11-21 12:41:32
    FAILED: codi\theorems\output\codi_down_sum_theorems_17.vam.out IN 599.8
    FAILED: codi\theorems\output\codi_down_sum_theorems_17.m4.out IN 600.04 ON 2012-11-21 12:41:32
codi\theorems\output\codi_down_sum_theorems_18 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_18
    FAILED: codi\theorems\output\codi_down_sum_theorems_18.p9.out IN 600.01 ON 2012-11-21 12:51:40
    FAILED: codi\theorems\output\codi_down_sum_theorems_18.vam.out IN 599.8
    FAILED: codi\theorems\output\codi_down_sum_theorems_18.m4.out IN 600.04 ON 2012-11-21 12:51:42
codi\theorems\output\codi_down_sum_theorems_19 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_19
    FAILED: codi\theorems\output\codi_down_sum_theorems_19.p9.out IN 600.00 ON 2012-11-21 13:01:50
    FAILED: codi\theorems\output\codi_down_sum_theorems_19.vam.out IN 599.727
    FAILED: codi\theorems\output\codi_down_sum_theorems_19.m4.out IN 600.02 ON 2012-11-21 13:01:50
codi\theorems\output\codi_down_sum_theorems_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_2.p9.out IN 2.53 ON 2012-11-21 11:19:09, PROOF LENGTH=110
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_2.vam.out IN 2.129
    FAILED: codi\theorems\output\codi_down_sum_theorems_2.m4.out IN 3.21 ON 2012-11-21 11:19:10
codi\theorems\output\codi_down_sum_theorems_20 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_20
    FAILED: codi\theorems\output\codi_down_sum_theorems_20.p9.out IN 600.01 ON 2012-11-21 13:12:00
    FAILED: codi\theorems\output\codi_down_sum_theorems_20.vam.out IN 599.737
    FAILED: codi\theorems\output\codi_down_sum_theorems_20.m4.out IN 600.12 ON 2012-11-21 13:11:57
codi\theorems\output\codi_down_sum_theorems_3 tested with: Mace4, Prover9, Vampire
    FAILED: codi\theorems\output\codi_down_sum_theorems_3.p9.out IN 600.00 ON 2012-11-21 11:29:20
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_3.vam.out IN 310.693
    FAILED: codi\theorems\output\codi_down_sum_theorems_3.m4.out IN 600.04 ON 2012-11-21 11:29:17
codi\theorems\output\codi_down_sum_theorems_4 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_4.p9.out IN 0.19 ON 2012-11-21 11:29:24, PROOF LENGTH=39
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_4.vam.out IN 0.057
    FAILED: codi\theorems\output\codi_down_sum_theorems_4.m4.out IN 1.91 ON 2012-11-21 11:29:25
codi\theorems\output\codi_down_sum_theorems_5 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_5
    FAILED: codi\theorems\output\codi_down_sum_theorems_5.p9.out IN 600.00 ON 2012-11-21 11:39:35
    FAILED: codi\theorems\output\codi_down_sum_theorems_5.vam.out IN 599.704
    FAILED: codi\theorems\output\codi_down_sum_theorems_5.m4.out IN 600.03 ON 2012-11-21 11:39:34
codi\theorems\output\codi_down_sum_theorems_6 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_6.p9.out IN 0.42 ON 2012-11-21 11:39:38, PROOF LENGTH=50
    FAILED: codi\theorems\output\codi_down_sum_theorems_6.vam.out IN 599.731
    FAILED: codi\theorems\output\codi_down_sum_theorems_6.m4.out IN 1.89 ON 2012-11-21 11:39:40
codi\theorems\output\codi_down_sum_theorems_7 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_7.p9.out IN 7.26 ON 2012-11-21 11:39:51, PROOF LENGTH=107
  SUCCESS: codi\theorems\output\codi_down_sum_theorems_7.vam.out IN 413.57
    FAILED: codi\theorems\output\codi_down_sum_theorems_7.m4.out IN 2.00 ON 2012-11-21 11:39:46
codi\theorems\output\codi_down_sum_theorems_8 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_8
    FAILED: codi\theorems\output\codi_down_sum_theorems_8.p9.out IN 600.00 ON 2012-11-21 11:49:58
    FAILED: codi\theorems\output\codi_down_sum_theorems_8.vam.out IN 599.703
    FAILED: codi\theorems\output\codi_down_sum_theorems_8.m4.out IN 600.03 ON 2012-11-21 11:49:57
codi\theorems\output\codi_down_sum_theorems_9 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_sum_theorems_9
    FAILED: codi\theorems\output\codi_down_sum_theorems_9.p9.out IN 600.01 ON 2012-11-21 12:00:05
    FAILED: codi\theorems\output\codi_down_sum_theorems_9.vam.out IN 599.832
    FAILED: codi\theorems\output\codi_down_sum_theorems_9.m4.out IN 600.04 ON 2012-11-21 12:00:04
codi\theorems\output\codi_down_theoremsep-e1_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e1_1.relevance1.p9.out IN 1.99 ON 2011-11-24 19:40:24, PROOF LENGTH=37
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e1_1.vam.out IN 0.511
    FAILED: codi\theorems\output\codi_down_theoremsep-e1_1.relevance1.m4.out IN 3.94 ON 2011-11-24 19:40:26
codi\theorems\output\codi_down_theoremsep-e2_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e2_1.relevance1.p9.out IN 113.01 ON 2011-11-24 19:42:29, PROOF LENGTH=153
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e2_1.vam.out IN 0.526
    FAILED: codi\theorems\output\codi_down_theoremsep-e2_1.relevance1.m4.out IN 113.91 ON 2011-11-24 19:42:29
codi\theorems\output\codi_down_theoremsep-e2_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e2_2.relevance1.p9.out IN 0.37 ON 2011-11-24 19:42:34, PROOF LENGTH=23
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e2_2.vam.out IN 0.011
    FAILED: codi\theorems\output\codi_down_theoremsep-e2_2.relevance1.m4.out IN 1.96 ON 2011-11-24 19:42:36
codi\theorems\output\codi_down_theoremsep-e2_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e2_3.relevance1.p9.out IN 15.95 ON 2011-11-24 19:42:56, PROOF LENGTH=34
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e2_3.vam.out IN 0.109
    FAILED: codi\theorems\output\codi_down_theoremsep-e2_3.relevance1.m4.out IN 1.85 ON 2011-11-24 19:42:42
codi\theorems\output\codi_down_theoremsep-e3_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e3_1.relevance1.p9.out IN 19.39 ON 2011-11-24 19:43:44, PROOF LENGTH=97
  SUCCESS: codi\theorems\output\codi_down_theoremsep-e3_1.vam.out IN 5.256
    FAILED: codi\theorems\output\codi_down_theoremsep-e3_1.relevance1.m4.out IN 19.96 ON 2011-11-24 19:43:44
codi\theorems\output\codi_down_theoremspo-e1_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremspo-e1_1.relevance1.p9.out IN 91.09 ON 2012-2-24 16:08:07, PROOF LENGTH=100
  SUCCESS: codi\theorems\output\codi_down_theoremspo-e1_1.vam.out IN 0.523
    FAILED: codi\theorems\output\codi_down_theoremspo-e1_1.relevance1.m4.out IN 91.84 ON 2012-2-24 16:08:08
codi\theorems\output\codi_down_theoremspo-e1_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremspo-e1_2.relevance1.p9.out IN 78.39 ON 2012-2-24 16:09:31, PROOF LENGTH=153
  SUCCESS: codi\theorems\output\codi_down_theoremspo-e1_2.vam.out IN 0.164
    FAILED: codi\theorems\output\codi_down_theoremspo-e1_2.relevance1.m4.out IN 79.47 ON 2012-2-24 16:09:32
codi\theorems\output\codi_down_theoremst1-t2_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst1-t2_1.relevance1.p9.out IN 0.24 ON 2011-11-24 19:38:45, PROOF LENGTH=57
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  SUCCESS: codi\theorems\output\codi_down_theoremst1-t2_1.vam.out IN 0.447
    FAILED: codi\theorems\output\codi_down_theoremst1-t2_1.relevance1.m4.out IN 1.93 ON 2011-11-24 19:38:47
codi\theorems\output\codi_down_theoremst1-t2_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst1-t2_2.relevance1.p9.out IN 0.18 ON 2011-11-24 19:38:51, PROOF LENGTH=46
  SUCCESS: codi\theorems\output\codi_down_theoremst1-t2_2.vam.out IN 6.097
    FAILED: codi\theorems\output\codi_down_theoremst1-t2_2.relevance1.m4.out IN 1.97 ON 2011-11-24 19:38:53
codi\theorems\output\codi_down_theoremst1-t2_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst1-t2_3.relevance1.p9.out IN 0.29 ON 2011-11-24 19:38:58, PROOF LENGTH=120
  SUCCESS: codi\theorems\output\codi_down_theoremst1-t2_3.vam.out IN 79.022
    FAILED: codi\theorems\output\codi_down_theoremst1-t2_3.relevance1.m4.out IN 1.95 ON 2011-11-24 19:39:00
codi\theorems\output\codi_down_theoremst11prime_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_down_theoremst11prime_1.relevance1.p9.out IN 0.68 ON 2012-2-23 18:28:03, PROOF LENGTH=114
    FAILED: codi\theorems\output\codi_down_theoremst11prime_1.relevance1.m4.out IN 1.96 ON 2012-2-23 18:28:04
codi\theorems\output\codi_down_theoremst11_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst11_1.relevance1.p9.out IN 0.22 ON 2012-2-23 17:48:45, PROOF LENGTH=63
  SUCCESS: codi\theorems\output\codi_down_theoremst11_1.vam.out IN 9.543
    FAILED: codi\theorems\output\codi_down_theoremst11_1.relevance1.m4.out IN 1.91 ON 2012-2-23 17:48:47
codi\theorems\output\codi_down_theoremst12 tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_down_theoremst12.p9.out IN 1.93 ON 2012-8-15 18:03:24
  SUCCESS: codi\theorems\output\codi_down_theoremst12.m4.out IN 0.01 ON 2012-8-15 18:03:22, MODEL SIZE=2
codi\theorems\output\codi_down_theoremst12_1 tested with: Mace4, Prover9
  UNKNOWN: codi\theorems\output\codi_down_theoremst12_1
    FAILED: codi\theorems\output\codi_down_theoremst12_1.p9.out IN 600.29 ON 2012-8-15 18:13:31
    FAILED: codi\theorems\output\codi_down_theoremst12_1.m4.out IN 600.04 ON 2012-8-15 18:13:36
codi\theorems\output\codi_down_theoremst3-t5 tested with: Mace4, Prover9, Vampire
    FAILED: codi\theorems\output\codi_down_theoremst3-t5.p9.out IN 1.92 ON 2012-11-19 19:24:44
    FAILED: codi\theorems\output\codi_down_theoremst3-t5.vam.out IN 599.704
  SUCCESS: codi\theorems\output\codi_down_theoremst3-t5.m4.out IN 0.01 ON 2012-11-19 19:24:42, MODEL SIZE=2
codi\theorems\output\codi_down_theoremst3-t5_1 tested with: Mace4, Prover9, Vampire
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_1.relevance1.p9.out IN 600.19 ON 2011-11-24 19:49:27
  SUCCESS: codi\theorems\output\codi_down_theoremst3-t5_1.vam.out IN 109.556
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_1.relevance1.m4.out IN 600.02 ON 2011-11-24 19:49:33
codi\theorems\output\codi_down_theoremst3-t5_2 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_theoremst3-t5_2
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_2.relevance1.p9.out IN 600.78 ON 2011-11-24 19:59:37
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_2.vam.out IN 599.638
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_2.relevance1.m4.out IN 600.01 ON 2011-11-24 19:59:42
codi\theorems\output\codi_down_theoremst3-t5_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst3-t5_3.relevance1.p9.out IN 0.06 ON 2011-11-24 19:59:45, PROOF LENGTH=66
  SUCCESS: codi\theorems\output\codi_down_theoremst3-t5_3.vam.out IN 2.464
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_3.relevance1.m4.out IN 1.95 ON 2011-11-24 19:59:46
codi\theorems\output\codi_down_theoremst3-t5_4 tested with: Mace4, Prover9, Vampire
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_4.relevance1.p9.out IN 600.05 ON 2011-11-24 20:09:51
  SUCCESS: codi\theorems\output\codi_down_theoremst3-t5_4.vam.out IN 214.442
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_4.relevance1.m4.out IN 600.01 ON 2011-11-24 20:09:57
codi\theorems\output\codi_down_theoremst3-t5_5 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_theoremst3-t5_5
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_5.relevance1.p9.out IN 600.07 ON 2011-11-24 20:20:00
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_5.vam.out IN 599.551
    FAILED: codi\theorems\output\codi_down_theoremst3-t5_5.relevance1.m4.out IN 600.01 ON 2011-11-24 20:20:07
codi\theorems\output\codi_down_theoremst3-t5_helper tested with: Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst3-t5_helper.vam.out IN 0.007
codi\theorems\output\codi_down_theoremst6-t10 tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_down_theoremst6-t10.p9.out IN 1.91 ON 2012-11-19 19:25:49
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10.m4.out IN 0.01 ON 2012-11-19 19:25:47, MODEL SIZE=2
codi\theorems\output\codi_down_theoremst6-t10_1 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_theoremst6-t10_1
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_1.p9.out IN 600.00 ON 2012-11-19 18:34:52
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_1.vam.out IN 599.662
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_1.m4.out IN 600.02 ON 2012-11-19 18:34:55
codi\theorems\output\codi_down_theoremst6-t10_10 tested with: Mace4, Prover9, Vampire
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_10.p9.out IN 600.01 ON 2012-11-19 19:15:55
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_10.vam.out IN 1.988
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_10.m4.out IN 600.02 ON 2012-11-19 19:15:59
codi\theorems\output\codi_down_theoremst6-t10_2 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_theoremst6-t10_2
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_2.p9.out IN 600.10 ON 2012-11-19 18:45:01
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_2.vam.out IN 599.703
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_2.m4.out IN 600.02 ON 2012-11-19 18:45:02
codi\theorems\output\codi_down_theoremst6-t10_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_3.p9.out IN 2.90 ON 2012-11-19 18:45:08, PROOF LENGTH=44
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_3.vam.out IN 24.486
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_3.m4.out IN 3.83 ON 2012-11-19 18:45:09
codi\theorems\output\codi_down_theoremst6-t10_4 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_4.p9.out IN 2.10 ON 2012-11-19 18:45:15, PROOF LENGTH=41
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_4.vam.out IN 215.469
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_4.m4.out IN 3.87 ON 2012-11-19 18:45:17
codi\theorems\output\codi_down_theoremst6-t10_5 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_theoremst6-t10_5
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_5.p9.out IN 600.01 ON 2012-11-19 18:55:23
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_5.vam.out IN 599.8
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_5.m4.out IN 600.03 ON 2012-11-19 18:55:27
codi\theorems\output\codi_down_theoremst6-t10_6 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_down_theoremst6-t10_6
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_6.p9.out IN 600.01 ON 2012-11-19 19:05:32
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_6.vam.out IN 599.73
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    FAILED: codi\theorems\output\codi_down_theoremst6-t10_6.m4.out IN 600.05 ON 2012-11-19 19:05:35
codi\theorems\output\codi_down_theoremst6-t10_7 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_7.p9.out IN 0.03 ON 2012-11-19 19:05:38, PROOF LENGTH=12
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_7.vam.out IN 0.406
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_7.m4.out IN 1.90 ON 2012-11-19 19:05:40
codi\theorems\output\codi_down_theoremst6-t10_8 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_8.p9.out IN 0.02 ON 2012-11-19 19:05:44, PROOF LENGTH=24
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_8.vam.out IN 0.105
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_8.m4.out IN 1.56 ON 2012-11-19 19:05:45
codi\theorems\output\codi_down_theoremst6-t10_9 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_9.p9.out IN 0.03 ON 2012-11-19 19:05:48, PROOF LENGTH=24
  SUCCESS: codi\theorems\output\codi_down_theoremst6-t10_9.vam.out IN 0.109
    FAILED: codi\theorems\output\codi_down_theoremst6-t10_9.m4.out IN 1.57 ON 2012-11-19 19:05:49
codi\theorems\output\codi_down_theoremsz-a1_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_down_theoremsz-a1_1.relevance1.p9.out IN 0.01 ON 2011-11-16 18:25:23, PROOF LENGTH=5
    FAILED: codi\theorems\output\codi_down_theoremsz-a1_1.relevance1.m4.out IN 1.95 ON 2011-11-16 18:25:25
codi\theorems\output\codi_int_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_int_theorems.p9.out IN 1.88 ON 2012-11-19 18:53:01
  SUCCESS: codi\theorems\output\codi_int_theorems.m4.out IN 0.01 ON 2012-11-19 18:52:59, MODEL SIZE=2
codi\theorems\output\codi_int_theoremst11prime tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_int_theoremst11prime.p9.out IN 1.95 ON 2012-11-19 18:23:08
  SUCCESS: codi\theorems\output\codi_int_theoremst11prime.m4.out IN 0.01 ON 2012-11-19 18:23:06, MODEL SIZE=2
codi\theorems\output\codi_int_theoremst11prime_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theoremst11prime_1.p9.out IN 204.19 ON 2012-11-19 18:26:38, PROOF LENGTH=56
    FAILED: codi\theorems\output\codi_int_theoremst11prime_1.m4.out IN 200.07 ON 2012-11-19 18:26:39
codi\theorems\output\codi_int_theoremst11prime_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theoremst11prime_2.p9.out IN 283.48 ON 2012-11-19 18:31:28, PROOF LENGTH=52
    FAILED: codi\theorems\output\codi_int_theoremst11prime_2.m4.out IN 279.00 ON 2012-11-19 18:31:29
codi\theorems\output\codi_int_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_1.p9.out IN 298.83 ON 2012-11-19 18:39:59, PROOF LENGTH=46
    FAILED: codi\theorems\output\codi_int_theorems_1.m4.out IN 294.59 ON 2012-11-19 18:39:59
codi\theorems\output\codi_int_theorems_10 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_10.p9.out IN 0.44 ON 2012-11-19 18:48:05, PROOF LENGTH=42
    FAILED: codi\theorems\output\codi_int_theorems_10.m4.out IN 1.86 ON 2012-11-19 18:48:06
codi\theorems\output\codi_int_theorems_11 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_11.p9.out IN 1.74 ON 2012-11-19 18:48:12, PROOF LENGTH=36
    FAILED: codi\theorems\output\codi_int_theorems_11.m4.out IN 1.28 ON 2012-11-19 18:48:11
codi\theorems\output\codi_int_theorems_12 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_12.p9.out IN 1.76 ON 2012-11-19 18:48:16, PROOF LENGTH=32
    FAILED: codi\theorems\output\codi_int_theorems_12.m4.out IN 1.42 ON 2012-11-19 18:48:16
codi\theorems\output\codi_int_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_2.p9.out IN 296.96 ON 2012-11-19 18:45:02, PROOF LENGTH=46
    FAILED: codi\theorems\output\codi_int_theorems_2.m4.out IN 292.99 ON 2012-11-19 18:45:02
codi\theorems\output\codi_int_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_3.p9.out IN 0.01 ON 2012-11-19 18:45:06, PROOF LENGTH=7
    FAILED: codi\theorems\output\codi_int_theorems_3.m4.out IN 1.84 ON 2012-11-19 18:45:08
codi\theorems\output\codi_int_theorems_4 tested with: Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_4.int-t8a.manual.p9.out IN 0.33 ON 2011-5-3 17:38:17, PROOF LENGTH=80
codi\theorems\output\codi_int_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_5.p9.out IN 0.23 ON 2012-11-19 18:45:18, PROOF LENGTH=40
    FAILED: codi\theorems\output\codi_int_theorems_5.m4.out IN 1.88 ON 2012-11-19 18:45:20
codi\theorems\output\codi_int_theorems_6 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_6.p9.out IN 3.77 ON 2012-11-19 18:45:28, PROOF LENGTH=47
    FAILED: codi\theorems\output\codi_int_theorems_6.m4.out IN 3.76 ON 2012-11-19 18:45:28
codi\theorems\output\codi_int_theorems_7 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_7.p9.out IN 49.63 ON 2012-11-19 18:46:22, PROOF LENGTH=92
    FAILED: codi\theorems\output\codi_int_theorems_7.m4.out IN 47.91 ON 2012-11-19 18:46:22
codi\theorems\output\codi_int_theorems_8 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_8.p9.out IN 4.01 ON 2012-11-19 18:46:30, PROOF LENGTH=44
    FAILED: codi\theorems\output\codi_int_theorems_8.m4.out IN 5.70 ON 2012-11-19 18:46:32
codi\theorems\output\codi_int_theorems_9 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_int_theorems_9.p9.out IN 82.88 ON 2012-11-19 18:47:59, PROOF LENGTH=34
    FAILED: codi\theorems\output\codi_int_theorems_9.m4.out IN 82.10 ON 2012-11-19 18:48:00
codi\theorems\output\codi_linear_int tested with: Prover9
  SUCCESS: codi\theorems\output\codi_linear_int.int-t11'i.manual.p9.out IN 0.01 ON 2011-9-1 11:39:40, PROOF LENGTH=19
codi\theorems\output\codi_plp_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_plp_theorems.p9.out IN 1.96 ON 2012-8-21 13:59:55
  SUCCESS: codi\theorems\output\codi_plp_theorems.m4.out IN 0.01 ON 2012-8-21 13:59:53, MODEL SIZE=2
codi\theorems\output\codi_plp_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_plp_theorems_1.p9.out IN 2.89 ON 2012-8-21 14:00:04, PROOF LENGTH=104
    FAILED: codi\theorems\output\codi_plp_theorems_1.m4.out IN 3.91 ON 2012-8-21 14:00:05
codi\theorems\output\codi_plp_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_plp_theorems_2.p9.out IN 0.03 ON 2012-8-21 14:00:09, PROOF LENGTH=31
    FAILED: codi\theorems\output\codi_plp_theorems_2.m4.out IN 0.86 ON 2012-8-21 14:00:10
codi\theorems\output\codi_plp_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_plp_theorems_3.p9.out IN 0.03 ON 2012-8-21 14:00:13, PROOF LENGTH=21
    FAILED: codi\theorems\output\codi_plp_theorems_3.m4.out IN 0.84 ON 2012-8-21 14:00:14
codi\theorems\output\codi_pl_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_pl_theorems.p9.out IN 1.95 ON 2012-10-23 21:30:22
  SUCCESS: codi\theorems\output\codi_pl_theorems.m4.out IN 0.01 ON 2012-10-23 21:30:20, MODEL SIZE=2
codi\theorems\output\codi_pl_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_pl_theorems_1.p9.out IN 0.03 ON 2012-10-23 21:30:28, PROOF LENGTH=20
    FAILED: codi\theorems\output\codi_pl_theorems_1.m4.out IN 0.86 ON 2012-10-23 21:30:29
codi\theorems\output\codi_pl_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_pl_theorems_2.p9.out IN 0.04 ON 2012-10-23 21:30:32, PROOF LENGTH=29
    FAILED: codi\theorems\output\codi_pl_theorems_2.m4.out IN 0.84 ON 2012-10-23 21:30:33
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codi\theorems\output\codi_pl_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_pl_theorems_3.p9.out IN 0.03 ON 2012-10-23 21:30:36, PROOF LENGTH=12
    FAILED: codi\theorems\output\codi_pl_theorems_3.m4.out IN 0.84 ON 2012-10-23 21:30:37
codi\theorems\output\codi_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_theorems.p9.out IN 1.91 ON 2012-5-28 20:06:07
  SUCCESS: codi\theorems\output\codi_theorems.m4.out IN 0.01 ON 2012-5-28 20:06:05, MODEL SIZE=2
codi\theorems\output\codi_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_1.p9.out IN 0.03 ON 2012-5-28 20:06:13, PROOF LENGTH=15
    FAILED: codi\theorems\output\codi_theorems_1.m4.out IN 1.96 ON 2012-5-28 20:06:15
codi\theorems\output\codi_theorems_10 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_10.p9.out IN 16.60 ON 2012-5-28 20:12:12, PROOF LENGTH=52
    FAILED: codi\theorems\output\codi_theorems_10.m4.out IN 17.64 ON 2012-5-28 20:12:13
codi\theorems\output\codi_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_2.p9.out IN 0.02 ON 2012-5-28 20:06:19, PROOF LENGTH=11
    FAILED: codi\theorems\output\codi_theorems_2.m4.out IN 1.94 ON 2012-5-28 20:06:21
codi\theorems\output\codi_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_3.p9.out IN 0.17 ON 2012-5-28 20:06:25, PROOF LENGTH=26
    FAILED: codi\theorems\output\codi_theorems_3.m4.out IN 1.94 ON 2012-5-28 20:06:27
codi\theorems\output\codi_theorems_4 tested with: Prover9
  SUCCESS: codi\theorems\output\codi_theorems_4.cd-t4.manual.p9.out IN 5.27 ON 2011-10-21 16:19:55, PROOF LENGTH=46
codi\theorems\output\codi_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_5.p9.out IN 0.03 ON 2012-5-28 20:11:01, PROOF LENGTH=21
    FAILED: codi\theorems\output\codi_theorems_5.m4.out IN 1.94 ON 2012-5-28 20:11:03
codi\theorems\output\codi_theorems_6 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_6.p9.out IN 5.55 ON 2012-5-28 20:11:13, PROOF LENGTH=53
    FAILED: codi\theorems\output\codi_theorems_6.m4.out IN 5.85 ON 2012-5-28 20:11:13
codi\theorems\output\codi_theorems_7 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_7.p9.out IN 0.02 ON 2012-5-28 20:11:17, PROOF LENGTH=21
    FAILED: codi\theorems\output\codi_theorems_7.m4.out IN 1.97 ON 2012-5-28 20:11:19
codi\theorems\output\codi_theorems_8 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_8.p9.out IN 16.87 ON 2012-5-28 20:11:40, PROOF LENGTH=52
    FAILED: codi\theorems\output\codi_theorems_8.m4.out IN 17.64 ON 2012-5-28 20:11:41
codi\theorems\output\codi_theorems_9 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\codi_theorems_9.p9.out IN 5.51 ON 2012-5-28 20:11:51, PROOF LENGTH=53
    FAILED: codi\theorems\output\codi_theorems_9.m4.out IN 5.86 ON 2012-5-28 20:11:51
codi\theorems\output\codi_updown_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\codi_updown_theorems.p9.out IN 1.88 ON 2012-11-20 19:18:34
  SUCCESS: codi\theorems\output\codi_updown_theorems.m4.out IN 0.01 ON 2012-11-20 19:18:32, MODEL SIZE=2
codi\theorems\output\codi_updown_theorems_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_updown_theorems_1.p9.out IN 0.40 ON 2012-11-20 19:18:41, PROOF LENGTH=47
    FAILED: codi\theorems\output\codi_updown_theorems_1.vam.out IN 1046.59
    FAILED: codi\theorems\output\codi_updown_theorems_1.m4.out IN 1.93 ON 2012-11-20 19:18:43
codi\theorems\output\codi_updown_theorems_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_updown_theorems_2.p9.out IN 0.29 ON 2012-11-20 19:18:47, PROOF LENGTH=66
    FAILED: codi\theorems\output\codi_updown_theorems_2.vam.out IN 599.811
    FAILED: codi\theorems\output\codi_updown_theorems_2.m4.out IN 1.94 ON 2012-11-20 19:18:49
codi\theorems\output\codi_updown_theorems_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_updown_theorems_3.p9.out IN 67.17 ON 2012-11-20 19:20:01, PROOF LENGTH=67
    FAILED: codi\theorems\output\codi_updown_theorems_3.vam.out IN 599.702
    FAILED: codi\theorems\output\codi_updown_theorems_3.m4.out IN 67.93 ON 2012-11-20 19:20:03
codi\theorems\output\codi_updown_theorems_4 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_updown_theorems_4.p9.out IN 0.03 ON 2012-11-20 19:20:07, PROOF LENGTH=9
  SUCCESS: codi\theorems\output\codi_updown_theorems_4.vam.out IN 0.018
    FAILED: codi\theorems\output\codi_updown_theorems_4.m4.out IN 1.90 ON 2012-11-20 19:20:09
codi\theorems\output\codi_updown_theorems_5 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\codi_updown_theorems_5.p9.out IN 21.26 ON 2012-11-20 19:20:35, PROOF LENGTH=58
  SUCCESS: codi\theorems\output\codi_updown_theorems_5.vam.out IN 22.018
    FAILED: codi\theorems\output\codi_updown_theorems_5.m4.out IN 2.20 ON 2012-11-20 19:20:16
codi\theorems\output\codi_updown_theorems_6 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_updown_theorems_6
    FAILED: codi\theorems\output\codi_updown_theorems_6.p9.out IN 600.01 ON 2012-11-20 19:30:43
    FAILED: codi\theorems\output\codi_updown_theorems_6.vam.out IN 599.8
    FAILED: codi\theorems\output\codi_updown_theorems_6.m4.out IN 600.04 ON 2012-11-20 19:30:43
codi\theorems\output\codi_updown_theorems_7 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_updown_theorems_7
    FAILED: codi\theorems\output\codi_updown_theorems_7.p9.out IN 600.01 ON 2012-11-20 19:41:06
    FAILED: codi\theorems\output\codi_updown_theorems_7.vam.out IN 599.7
    FAILED: codi\theorems\output\codi_updown_theorems_7.m4.out IN 600.04 ON 2012-11-20 19:41:00
codi\theorems\output\codi_updown_theorems_8 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_updown_theorems_8
    FAILED: codi\theorems\output\codi_updown_theorems_8.p9.out IN 600.00 ON 2012-11-20 19:52:02
    FAILED: codi\theorems\output\codi_updown_theorems_8.vam.out IN 599.8
    FAILED: codi\theorems\output\codi_updown_theorems_8.m4.out IN 600.05 ON 2012-11-20 19:51:42
codi\theorems\output\codi_updown_theorems_9 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codi\theorems\output\codi_updown_theorems_9
    FAILED: codi\theorems\output\codi_updown_theorems_9.p9.out IN 600.01 ON 2012-11-20 20:02:23
    FAILED: codi\theorems\output\codi_updown_theorems_9.vam.out IN 599.7
    FAILED: codi\theorems\output\codi_updown_theorems_9.m4.out IN 600.05 ON 2012-11-20 20:02:25
codi\theorems\output\con_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\con_theorems.p9.out IN 1.89 ON 2012-11-4 12:01:06
  SUCCESS: codi\theorems\output\con_theorems.m4.out IN 0.01 ON 2012-11-4 12:01:04, MODEL SIZE=2
codi\theorems\output\con_theorems_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\con_theorems_1.p9.out IN 0.02 ON 2012-11-4 12:01:12, PROOF LENGTH=9
  SUCCESS: codi\theorems\output\con_theorems_1.vam.out IN 0.005
    FAILED: codi\theorems\output\con_theorems_1.m4.out IN 1.94 ON 2012-11-4 12:01:14
codi\theorems\output\con_theorems_2 tested with: Mace4, Prover9, Vampire
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  SUCCESS: codi\theorems\output\con_theorems_2.p9.out IN 0.03 ON 2012-11-4 12:01:18, PROOF LENGTH=22
  SUCCESS: codi\theorems\output\con_theorems_2.vam.out IN 0.006
    FAILED: codi\theorems\output\con_theorems_2.m4.out IN 1.95 ON 2012-11-4 12:01:20
codi\theorems\output\con_theorems_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\con_theorems_3.p9.out IN 0.02 ON 2012-11-4 12:01:24, PROOF LENGTH=8
  SUCCESS: codi\theorems\output\con_theorems_3.vam.out IN 0.015
    FAILED: codi\theorems\output\con_theorems_3.m4.out IN 1.94 ON 2012-11-4 12:01:26
codi\theorems\output\con_theorems_4 tested with: Mace4, Prover9, Paradox3, Vampire
  UNKNOWN: codi\theorems\output\con_theorems_4
    FAILED: codi\theorems\output\con_theorems_4.p9.out IN 600.46 ON 2012-11-4 12:11:31
    FAILED: codi\theorems\output\con_theorems_4.vam.out IN 599.7
    FAILED: codi\theorems\output\con_theorems_4.m4.out IN 600.04 ON 2012-11-4 12:11:34
    FAILED: codi\theorems\output\con_theorems_4.tptp.out, ATTEMPTED UP TO MODEL SIZE=50
codi\theorems\output\con_theorems_5 tested with: Mace4, Prover9, Paradox3, Vampire
  UNKNOWN: codi\theorems\output\con_theorems_5
    FAILED: codi\theorems\output\con_theorems_5.p9.out IN 600.05 ON 2012-11-4 12:21:38
    FAILED: codi\theorems\output\con_theorems_5.vam.out IN 599.718
    FAILED: codi\theorems\output\con_theorems_5.m4.out IN 600.04 ON 2012-11-4 12:21:39
    FAILED: codi\theorems\output\con_theorems_5.tptp.out, ATTEMPTED UP TO MODEL SIZE=49
codi\theorems\output\con_theorems_6 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\con_theorems_6.p9.out IN 0.10 ON 2012-11-4 12:21:43, PROOF LENGTH=43
  SUCCESS: codi\theorems\output\con_theorems_6.vam.out IN 0.015
    FAILED: codi\theorems\output\con_theorems_6.m4.out IN 1.92 ON 2012-11-4 12:21:44
codi\theorems\output\ep_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\ep_theorems.p9.out IN 1.95 ON 2011-10-20 19:37:51
  SUCCESS: codi\theorems\output\ep_theorems.m4.out IN 0.01 ON 2011-10-20 19:37:49, MODEL SIZE=3
codi\theorems\output\ep_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_1.relevance1.p9.out IN 0.01 ON 2011-10-20 19:51:29, PROOF LENGTH=11
    FAILED: codi\theorems\output\ep_theorems_1.relevance1.m4.out IN 1.03 ON 2011-10-20 19:51:30
codi\theorems\output\ep_theorems_10 tested with: Mace4, Prover9
    FAILED: codi\theorems\output\ep_theorems_10.relevance1.p9.out IN 1.97 ON 2011-10-20 19:52:11
  SUCCESS: codi\theorems\output\ep_theorems_10.relevance1.m4.out IN 0.02 ON 2011-10-20 19:52:09, MODEL SIZE=3
codi\theorems\output\ep_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_2.relevance1.p9.out IN 0.02 ON 2011-10-20 19:51:33, PROOF LENGTH=11
    FAILED: codi\theorems\output\ep_theorems_2.relevance1.m4.out IN 1.04 ON 2011-10-20 19:51:34
codi\theorems\output\ep_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_3.relevance1.p9.out IN 0.12 ON 2011-10-20 19:51:37, PROOF LENGTH=20
    FAILED: codi\theorems\output\ep_theorems_3.relevance1.m4.out IN 1.10 ON 2011-10-20 19:51:38
codi\theorems\output\ep_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_4.relevance1.p9.out IN 0.03 ON 2011-10-20 19:51:42, PROOF LENGTH=10
    FAILED: codi\theorems\output\ep_theorems_4.relevance1.m4.out IN 1.08 ON 2011-10-20 19:51:43
codi\theorems\output\ep_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_5.relevance1.p9.out IN 0.03 ON 2011-10-20 19:51:46, PROOF LENGTH=13
    FAILED: codi\theorems\output\ep_theorems_5.relevance1.m4.out IN 1.05 ON 2011-10-20 19:51:47
codi\theorems\output\ep_theorems_6 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_6.relevance1.p9.out IN 0.03 ON 2011-10-20 19:51:50, PROOF LENGTH=10
    FAILED: codi\theorems\output\ep_theorems_6.relevance1.m4.out IN 1.05 ON 2011-10-20 19:51:51
codi\theorems\output\ep_theorems_7 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_7.relevance1.p9.out IN 0.03 ON 2011-10-20 19:51:54, PROOF LENGTH=13
    FAILED: codi\theorems\output\ep_theorems_7.relevance1.m4.out IN 1.07 ON 2011-10-20 19:51:55
codi\theorems\output\ep_theorems_8 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_8.relevance1.p9.out IN 0.03 ON 2011-10-20 19:51:59, PROOF LENGTH=15
    FAILED: codi\theorems\output\ep_theorems_8.relevance1.m4.out IN 1.04 ON 2011-10-20 19:52:00
codi\theorems\output\ep_theorems_9 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\ep_theorems_9.relevance1.p9.out IN 0.05 ON 2011-10-20 19:52:03, PROOF LENGTH=30
    FAILED: codi\theorems\output\ep_theorems_9.relevance1.m4.out IN 1.96 ON 2011-10-20 19:52:05
codi\theorems\output\icon_theorems tested with: Mace4, Prover9
    FAILED: codi\theorems\output\icon_theorems.p9.out IN 1.93 ON 2012-3-15 10:25:22
  SUCCESS: codi\theorems\output\icon_theorems.m4.out IN 0.02 ON 2012-3-15 10:25:20, MODEL SIZE=2
codi\theorems\output\icon_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\icon_theorems_1.p9.out IN 0.07 ON 2012-3-15 10:25:30, PROOF LENGTH=12
    FAILED: codi\theorems\output\icon_theorems_1.m4.out IN 1.86 ON 2012-3-15 10:25:32
codi\theorems\output\icon_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\icon_theorems_2.p9.out IN 2.95 ON 2012-3-15 10:25:39, PROOF LENGTH=80
    FAILED: codi\theorems\output\icon_theorems_2.m4.out IN 3.92 ON 2012-3-15 10:25:40
codi\theorems\output\icon_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\icon_theorems_3.p9.out IN 0.06 ON 2012-3-15 10:25:44, PROOF LENGTH=12
    FAILED: codi\theorems\output\icon_theorems_3.m4.out IN 1.84 ON 2012-3-15 10:25:46
codi\theorems\output\icon_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\icon_theorems_4.p9.out IN 0.68 ON 2012-3-15 10:25:51, PROOF LENGTH=36
    FAILED: codi\theorems\output\icon_theorems_4.m4.out IN 1.93 ON 2012-3-15 10:25:52
codi\theorems\output\inc_theorems_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\inc_theorems_1.relevance1.p9.out IN 0.05 ON 2011-10-20 20:52:39, PROOF LENGTH=29
  SUCCESS: codi\theorems\output\inc_theorems_1.vam.out IN 0.057
    FAILED: codi\theorems\output\inc_theorems_1.relevance1.m4.out IN 1.95 ON 2011-10-20 20:52:41
codi\theorems\output\inc_theorems_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\inc_theorems_2.relevance1.p9.out IN 5.12 ON 2011-10-20 20:52:50, PROOF LENGTH=53
  SUCCESS: codi\theorems\output\inc_theorems_2.vam.out IN 0.064
    FAILED: codi\theorems\output\inc_theorems_2.relevance1.m4.out IN 5.92 ON 2011-10-20 20:52:51
codi\theorems\output\inc_theorems_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\inc_theorems_3.relevance1.p9.out IN 0.07 ON 2011-10-20 20:52:56, PROOF LENGTH=44
  SUCCESS: codi\theorems\output\inc_theorems_3.vam.out IN 0.077
    FAILED: codi\theorems\output\inc_theorems_3.relevance1.m4.out IN 1.96 ON 2011-10-20 20:52:57
codi\theorems\output\inc_theorems_4 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\inc_theorems_4.relevance1.p9.out IN 0.26 ON 2011-10-20 20:53:02, PROOF LENGTH=27
  SUCCESS: codi\theorems\output\inc_theorems_4.vam.out IN 0.453
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    FAILED: codi\theorems\output\inc_theorems_4.relevance1.m4.out IN 1.96 ON 2011-10-20 20:53:04
codi\theorems\output\inc_theorems_5 tested with: Mace4, Prover9, Vampire
  SUCCESS: codi\theorems\output\inc_theorems_5.relevance1.p9.out IN 0.01 ON 2011-10-20 20:53:08, PROOF LENGTH=18
  SUCCESS: codi\theorems\output\inc_theorems_5.vam.out IN 0.07
    FAILED: codi\theorems\output\inc_theorems_5.relevance1.m4.out IN 0.69 ON 2011-10-20 20:53:09
codi\theorems\output\inc_theorems_6 tested with: Mace4, Prover9, Vampire
    FAILED: codi\theorems\output\inc_theorems_6.relevance1.p9.out IN 600.01 ON 2011-10-20 21:02:04
  SUCCESS: codi\theorems\output\inc_theorems_6.vam.out IN 1.886
    FAILED: codi\theorems\output\inc_theorems_6.relevance1.m4.out IN 600.01 ON 2011-10-20 21:02:11
codi\theorems\output\int_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\int_theorems_1.relevance1.p9.out IN 0.03 ON 2011-5-2 16:40:47, PROOF LENGTH=10
    FAILED: codi\theorems\output\int_theorems_1.relevance1.m4.out IN 1.20 ON 2011-5-2 16:40:49
codi\theorems\output\int_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\int_theorems_2.relevance1.p9.out IN 0.03 ON 2011-5-2 16:40:52, PROOF LENGTH=10
    FAILED: codi\theorems\output\int_theorems_2.relevance1.m4.out IN 1.21 ON 2011-5-2 16:40:53
codi\theorems\output\int_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\int_theorems_3.relevance1.p9.out IN 0.03 ON 2011-5-2 16:40:56, PROOF LENGTH=9
    FAILED: codi\theorems\output\int_theorems_3.relevance1.m4.out IN 1.22 ON 2011-5-2 16:40:57
codi\theorems\output\int_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\int_theorems_4.relevance1.p9.out IN 0.02 ON 2011-5-2 16:41:00, PROOF LENGTH=11
    FAILED: codi\theorems\output\int_theorems_4.relevance1.m4.out IN 1.26 ON 2011-5-2 16:41:01
codi\theorems\output\int_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\int_theorems_5.relevance1.p9.out IN 0.03 ON 2011-5-2 16:41:04, PROOF LENGTH=11
    FAILED: codi\theorems\output\int_theorems_5.relevance1.m4.out IN 1.96 ON 2011-5-2 16:41:06
codi\theorems\output\po_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\po_theorems_1.relevance1.p9.out IN 0.01 ON 2011-10-20 21:03:45, PROOF LENGTH=16
    FAILED: codi\theorems\output\po_theorems_1.relevance1.m4.out IN 0.54 ON 2011-10-20 21:03:46
codi\theorems\output\po_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\po_theorems_2.relevance1.p9.out IN 0.01 ON 2011-10-20 21:03:49, PROOF LENGTH=10
    FAILED: codi\theorems\output\po_theorems_2.relevance1.m4.out IN 1.96 ON 2011-10-20 21:03:51
codi\theorems\output\po_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\po_theorems_3.relevance1.p9.out IN 0.01 ON 2011-10-20 21:03:56, PROOF LENGTH=19
    FAILED: codi\theorems\output\po_theorems_3.relevance1.m4.out IN 0.55 ON 2011-10-20 21:03:56
codi\theorems\output\sc_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\sc_theorems_1.relevance1.p9.out IN 0.02 ON 2011-10-20 21:25:19, PROOF LENGTH=11
    FAILED: codi\theorems\output\sc_theorems_1.relevance1.m4.out IN 1.95 ON 2011-10-20 21:25:21
codi\theorems\output\sc_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\sc_theorems_2.relevance1.p9.out IN 0.02 ON 2011-10-20 21:25:25, PROOF LENGTH=18
    FAILED: codi\theorems\output\sc_theorems_2.relevance1.m4.out IN 0.86 ON 2011-10-20 21:25:26
codi\theorems\output\sc_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\sc_theorems_3.relevance1.p9.out IN 0.03 ON 2011-10-20 21:25:29, PROOF LENGTH=18
    FAILED: codi\theorems\output\sc_theorems_3.relevance1.m4.out IN 0.87 ON 2011-10-20 21:25:30
codi\theorems\output\sc_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\sc_theorems_4.relevance1.p9.out IN 0.16 ON 2011-10-20 21:25:34, PROOF LENGTH=33
    FAILED: codi\theorems\output\sc_theorems_4.relevance1.m4.out IN 1.95 ON 2011-10-20 21:25:36
codi\theorems\output\sc_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\sc_theorems_5.relevance1.p9.out IN 0.02 ON 2011-10-20 21:25:40, PROOF LENGTH=16
    FAILED: codi\theorems\output\sc_theorems_5.relevance1.m4.out IN 0.85 ON 2011-10-20 21:25:41
codi\theorems\output\sc_theorems_6 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\sc_theorems_6.relevance1.p9.out IN 106.62 ON 2011-10-20 21:27:31, PROOF LENGTH=54
    FAILED: codi\theorems\output\sc_theorems_6.relevance1.m4.out IN 107.56 ON 2011-10-20 21:27:32
codi\theorems\output\sc_theorems_7 tested with: Mace4, Prover9
  SUCCESS: codi\theorems\output\sc_theorems_7.relevance1.p9.out IN 0.02 ON 2011-10-20 21:27:37, PROOF LENGTH=15
    FAILED: codi\theorems\output\sc_theorems_7.relevance1.m4.out IN 1.97 ON 2011-10-20 21:27:39


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mt
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mt\consistency\output\rcc_basic_strict_nontrivial_2 tested with: Paradox3
  SUCCESS: mt\consistency\output\rcc_basic_strict_nontrivial_2.tptp.out, MODEL SIZE=3
mt\consistency\output\rcc_basic_strict_nontrivial_4 tested with: Paradox3
  SUCCESS: mt\consistency\output\rcc_basic_strict_nontrivial_4.tptp.out, MODEL SIZE=7
mt\consistency\output\rcc_basic_strict_nontrivial_4atoms tested with: Paradox3
  SUCCESS: mt\consistency\output\rcc_basic_strict_nontrivial_4atoms.tptp.out, MODEL SIZE=15
mt\consistency\output\rt0_13distinct tested with: Paradox3
  SUCCESS: mt\consistency\output\rt0_13distinct.tptp.out, MODEL SIZE=13
mt\consistency\output\rtminus_13distinct tested with: Mace4, Prover9, Paradox3
    FAILED: mt\consistency\output\rtminus_13distinct.p9.out IN 600.00 ON 2012-11-2 16:51:13
    FAILED: mt\consistency\output\rtminus_13distinct.m4.out IN 63.26 ON 2012-11-2 16:42:07
  SUCCESS: mt\consistency\output\rtminus_13distinct.tptp.out, MODEL SIZE=13
mt\consistency\output\rtminus_13elements tested with: Paradox3
  SUCCESS: mt\consistency\output\rtminus_13elements.tptp.out, MODEL SIZE=13
mt\output\rcc tested with: Mace4, Prover9
  UNKNOWN: mt\output\rcc
    FAILED: mt\output\rcc.p9.out IN 600.00 ON 2012-11-2 19:03:02
    FAILED: mt\output\rcc.m4.out IN 505.27 ON 2012-11-2 19:01:19
mt\output\rcc_basic tested with: Mace4, Prover9
  UNKNOWN: mt\output\rcc_basic
    FAILED: mt\output\rcc_basic.p9.out IN 600.00 ON 2012-11-2 19:01:50
    FAILED: mt\output\rcc_basic.m4.out IN 600.01 ON 2012-11-2 19:01:51
mt\output\rcc_basic_strict tested with: Mace4, Prover9
    FAILED: mt\output\rcc_basic_strict.p9.out IN 1.96 ON 2012-11-2 19:36:29
  SUCCESS: mt\output\rcc_basic_strict.m4.out IN 0.03 ON 2012-11-2 19:36:27, MODEL SIZE=3
mt\output\rcc_strict tested with: Mace4, Prover9
  UNKNOWN: mt\output\rcc_strict
    FAILED: mt\output\rcc_strict.p9.out IN 600.00 ON 2012-11-2 19:02:33
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    FAILED: mt\output\rcc_strict.m4.out IN 497.94 ON 2012-11-2 19:00:42
mt\output\rt13 tested with: Paradox3
  SUCCESS: mt\output\rt13.tptp.out, MODEL SIZE=13
mt\output\rt13_distinct tested with: Paradox3
  SUCCESS: mt\output\rt13_distinct.tptp.out, MODEL SIZE=13
mt\output\rtminus tested with: Mace4, Prover9
  UNKNOWN: mt\output\rtminus
    FAILED: mt\output\rtminus.p9.out IN 600.28 ON 2012-11-2 17:41:49
    FAILED: mt\output\rtminus.m4.out IN 600.01 ON 2012-11-2 17:42:01

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inch
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inch\consistency\output\inch_calculus_extended_full tested with: Mace4, Prover9, Paradox3
    FAILED: inch\consistency\output\inch_calculus_extended_full.p9.out IN 600.11 ON 2012-7-5 18:34:02
    FAILED: inch\consistency\output\inch_calculus_extended_full.m4.out IN 600.04 ON 2012-7-5 18:34:07
  SUCCESS: inch\consistency\output\inch_calculus_extended_full.tptp.out, MODEL SIZE=5
inch\consistency\output\inch_calculus_extended_noti-e3 tested with: Mace4, Prover9
    FAILED: inch\consistency\output\inch_calculus_extended_noti-e3.p9.out IN 67.87 ON 2012-6-29 03:38:10
  SUCCESS: inch\consistency\output\inch_calculus_extended_noti-e3.m4.out IN 65.62 ON 2012-6-29 03:38:08, MODEL SIZE=5
inch\consistency\output\inch_calculus_notc-a5 tested with: Mace4, Prover9, Paradox3
    FAILED: inch\consistency\output\inch_calculus_notc-a5.p9.out
    FAILED: inch\consistency\output\inch_calculus_notc-a5.m4.out
  SUCCESS: inch\consistency\output\inch_calculus_notc-a5.tptp.out, MODEL SIZE=5
inch\consistency\output\inch_calculus_no_intersection tested with: Mace4, Prover9, Paradox3
    FAILED: inch\consistency\output\inch_calculus_no_intersection.p9.out
    FAILED: inch\consistency\output\inch_calculus_no_intersection.m4.out
  SUCCESS: inch\consistency\output\inch_calculus_no_intersection.tptp.out, MODEL SIZE=4
inch\consistency\output\inch_original_noti-pa7 tested with: Paradox3
  SUCCESS: inch\consistency\output\inch_original_noti-pa7.tptp.out, MODEL SIZE=6
inch\output\inch_calculus tested with: Mace4, Prover9
    FAILED: inch\output\inch_calculus.p9.out IN 1.95 ON 2012-5-20 16:29:58
  SUCCESS: inch\output\inch_calculus.m4.out IN 1.23 ON 2012-5-20 16:29:57, MODEL SIZE=2
inch\output\inch_original tested with: Mace4, Prover9
    FAILED: inch\output\inch_original.p9.out IN 1.96 ON 2012-5-20 18:30:53
  SUCCESS: inch\output\inch_original.m4.out IN 0.14 ON 2012-5-20 18:30:52, MODEL SIZE=2
inch\output\inch_weak tested with: Mace4, Prover9
    FAILED: inch\output\inch_weak.p9.out IN 1.95 ON 2012-5-20 16:25:31
  SUCCESS: inch\output\inch_weak.m4.out IN 0.01 ON 2012-5-20 16:25:29, MODEL SIZE=2
inch\output\inch_weak_closed tested with: Mace4, Prover9
    FAILED: inch\output\inch_weak_closed.p9.out IN 1.97 ON 2012-5-20 16:25:40
  SUCCESS: inch\output\inch_weak_closed.m4.out IN 0.03 ON 2012-5-20 16:25:38, MODEL SIZE=2
inch\theorems\output\codi_down_c-e4_i-m10 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_down_c-e4_i-m10.p9.out IN 1.96 ON 2012-5-28 18:32:03
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m10.m4.out IN 0.02 ON 2012-5-28 18:32:01, MODEL SIZE=2
inch\theorems\output\codi_down_c-e4_i-m10_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m10_1.p9.out IN 0.03 ON 2012-5-28 18:32:10, PROOF LENGTH=20
    FAILED: inch\theorems\output\codi_down_c-e4_i-m10_1.m4.out IN 1.88 ON 2012-5-28 18:32:12
inch\theorems\output\codi_down_c-e4_i-m10_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m10_2.p9.out IN 0.04 ON 2012-5-28 18:32:16, PROOF LENGTH=21
    FAILED: inch\theorems\output\codi_down_c-e4_i-m10_2.m4.out IN 1.89 ON 2012-5-28 18:32:18
inch\theorems\output\codi_down_c-e4_i-m7 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_down_c-e4_i-m7.p9.out IN 1.82 ON 2012-11-20 18:59:33
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m7.m4.out IN 0.01 ON 2012-11-20 18:59:31, MODEL SIZE=2
inch\theorems\output\codi_down_c-e4_i-m7_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m7_1.p9.out IN 0.07 ON 2012-11-20 18:59:40, PROOF LENGTH=31
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m7_1.vam.out IN 0.006
    FAILED: inch\theorems\output\codi_down_c-e4_i-m7_1.m4.out IN 1.92 ON 2012-11-20 18:59:42
inch\theorems\output\codi_down_c-e4_i-m7_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m7_2.p9.out IN 151.68 ON 2012-11-20 19:02:18, PROOF LENGTH=29
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m7_2.vam.out IN 5.16
    FAILED: inch\theorems\output\codi_down_c-e4_i-m7_2.m4.out IN 151.39 ON 2012-11-20 19:02:20
inch\theorems\output\codi_down_c-e4_i-m7_3 tested with: Mace4, Prover9, Vampire
  UNKNOWN: inch\theorems\output\codi_down_c-e4_i-m7_3
    FAILED: inch\theorems\output\codi_down_c-e4_i-m7_3.p9.out IN 600.01 ON 2012-11-20 19:12:27
    FAILED: inch\theorems\output\codi_down_c-e4_i-m7_3.vam.out IN 599.802
    FAILED: inch\theorems\output\codi_down_c-e4_i-m7_3.m4.out IN 600.02 ON 2012-11-20 19:12:27
inch\theorems\output\codi_down_c-e4_i-m8 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_down_c-e4_i-m8.p9.out IN 1.86 ON 2012-5-28 18:10:49
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m8.m4.out IN 0.01 ON 2012-5-28 18:10:48, MODEL SIZE=2
inch\theorems\output\codi_down_c-e4_i-m8_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m8_1.p9.out IN 24.73 ON 2012-5-28 18:11:20, PROOF LENGTH=113
    FAILED: inch\theorems\output\codi_down_c-e4_i-m8_1.m4.out IN 25.52 ON 2012-5-28 18:11:21
inch\theorems\output\codi_down_c-e4_i-m8_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m8_2.p9.out IN 55.91 ON 2012-5-28 18:12:21, PROOF LENGTH=94
    FAILED: inch\theorems\output\codi_down_c-e4_i-m8_2.m4.out IN 57.18 ON 2012-5-28 18:12:23
inch\theorems\output\codi_down_c-e4_i-m9 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_down_c-e4_i-m9.p9.out IN 1.96 ON 2012-5-28 18:22:56
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m9.m4.out IN 0.01 ON 2012-5-28 18:22:54, MODEL SIZE=2
inch\theorems\output\codi_down_c-e4_i-m9_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m9_1.p9.out IN 0.02 ON 2012-5-28 18:23:02, PROOF LENGTH=6
    FAILED: inch\theorems\output\codi_down_c-e4_i-m9_1.m4.out IN 1.89 ON 2012-5-28 18:23:04
inch\theorems\output\codi_down_c-e4_i-m9_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m9_2.p9.out IN 0.05 ON 2012-5-28 18:23:08, PROOF LENGTH=35
    FAILED: inch\theorems\output\codi_down_c-e4_i-m9_2.m4.out IN 1.90 ON 2012-5-28 18:23:10
inch\theorems\output\codi_down_c-e4_i-m9_3 tested with: Mace4, Prover9
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  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m9_3.p9.out IN 0.02 ON 2012-5-28 18:23:14, PROOF LENGTH=7
    FAILED: inch\theorems\output\codi_down_c-e4_i-m9_3.m4.out IN 0.44 ON 2012-5-28 18:23:14
inch\theorems\output\codi_down_c-e4_i-m9_4 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m9_4.p9.out IN 0.02 ON 2012-5-28 18:23:18, PROOF LENGTH=9
    FAILED: inch\theorems\output\codi_down_c-e4_i-m9_4.m4.out IN 1.93 ON 2012-5-28 18:23:20
inch\theorems\output\codi_down_c-e4_i-m9_5 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m9_5.p9.out IN 0.03 ON 2012-5-28 18:23:24, PROOF LENGTH=8
    FAILED: inch\theorems\output\codi_down_c-e4_i-m9_5.m4.out IN 0.44 ON 2012-5-28 18:23:24
inch\theorems\output\codi_down_c-e4_i-m9_6 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_down_c-e4_i-m9_6.p9.out IN 0.03 ON 2012-5-28 18:23:28, PROOF LENGTH=22
    FAILED: inch\theorems\output\codi_down_c-e4_i-m9_6.m4.out IN 0.44 ON 2012-5-28 18:23:28
inch\theorems\output\codi_linear_c-e4_i-e1 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_c-e4_i-e1.p9.out IN 1.93 ON 2012-5-28 16:23:47
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-e1.m4.out IN 0.00 ON 2012-5-28 16:23:45, MODEL SIZE=2
inch\theorems\output\codi_linear_c-e4_i-e1_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-e1_1.p9.out IN 0.01 ON 2012-5-28 16:23:52, PROOF LENGTH=25
    FAILED: inch\theorems\output\codi_linear_c-e4_i-e1_1.m4.out IN 0.34 ON 2012-5-28 16:23:52
inch\theorems\output\codi_linear_c-e4_i-m3 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m3.p9.out IN 1.97 ON 2012-5-28 17:06:39
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m3.m4.out IN 0.01 ON 2012-5-28 17:06:37, MODEL SIZE=2
inch\theorems\output\codi_linear_c-e4_i-m3_1 tested with: Mace4, Prover9, Vampire
  UNKNOWN: inch\theorems\output\codi_linear_c-e4_i-m3_1
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m3_1.p9.out IN 600.05 ON 2012-5-28 17:16:45
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m3_1.vam.out IN 599.802
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m3_1.m4.out IN 600.01 ON 2012-5-28 17:16:51
inch\theorems\output\codi_linear_c-e4_i-m3_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m3_2.p9.out IN 1.22 ON 2012-5-28 17:16:56, PROOF LENGTH=36
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m3_2.vam.out IN 2.055
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m3_2.m4.out IN 1.80 ON 2012-5-28 17:16:57
inch\theorems\output\codi_linear_c-e4_i-m4 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m4.p9.out IN 1.95 ON 2012-5-28 17:06:45
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m4.m4.out IN 0.02 ON 2012-5-28 17:06:43, MODEL SIZE=2
inch\theorems\output\codi_linear_c-e4_i-m4_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m4_1.p9.out IN 0.01 ON 2012-5-28 17:06:50, PROOF LENGTH=20
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m4_1.m4.out IN 1.83 ON 2012-5-28 17:06:52
inch\theorems\output\codi_linear_c-e4_i-m4_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m4_2.p9.out IN 0.02 ON 2012-5-28 17:06:56, PROOF LENGTH=21
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m4_2.m4.out IN 1.82 ON 2012-5-28 17:06:58
inch\theorems\output\codi_linear_c-e4_i-m5 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m5.p9.out IN 1.96 ON 2012-5-28 17:07:08
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m5.m4.out IN 0.00 ON 2012-5-28 17:07:06, MODEL SIZE=2
inch\theorems\output\codi_linear_c-e4_i-m5_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m5_1.p9.out IN 0.02 ON 2012-5-28 17:07:13, PROOF LENGTH=23
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m5_1.m4.out IN 1.87 ON 2012-5-28 17:07:15
inch\theorems\output\codi_linear_c-e4_i-m5_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m5_2.p9.out IN 0.02 ON 2012-5-28 17:07:19, PROOF LENGTH=19
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m5_2.m4.out IN 1.84 ON 2012-5-28 17:07:21
inch\theorems\output\codi_linear_c-e4_i-m6 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m6.p9.out IN 1.97 ON 2012-5-28 17:08:54
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m6.m4.out IN 0.01 ON 2012-5-28 17:08:52, MODEL SIZE=2
inch\theorems\output\codi_linear_c-e4_i-m6_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m6_1.p9.out IN 0.06 ON 2012-5-28 17:09:00, PROOF LENGTH=59
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m6_1.m4.out IN 1.85 ON 2012-5-28 17:09:02
inch\theorems\output\codi_linear_c-e4_i-m6_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-m6_2.p9.out IN 2.09 ON 2012-5-28 17:09:08, PROOF LENGTH=32
    FAILED: inch\theorems\output\codi_linear_c-e4_i-m6_2.m4.out IN 3.78 ON 2012-5-28 17:09:10
inch\theorems\output\codi_linear_c-e4_i-pa4 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_c-e4_i-pa4.p9.out IN 1.94 ON 2012-5-28 16:23:30
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-pa4.m4.out IN 0.02 ON 2012-5-28 16:23:28, MODEL SIZE=2
inch\theorems\output\codi_linear_c-e4_i-pa4_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-pa4_1.p9.out IN 0.26 ON 2012-5-28 16:23:36, PROOF LENGTH=17
    FAILED: inch\theorems\output\codi_linear_c-e4_i-pa4_1.m4.out IN 1.89 ON 2012-5-28 16:23:38
inch\theorems\output\codi_linear_c-e4_i-pa5 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_c-e4_i-pa5.p9.out IN 1.89 ON 2012-5-28 16:22:42
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-pa5.m4.out IN 0.01 ON 2012-5-28 16:22:40, MODEL SIZE=2
inch\theorems\output\codi_linear_c-e4_i-pa5_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_c-e4_i-pa5_1.p9.out IN 0.41 ON 2012-5-28 16:22:49, PROOF LENGTH=27
    FAILED: inch\theorems\output\codi_linear_c-e4_i-pa5_1.m4.out IN 1.91 ON 2012-5-28 16:22:50
inch\theorems\output\codi_linear_i-e1 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_i-e1.p9.out IN 1.94 ON 2012-5-28 14:11:08
  SUCCESS: inch\theorems\output\codi_linear_i-e1.m4.out IN 0.01 ON 2012-5-28 14:11:06, MODEL SIZE=2
inch\theorems\output\codi_linear_i-e1_1 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_i-e1_1.p9.out IN 1.96 ON 2012-5-28 14:11:16
  SUCCESS: inch\theorems\output\codi_linear_i-e1_1.m4.out IN 0.01 ON 2012-5-28 14:11:14, MODEL SIZE=2
inch\theorems\output\codi_linear_i-m2 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_i-m2.p9.out IN 1.96 ON 2012-5-28 14:25:14
  SUCCESS: inch\theorems\output\codi_linear_i-m2.m4.out IN 0.01 ON 2012-5-28 14:25:12, MODEL SIZE=2
inch\theorems\output\codi_linear_i-m2_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_i-m2_1.p9.out IN 0.04 ON 2012-5-28 14:25:20, PROOF LENGTH=12
    FAILED: inch\theorems\output\codi_linear_i-m2_1.m4.out IN 0.91 ON 2012-5-28 14:25:21
inch\theorems\output\codi_linear_i-m2_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_i-m2_2.p9.out IN 0.03 ON 2012-5-28 14:25:24, PROOF LENGTH=18
    FAILED: inch\theorems\output\codi_linear_i-m2_2.m4.out IN 0.90 ON 2012-5-28 14:25:25
inch\theorems\output\codi_linear_i-m3 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_i-m3.p9.out IN 1.98 ON 2012-5-28 16:54:25
  SUCCESS: inch\theorems\output\codi_linear_i-m3.m4.out IN 0.01 ON 2012-5-28 16:54:23, MODEL SIZE=2
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inch\theorems\output\codi_linear_i-m3_1 tested with: Mace4, Prover9
  UNKNOWN: inch\theorems\output\codi_linear_i-m3_1
    FAILED: inch\theorems\output\codi_linear_i-m3_1.p9.out IN 600.06 ON 2012-5-28 17:04:31
    FAILED: inch\theorems\output\codi_linear_i-m3_1.m4.out IN 600.02 ON 2012-5-28 17:04:33
inch\theorems\output\codi_linear_i-m3_2 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_i-m3_2.p9.out IN 1.95 ON 2012-5-28 17:04:38
  SUCCESS: inch\theorems\output\codi_linear_i-m3_2.m4.out IN 0.02 ON 2012-5-28 17:04:36, MODEL SIZE=3
inch\theorems\output\codi_linear_i-pa3 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_i-pa3.p9.out IN 1.94 ON 2012-5-28 14:07:25
  SUCCESS: inch\theorems\output\codi_linear_i-pa3.m4.out IN 0.02 ON 2012-5-28 14:07:23, MODEL SIZE=2
inch\theorems\output\codi_linear_i-pa3_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_i-pa3_1.p9.out IN 0.03 ON 2012-5-28 14:07:31, PROOF LENGTH=20
    FAILED: inch\theorems\output\codi_linear_i-pa3_1.m4.out IN 1.93 ON 2012-5-28 14:07:33
inch\theorems\output\codi_linear_i-pa6 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_i-pa6.p9.out IN 1.94 ON 2012-5-28 14:14:12
  SUCCESS: inch\theorems\output\codi_linear_i-pa6.m4.out IN 0.01 ON 2012-5-28 14:14:10, MODEL SIZE=2
inch\theorems\output\codi_linear_i-pa6_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_i-pa6_1.p9.out IN 0.15 ON 2012-5-28 14:14:18, PROOF LENGTH=59
    FAILED: inch\theorems\output\codi_linear_i-pa6_1.m4.out IN 1.93 ON 2012-5-28 14:14:20
inch\theorems\output\codi_linear_i-pa7 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\codi_linear_i-pa7.p9.out IN 1.94 ON 2012-5-28 14:18:12
  SUCCESS: inch\theorems\output\codi_linear_i-pa7.m4.out IN 0.01 ON 2012-5-28 14:18:10, MODEL SIZE=2
inch\theorems\output\codi_linear_i-pa7_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\codi_linear_i-pa7_1.p9.out IN 255.44 ON 2012-5-28 14:22:34, PROOF LENGTH=42
    FAILED: inch\theorems\output\codi_linear_i-pa7_1.m4.out IN 254.02 ON 2012-5-28 14:22:35
inch\theorems\output\inch_calculus_c-e4 tested with: Mace4, Prover9
  UNKNOWN: inch\theorems\output\inch_calculus_c-e4
    FAILED: inch\theorems\output\inch_calculus_c-e4.p9.out IN 600.00 ON 2012-6-29 12:50:06
    FAILED: inch\theorems\output\inch_calculus_c-e4.m4.out IN 600.04 ON 2012-6-29 12:50:17
inch\theorems\output\inch_calculus_c-e4_1 tested with: Mace4, Prover9, Vampire
  UNKNOWN: inch\theorems\output\inch_calculus_c-e4_1
    FAILED: inch\theorems\output\inch_calculus_c-e4_1.p9.out IN 600.02 ON 2012-6-29 13:00:26
    FAILED: inch\theorems\output\inch_calculus_c-e4_1.vam.out IN 599.502
    FAILED: inch\theorems\output\inch_calculus_c-e4_1.m4.out IN 600.08 ON 2012-6-29 13:00:25
inch\theorems\output\inch_calculus_c-e4_2 tested with: Mace4, Prover9, Vampire
    FAILED: inch\theorems\output\inch_calculus_c-e4_2.p9.out IN 600.01 ON 2012-6-29 13:10:35
  SUCCESS: inch\theorems\output\inch_calculus_c-e4_2.vam.out IN 18.517
    FAILED: inch\theorems\output\inch_calculus_c-e4_2.m4.out IN 600.04 ON 2012-6-29 13:10:34
inch\theorems\output\inch_calculus_extended_theorems tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_calculus_extended_theorems.p9.out IN 1.97 ON 2012-6-29 02:43:58
  SUCCESS: inch\theorems\output\inch_calculus_extended_theorems.m4.out IN 0.01 ON 2012-6-29 02:43:56, MODEL SIZE=2
inch\theorems\output\inch_calculus_extended_theorems_1 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_calculus_extended_theorems_1.p9.out IN 67.88 ON 2012-6-29 02:45:11
  SUCCESS: inch\theorems\output\inch_calculus_extended_theorems_1.m4.out IN 65.78 ON 2012-6-29 02:45:10, MODEL SIZE=5
inch\theorems\output\inch_calculus_intersection tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_calculus_intersection.p9.out IN 37.15 ON 2012-6-21 15:30:51
  SUCCESS: inch\theorems\output\inch_calculus_intersection.m4.out IN 33.01 ON 2012-6-21 15:30:51, MODEL SIZE=2
inch\theorems\output\inch_calculus_intersection_1 tested with: Mace4, Prover9, Vampire
  UNKNOWN: inch\theorems\output\inch_calculus_intersection_1
    FAILED: inch\theorems\output\inch_calculus_intersection_1.p9.out IN 600.86 ON 2012-6-21 15:41:00
    FAILED: inch\theorems\output\inch_calculus_intersection_1.vam.out IN 599.7
    FAILED: inch\theorems\output\inch_calculus_intersection_1.m4.out IN 600.02 ON 2012-6-21 15:41:01
inch\theorems\output\inch_calculus_intersection_2 tested with: Mace4, Prover9, Vampire
    FAILED: inch\theorems\output\inch_calculus_intersection_2.p9.out IN 600.65 ON 2012-6-21 15:51:07
  SUCCESS: inch\theorems\output\inch_calculus_intersection_2.vam.out IN 119.085
    FAILED: inch\theorems\output\inch_calculus_intersection_2.m4.out IN 600.02 ON 2012-6-21 15:51:09
inch\theorems\output\inch_calculus_theorems tested with: Mace4, Prover9
  UNKNOWN: inch\theorems\output\inch_calculus_theorems
    FAILED: inch\theorems\output\inch_calculus_theorems.p9.out IN 600.00 ON 2012-11-21 11:25:10
    FAILED: inch\theorems\output\inch_calculus_theorems.m4.out IN 600.02 ON 2012-11-21 11:25:24
inch\theorems\output\inch_calculus_theorems_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: inch\theorems\output\inch_calculus_theorems_1.p9.out IN 182.15 ON 2012-6-29 03:42:05, PROOF LENGTH=34
    FAILED: inch\theorems\output\inch_calculus_theorems_1.vam.out IN 599.631
    FAILED: inch\theorems\output\inch_calculus_theorems_1.m4.out IN 182.57 ON 2012-6-29 03:42:06
inch\theorems\output\inch_calculus_theorems_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: inch\theorems\output\inch_calculus_theorems_2.p9.out IN 0.03 ON 2012-6-29 03:42:10, PROOF LENGTH=20
  SUCCESS: inch\theorems\output\inch_calculus_theorems_2.vam.out IN 0.067
    FAILED: inch\theorems\output\inch_calculus_theorems_2.m4.out IN 1.96 ON 2012-6-29 03:42:12
inch\theorems\output\inch_calculus_theorems_3 tested with: Mace4, Prover9, Vampire
    FAILED: inch\theorems\output\inch_calculus_theorems_3.p9.out IN 600.00 ON 2012-6-29 03:52:17
  SUCCESS: inch\theorems\output\inch_calculus_theorems_3.vam.out IN 29.717
    FAILED: inch\theorems\output\inch_calculus_theorems_3.m4.out IN 600.07 ON 2012-6-29 03:52:19
inch\theorems\output\inch_cd-a1 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_cd-a1.p9.out IN 1.91 ON 2012-5-22 11:05:31
  SUCCESS: inch\theorems\output\inch_cd-a1.m4.out IN 0.02 ON 2012-5-22 11:05:29, MODEL SIZE=2
inch\theorems\output\inch_cd-a1_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_cd-a1_1.p9.out IN 0.02 ON 2012-5-22 11:05:36, PROOF LENGTH=30
    FAILED: inch\theorems\output\inch_cd-a1_1.m4.out IN 0.88 ON 2012-5-22 11:05:37
inch\theorems\output\inch_cont_basic tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_cont_basic.p9.out IN 1.95 ON 2012-5-20 16:30:07
  SUCCESS: inch\theorems\output\inch_cont_basic.m4.out IN 0.01 ON 2012-5-20 16:30:05, MODEL SIZE=2
inch\theorems\output\inch_cont_basic_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_cont_basic_1.p9.out IN 0.02 ON 2012-5-22 10:48:02, PROOF LENGTH=23
    FAILED: inch\theorems\output\inch_cont_basic_1.m4.out IN 1.04 ON 2012-5-22 10:48:03
inch\theorems\output\inch_cont_basic_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_cont_basic_2.p9.out IN 0.02 ON 2012-5-22 10:48:06, PROOF LENGTH=20
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    FAILED: inch\theorems\output\inch_cont_basic_2.m4.out IN 1.89 ON 2012-5-22 10:48:08
inch\theorems\output\inch_cont_basic_3 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_cont_basic_3.p9.out IN 0.02 ON 2012-5-22 10:48:12, PROOF LENGTH=26
    FAILED: inch\theorems\output\inch_cont_basic_3.m4.out IN 0.84 ON 2012-5-22 10:48:13
inch\theorems\output\inch_cont_basic_4 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_cont_basic_4.p9.out IN 0.02 ON 2012-5-22 10:48:16, PROOF LENGTH=26
    FAILED: inch\theorems\output\inch_cont_basic_4.m4.out IN 0.85 ON 2012-5-22 10:48:17
inch\theorems\output\inch_cont_basic_5 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_cont_basic_5.p9.out IN 0.02 ON 2012-5-22 10:48:20, PROOF LENGTH=21
    FAILED: inch\theorems\output\inch_cont_basic_5.m4.out IN 1.77 ON 2012-5-22 10:48:22
inch\theorems\output\inch_dim_linear tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_dim_linear.p9.out IN 1.96 ON 2012-5-22 10:46:38
  SUCCESS: inch\theorems\output\inch_dim_linear.m4.out IN 0.01 ON 2012-5-22 10:46:36, MODEL SIZE=2
inch\theorems\output\inch_dim_linear_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_dim_linear_1.p9.out IN 0.01 ON 2012-5-22 10:46:42, PROOF LENGTH=8
    FAILED: inch\theorems\output\inch_dim_linear_1.m4.out IN 0.87 ON 2012-5-22 10:46:43
inch\theorems\output\inch_dim_linear_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_dim_linear_2.p9.out IN 0.01 ON 2012-5-22 10:46:46, PROOF LENGTH=8
    FAILED: inch\theorems\output\inch_dim_linear_2.m4.out IN 0.88 ON 2012-5-22 10:46:47
inch\theorems\output\inch_dim_linear_3 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_dim_linear_3.p9.out IN 0.03 ON 2012-5-22 10:46:50, PROOF LENGTH=24
    FAILED: inch\theorems\output\inch_dim_linear_3.m4.out IN 1.92 ON 2012-5-22 10:46:52
inch\theorems\output\inch_dim_linear_4 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_dim_linear_4.p9.out IN 0.02 ON 2012-5-22 10:46:56, PROOF LENGTH=20
    FAILED: inch\theorems\output\inch_dim_linear_4.m4.out IN 1.69 ON 2012-5-22 10:46:58
inch\theorems\output\inch_dim_linear_5 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_dim_linear_5.p9.out IN 0.02 ON 2012-5-22 10:47:02, PROOF LENGTH=24
    FAILED: inch\theorems\output\inch_dim_linear_5.m4.out IN 0.89 ON 2012-5-22 10:47:03
inch\theorems\output\inch_original_dif-a3 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_original_dif-a3.p9.out IN 3.92 ON 2012-6-26 12:18:09
  SUCCESS: inch\theorems\output\inch_original_dif-a3.m4.out IN 2.52 ON 2012-6-26 12:18:08, MODEL SIZE=2
inch\theorems\output\inch_original_dif-a3_1 tested with: Mace4, Prover9
  UNKNOWN: inch\theorems\output\inch_original_dif-a3_1
    FAILED: inch\theorems\output\inch_original_dif-a3_1.p9.out IN 600.22 ON 2012-6-26 12:28:15
    FAILED: inch\theorems\output\inch_original_dif-a3_1.m4.out IN 600.07 ON 2012-6-26 12:28:23
inch\theorems\output\inch_original_dif-a3_2 tested with: Mace4, Prover9
  UNKNOWN: inch\theorems\output\inch_original_dif-a3_2
    FAILED: inch\theorems\output\inch_original_dif-a3_2.p9.out IN 600.51 ON 2012-6-26 12:38:27
    FAILED: inch\theorems\output\inch_original_dif-a3_2.m4.out IN 600.07 ON 2012-6-26 12:38:36
inch\theorems\output\inch_original_theorems tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_original_theorems.p9.out IN 1.95 ON 2012-5-20 18:34:34
  SUCCESS: inch\theorems\output\inch_original_theorems.m4.out IN 0.01 ON 2012-5-20 18:34:32, MODEL SIZE=2
inch\theorems\output\inch_original_theorems_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_1.p9.out IN 0.01 ON 2012-7-5 17:57:00, PROOF LENGTH=9
    FAILED: inch\theorems\output\inch_original_theorems_1.m4.out IN 1.96 ON 2012-7-5 17:57:02
inch\theorems\output\inch_original_theorems_10 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_10.p9.out IN 2.45 ON 2012-7-5 18:08:19, PROOF LENGTH=36
    FAILED: inch\theorems\output\inch_original_theorems_10.m4.out IN 3.91 ON 2012-7-5 18:08:21
inch\theorems\output\inch_original_theorems_11 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_11.p9.out IN 0.01 ON 2012-7-5 18:08:25, PROOF LENGTH=7
    FAILED: inch\theorems\output\inch_original_theorems_11.m4.out IN 1.95 ON 2012-7-5 18:08:27
inch\theorems\output\inch_original_theorems_12 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_12.p9.out IN 0.02 ON 2012-7-5 18:08:31, PROOF LENGTH=12
    FAILED: inch\theorems\output\inch_original_theorems_12.m4.out IN 1.96 ON 2012-7-5 18:08:33
inch\theorems\output\inch_original_theorems_13 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_13.p9.out IN 0.03 ON 2012-7-5 18:08:37, PROOF LENGTH=19
    FAILED: inch\theorems\output\inch_original_theorems_13.m4.out IN 1.93 ON 2012-7-5 18:08:39
inch\theorems\output\inch_original_theorems_14 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_14.p9.out IN 0.01 ON 2012-7-5 18:08:43, PROOF LENGTH=12
    FAILED: inch\theorems\output\inch_original_theorems_14.m4.out IN 1.95 ON 2012-7-5 18:08:45
inch\theorems\output\inch_original_theorems_15 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_15.p9.out IN 0.02 ON 2012-7-5 18:08:49, PROOF LENGTH=13
    FAILED: inch\theorems\output\inch_original_theorems_15.m4.out IN 1.94 ON 2012-7-5 18:08:51
inch\theorems\output\inch_original_theorems_16 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_16.p9.out IN 0.02 ON 2012-7-5 18:08:55, PROOF LENGTH=19
    FAILED: inch\theorems\output\inch_original_theorems_16.m4.out IN 1.95 ON 2012-7-5 18:08:57
inch\theorems\output\inch_original_theorems_17 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_17.p9.out IN 0.01 ON 2012-7-5 18:09:01, PROOF LENGTH=11
    FAILED: inch\theorems\output\inch_original_theorems_17.m4.out IN 1.94 ON 2012-7-5 18:09:03
inch\theorems\output\inch_original_theorems_18 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_18.p9.out IN 0.06 ON 2012-7-5 18:09:07, PROOF LENGTH=20
    FAILED: inch\theorems\output\inch_original_theorems_18.m4.out IN 1.94 ON 2012-7-5 18:09:09
inch\theorems\output\inch_original_theorems_19 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_19.p9.out IN 0.01 ON 2012-7-5 18:09:13, PROOF LENGTH=12
    FAILED: inch\theorems\output\inch_original_theorems_19.m4.out IN 1.94 ON 2012-7-5 18:09:15
inch\theorems\output\inch_original_theorems_2 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_2.p9.out IN 0.01 ON 2012-7-5 17:57:06, PROOF LENGTH=9
    FAILED: inch\theorems\output\inch_original_theorems_2.m4.out IN 1.93 ON 2012-7-5 17:57:08
inch\theorems\output\inch_original_theorems_20 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_20.p9.out IN 0.01 ON 2012-7-5 18:09:19, PROOF LENGTH=17
    FAILED: inch\theorems\output\inch_original_theorems_20.m4.out IN 1.94 ON 2012-7-5 18:09:21
inch\theorems\output\inch_original_theorems_21 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_21.p9.out IN 0.02 ON 2012-7-5 18:09:25, PROOF LENGTH=9
    FAILED: inch\theorems\output\inch_original_theorems_21.m4.out IN 1.94 ON 2012-7-5 18:09:27
inch\theorems\output\inch_original_theorems_22 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_22.p9.out IN 0.03 ON 2012-7-5 18:09:31, PROOF LENGTH=12
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    FAILED: inch\theorems\output\inch_original_theorems_22.m4.out IN 1.94 ON 2012-7-5 18:09:33
inch\theorems\output\inch_original_theorems_23 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_23.p9.out IN 0.02 ON 2012-7-5 18:09:37, PROOF LENGTH=11
    FAILED: inch\theorems\output\inch_original_theorems_23.m4.out IN 1.94 ON 2012-7-5 18:09:39
inch\theorems\output\inch_original_theorems_24 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_24.p9.out IN 0.01 ON 2012-7-5 18:09:43, PROOF LENGTH=9
    FAILED: inch\theorems\output\inch_original_theorems_24.m4.out IN 1.95 ON 2012-7-5 18:09:45
inch\theorems\output\inch_original_theorems_25 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_25.p9.out IN 1.13 ON 2012-7-5 18:09:50, PROOF LENGTH=32
    FAILED: inch\theorems\output\inch_original_theorems_25.m4.out IN 1.87 ON 2012-7-5 18:09:51
inch\theorems\output\inch_original_theorems_3 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_3.p9.out IN 0.01 ON 2012-7-5 17:57:12, PROOF LENGTH=9
    FAILED: inch\theorems\output\inch_original_theorems_3.m4.out IN 1.94 ON 2012-7-5 17:57:14
inch\theorems\output\inch_original_theorems_4 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_4.p9.out IN 0.02 ON 2012-7-5 17:57:18, PROOF LENGTH=18
    FAILED: inch\theorems\output\inch_original_theorems_4.m4.out IN 1.93 ON 2012-7-5 17:57:20
inch\theorems\output\inch_original_theorems_5 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_5.p9.out IN 0.00 ON 2012-7-5 17:57:24, PROOF LENGTH=2
    FAILED: inch\theorems\output\inch_original_theorems_5.m4.out IN 1.96 ON 2012-7-5 17:57:26
inch\theorems\output\inch_original_theorems_6 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_6.p9.out IN 0.01 ON 2012-7-5 17:57:30, PROOF LENGTH=7
    FAILED: inch\theorems\output\inch_original_theorems_6.m4.out IN 1.95 ON 2012-7-5 17:57:32
inch\theorems\output\inch_original_theorems_7 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_7.p9.out IN 0.03 ON 2012-7-5 17:57:36, PROOF LENGTH=14
    FAILED: inch\theorems\output\inch_original_theorems_7.m4.out IN 1.92 ON 2012-7-5 17:57:38
inch\theorems\output\inch_original_theorems_8 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_original_theorems_8.p9.out IN 0.08 ON 2012-7-5 17:57:42, PROOF LENGTH=19
    FAILED: inch\theorems\output\inch_original_theorems_8.m4.out IN 1.90 ON 2012-7-5 17:57:44
inch\theorems\output\inch_original_theorems_9 tested with: Mace4, Prover9
  UNKNOWN: inch\theorems\output\inch_original_theorems_9
    FAILED: inch\theorems\output\inch_original_theorems_9.p9.out IN 600.28 ON 2012-7-5 18:07:51
    FAILED: inch\theorems\output\inch_original_theorems_9.m4.out IN 600.02 ON 2012-7-5 18:08:14
inch\theorems\output\inch_weak_closed_z-a1 tested with: Mace4, Prover9
  UNKNOWN: inch\theorems\output\inch_weak_closed_z-a1
    FAILED: inch\theorems\output\inch_weak_closed_z-a1.p9.out IN 600.00 ON 2012-5-22 11:16:08
    FAILED: inch\theorems\output\inch_weak_closed_z-a1.m4.out IN 600.03 ON 2012-5-22 11:16:20
inch\theorems\output\inch_weak_closed_z-a1_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_weak_closed_z-a1_1.p9.out IN 0.50 ON 2012-5-22 11:16:24, PROOF LENGTH=27
    FAILED: inch\theorems\output\inch_weak_closed_z-a1_1.m4.out IN 1.90 ON 2012-5-22 11:16:25
inch\theorems\output\inch_weak_i-e1_d-a6 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_weak_i-e1_d-a6.p9.out IN 1.87 ON 2012-5-28 13:57:49
  SUCCESS: inch\theorems\output\inch_weak_i-e1_d-a6.m4.out IN 0.00 ON 2012-5-28 13:57:47, MODEL SIZE=2
inch\theorems\output\inch_weak_i-e1_d-a6_1 tested with: Mace4, Prover9
  SUCCESS: inch\theorems\output\inch_weak_i-e1_d-a6_1.p9.out IN 0.03 ON 2012-5-28 13:57:54, PROOF LENGTH=15
    FAILED: inch\theorems\output\inch_weak_i-e1_d-a6_1.m4.out IN 0.48 ON 2012-5-28 13:57:55
inch\theorems\output\inch_weak_z-a1 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_weak_z-a1.p9.out IN 1.94 ON 2012-5-22 11:17:45
  SUCCESS: inch\theorems\output\inch_weak_z-a1.m4.out IN 0.87 ON 2012-5-22 11:17:44, MODEL SIZE=2
inch\theorems\output\inch_weak_z-a1_1 tested with: Mace4, Prover9
    FAILED: inch\theorems\output\inch_weak_z-a1_1.p9.out IN 1.94 ON 2012-5-22 11:17:51
  SUCCESS: inch\theorems\output\inch_weak_z-a1_1.m4.out IN 0.41 ON 2012-5-22 11:17:50, MODEL SIZE=2

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codid
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codib\consistency\output\codib_down_nontrivial tested with: Paradox3
  SUCCESS: codib\consistency\output\codib_down_nontrivial.tptp.out, MODEL SIZE=9
codib\consistency\output\codib_down_nontrivial_icon tested with: Mace4, Prover9, Paradox3
    FAILED: codib\consistency\output\codib_down_nontrivial_icon.p9.out IN 600.01 ON 2012-9-4 00:40:49
    FAILED: codib\consistency\output\codib_down_nontrivial_icon.m4.out IN 600.21 ON 2012-9-4 00:40:47
  SUCCESS: codib\consistency\output\codib_down_nontrivial_icon.tptp.out, MODEL SIZE=5
codib\consistency\output\codib_down_nontrivial_simple tested with: Paradox3
  SUCCESS: codib\consistency\output\codib_down_nontrivial_simple.tptp.out, MODEL SIZE=8
codib\consistency\output\codib_updown_nontrivial tested with: Paradox3
  SUCCESS: codib\consistency\output\codib_updown_nontrivial.tptp.out, MODEL SIZE=10
codib\consistency\output\codib_updown_nontrivial_simple tested with: Paradox3
  SUCCESS: codib\consistency\output\codib_updown_nontrivial_simple.tptp.out, MODEL SIZE=9
codib\consistency\output\codid_down_nontrivial tested with: Paradox3
  SUCCESS: codib\consistency\output\codid_down_nontrivial.tptp.out, MODEL SIZE=9
codib\consistency\output\codid_down_nontrivial_simple tested with: Paradox3
  SUCCESS: codib\consistency\output\codid_down_nontrivial_simple.tptp.out, MODEL SIZE=8
codib\consistency\output\codid_updown_nontrivial_simple tested with: Paradox3
  SUCCESS: codib\consistency\output\codid_updown_nontrivial_simple.tptp.out, MODEL SIZE=9
codib\consistency\output\codi_bcont_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: codib\consistency\output\codi_bcont_nontrivial.p9.out IN 1.92 ON 2012-9-4 01:03:51
  SUCCESS: codib\consistency\output\codi_bcont_nontrivial.m4.out IN 0.03 ON 2012-9-4 01:03:49, MODEL SIZE=4
  SUCCESS: codib\consistency\output\codi_bcont_nontrivial.tptp.out, MODEL SIZE=4
codib\consistency\output\codi_bcont_nontrivial_icon tested with: Paradox3
  SUCCESS: codib\consistency\output\codi_bcont_nontrivial_icon.tptp.out, MODEL SIZE=5
codib\consistency\output\codi_bcont_nontrivial_simple tested with: Paradox3
  SUCCESS: codib\consistency\output\codi_bcont_nontrivial_simple.tptp.out, MODEL SIZE=8
codib\output\codib_down tested with: Mace4, Prover9
    FAILED: codib\output\codib_down.p9.out IN 1.92 ON 2012-7-13 13:52:33
  SUCCESS: codib\output\codib_down.m4.out IN 0.02 ON 2012-7-13 13:52:31, MODEL SIZE=2
codib\output\codib_updown tested with: Mace4, Prover9
    FAILED: codib\output\codib_updown.p9.out IN 1.93 ON 2012-7-9 14:40:18
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  SUCCESS: codib\output\codib_updown.m4.out IN 0.02 ON 2012-7-9 14:40:16, MODEL SIZE=2
codib\output\codi_bcont tested with: Mace4, Prover9
    FAILED: codib\output\codi_bcont.p9.out IN 1.93 ON 2012-7-13 11:55:06
  SUCCESS: codib\output\codi_bcont.m4.out IN 0.02 ON 2012-7-13 11:55:04, MODEL SIZE=2
codib\theorems\output\codib_boundary_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_boundary_theorems.p9.out IN 1.93 ON 2012-10-23 21:43:43
  SUCCESS: codib\theorems\output\codib_boundary_theorems.m4.out IN 0.01 ON 2012-10-23 21:43:41, MODEL SIZE=2
codib\theorems\output\codib_boundary_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_boundary_theorems_1.p9.out IN 0.04 ON 2012-10-23 21:43:50, PROOF LENGTH=17
    FAILED: codib\theorems\output\codib_boundary_theorems_1.m4.out IN 1.91 ON 2012-10-23 21:43:51
codib\theorems\output\codib_boundary_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_boundary_theorems_2.p9.out IN 0.04 ON 2012-10-23 21:43:56, PROOF LENGTH=12
    FAILED: codib\theorems\output\codib_boundary_theorems_2.m4.out IN 1.88 ON 2012-10-23 21:43:57
codib\theorems\output\codib_bp_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_bp_theorems.p9.out IN 1.91 ON 2012-10-23 21:38:19
  SUCCESS: codib\theorems\output\codib_bp_theorems.m4.out IN 0.02 ON 2012-10-23 21:38:17, MODEL SIZE=2
codib\theorems\output\codib_bp_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_bp_theorems_1.p9.out IN 0.08 ON 2012-10-23 21:38:28, PROOF LENGTH=22
    FAILED: codib\theorems\output\codib_bp_theorems_1.m4.out IN 1.92 ON 2012-10-23 21:38:30
codib\theorems\output\codib_bp_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_bp_theorems_2.p9.out IN 0.09 ON 2012-10-23 21:38:34, PROOF LENGTH=17
    FAILED: codib\theorems\output\codib_bp_theorems_2.m4.out IN 1.91 ON 2012-10-23 21:38:36
codib\theorems\output\codib_bp_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_bp_theorems_3.p9.out IN 0.10 ON 2012-10-23 21:38:40, PROOF LENGTH=18
    FAILED: codib\theorems\output\codib_bp_theorems_3.m4.out IN 1.93 ON 2012-10-23 21:38:42
codib\theorems\output\codib_bp_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_bp_theorems_4.p9.out IN 0.09 ON 2012-10-23 21:38:46, PROOF LENGTH=21
    FAILED: codib\theorems\output\codib_bp_theorems_4.m4.out IN 1.94 ON 2012-10-23 21:38:48
codib\theorems\output\codib_bp_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_bp_theorems_5.p9.out IN 0.07 ON 2012-10-23 21:38:52, PROOF LENGTH=27
    FAILED: codib\theorems\output\codib_bp_theorems_5.m4.out IN 1.91 ON 2012-10-23 21:38:54
codib\theorems\output\codib_down_9intersection_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_down_9intersection_theorems.p9.out IN 1.91 ON 2012-11-21 14:07:49
  SUCCESS: codib\theorems\output\codib_down_9intersection_theorems.m4.out IN 0.01 ON 2012-11-21 14:07:47, MODEL SIZE=2
codib\theorems\output\codib_down_9intersection_theorems_1 tested with: Mace4, Prover9, Vampire
  SUCCESS: codib\theorems\output\codib_down_9intersection_theorems_1.p9.out IN 0.28 ON 2012-11-21 14:07:56, PROOF LENGTH=16
    FAILED: codib\theorems\output\codib_down_9intersection_theorems_1.vam.out IN 599.707
    FAILED: codib\theorems\output\codib_down_9intersection_theorems_1.m4.out IN 1.91 ON 2012-11-21 14:07:58
codib\theorems\output\codib_down_9intersection_theorems_2 tested with: Mace4, Prover9, Vampire
  SUCCESS: codib\theorems\output\codib_down_9intersection_theorems_2.p9.out IN 0.91 ON 2012-11-21 14:08:03, PROOF LENGTH=16
  SUCCESS: codib\theorems\output\codib_down_9intersection_theorems_2.vam.out IN 0.46
    FAILED: codib\theorems\output\codib_down_9intersection_theorems_2.m4.out IN 1.91 ON 2012-11-21 14:08:04
codib\theorems\output\codib_down_9intersection_theorems_3 tested with: Mace4, Prover9, Vampire
  SUCCESS: codib\theorems\output\codib_down_9intersection_theorems_3.p9.out IN 0.31 ON 2012-11-21 14:08:09, PROOF LENGTH=15
  SUCCESS: codib\theorems\output\codib_down_9intersection_theorems_3.vam.out IN 0.278
    FAILED: codib\theorems\output\codib_down_9intersection_theorems_3.m4.out IN 1.92 ON 2012-11-21 14:08:10
codib\theorems\output\codib_down_9intersection_theorems_4 tested with: Mace4, Prover9, Vampire
  SUCCESS: codib\theorems\output\codib_down_9intersection_theorems_4.p9.out IN 0.36 ON 2012-11-21 14:08:15, PROOF LENGTH=15
  SUCCESS: codib\theorems\output\codib_down_9intersection_theorems_4.vam.out IN 0.236
    FAILED: codib\theorems\output\codib_down_9intersection_theorems_4.m4.out IN 1.89 ON 2012-11-21 14:08:16
codib\theorems\output\codib_down_9intersection_theorems_5 tested with: Mace4, Prover9, Vampire
  UNKNOWN: codib\theorems\output\codib_down_9intersection_theorems_5
    FAILED: codib\theorems\output\codib_down_9intersection_theorems_5.p9.out IN 600.01 ON 2012-11-21 14:18:25
    FAILED: codib\theorems\output\codib_down_9intersection_theorems_5.vam.out IN 599.703
    FAILED: codib\theorems\output\codib_down_9intersection_theorems_5.m4.out IN 600.04 ON 2012-11-21 14:18:25
codib\theorems\output\codib_icont_tcont_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_icont_tcont_theorems.p9.out IN 1.93 ON 2012-10-23 21:39:47
  SUCCESS: codib\theorems\output\codib_icont_tcont_theorems.m4.out IN 0.02 ON 2012-10-23 21:39:45, MODEL SIZE=2
codib\theorems\output\codib_icont_tcont_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_icont_tcont_theorems_1.p9.out IN 0.04 ON 2012-10-23 21:39:54, PROOF LENGTH=12
    FAILED: codib\theorems\output\codib_icont_tcont_theorems_1.m4.out IN 1.95 ON 2012-10-23 21:39:56
codib\theorems\output\codib_icont_tcont_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_icont_tcont_theorems_2.p9.out IN 0.26 ON 2012-10-23 21:40:01, PROOF LENGTH=13
    FAILED: codib\theorems\output\codib_icont_tcont_theorems_2.m4.out IN 1.93 ON 2012-10-23 21:40:02
codib\theorems\output\codib_icont_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_icont_theorems.p9.out IN 1.93 ON 2012-10-23 21:40:11
  SUCCESS: codib\theorems\output\codib_icont_theorems.m4.out IN 0.02 ON 2012-10-23 21:40:09, MODEL SIZE=2
codib\theorems\output\codib_icont_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_icont_theorems_1.p9.out IN 0.03 ON 2012-10-23 21:40:19, PROOF LENGTH=7
    FAILED: codib\theorems\output\codib_icont_theorems_1.m4.out IN 1.96 ON 2012-10-23 21:40:20
codib\theorems\output\codib_icont_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_icont_theorems_2.p9.out IN 0.04 ON 2012-10-23 21:40:25, PROOF LENGTH=15
    FAILED: codib\theorems\output\codib_icont_theorems_2.m4.out IN 1.93 ON 2012-10-23 21:40:26
codib\theorems\output\codib_icont_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_icont_theorems_3.p9.out IN 0.03 ON 2012-10-23 21:40:31, PROOF LENGTH=10
    FAILED: codib\theorems\output\codib_icont_theorems_3.m4.out IN 1.95 ON 2012-10-23 21:40:33
codib\theorems\output\codib_icont_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_icont_theorems_4.p9.out IN 0.03 ON 2012-10-23 21:40:37, PROOF LENGTH=13
    FAILED: codib\theorems\output\codib_icont_theorems_4.m4.out IN 1.95 ON 2012-10-23 21:40:39
codib\theorems\output\codib_icont_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_icont_theorems_5.p9.out IN 0.03 ON 2012-10-23 21:40:43, PROOF LENGTH=13
    FAILED: codib\theorems\output\codib_icont_theorems_5.m4.out IN 1.93 ON 2012-10-23 21:40:45
codib\theorems\output\codib_icont_theorems_6 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_icont_theorems_6.p9.out IN 0.05 ON 2012-10-23 21:40:49, PROOF LENGTH=20
    FAILED: codib\theorems\output\codib_icont_theorems_6.m4.out IN 1.93 ON 2012-10-23 21:40:51
codib\theorems\output\codib_stp_theorems tested with: Mace4, Prover9
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    FAILED: codib\theorems\output\codib_stp_theorems.p9.out IN 1.93 ON 2012-10-23 21:40:37
  SUCCESS: codib\theorems\output\codib_stp_theorems.m4.out IN 0.03 ON 2012-10-23 21:40:35, MODEL SIZE=2
codib\theorems\output\codib_stp_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_stp_theorems_1.p9.out IN 0.03 ON 2012-10-23 21:40:45, PROOF LENGTH=7
    FAILED: codib\theorems\output\codib_stp_theorems_1.m4.out IN 1.88 ON 2012-10-23 21:40:47
codib\theorems\output\codib_stp_theorems_2 tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_stp_theorems_2.p9.out IN 1.92 ON 2012-10-23 21:40:53
  SUCCESS: codib\theorems\output\codib_stp_theorems_2.m4.out IN 0.05 ON 2012-10-23 21:40:51, MODEL SIZE=5
codib\theorems\output\codib_stp_theorems_3 tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_stp_theorems_3.p9.out IN 1.91 ON 2012-10-23 21:40:59
  SUCCESS: codib\theorems\output\codib_stp_theorems_3.m4.out IN 0.03 ON 2012-10-23 21:40:57, MODEL SIZE=4
codib\theorems\output\codib_tcont_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_tcont_theorems.p9.out IN 1.95 ON 2012-10-23 21:38:48
  SUCCESS: codib\theorems\output\codib_tcont_theorems.m4.out IN 0.01 ON 2012-10-23 21:38:46, MODEL SIZE=2
codib\theorems\output\codib_tcont_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tcont_theorems_1.p9.out IN 0.03 ON 2012-10-23 21:38:55, PROOF LENGTH=7
    FAILED: codib\theorems\output\codib_tcont_theorems_1.m4.out IN 1.95 ON 2012-10-23 21:38:57
codib\theorems\output\codib_tcont_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tcont_theorems_2.p9.out IN 0.08 ON 2012-10-23 21:39:01, PROOF LENGTH=15
    FAILED: codib\theorems\output\codib_tcont_theorems_2.m4.out IN 1.95 ON 2012-10-23 21:39:03
codib\theorems\output\codib_tcont_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tcont_theorems_3.p9.out IN 0.02 ON 2012-10-23 21:39:07, PROOF LENGTH=10
    FAILED: codib\theorems\output\codib_tcont_theorems_3.m4.out IN 1.95 ON 2012-10-23 21:39:09
codib\theorems\output\codib_tcont_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tcont_theorems_4.p9.out IN 0.02 ON 2012-10-23 21:39:13, PROOF LENGTH=10
    FAILED: codib\theorems\output\codib_tcont_theorems_4.m4.out IN 1.95 ON 2012-10-23 21:39:15
codib\theorems\output\codib_tcont_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tcont_theorems_5.p9.out IN 0.05 ON 2012-10-23 21:39:19, PROOF LENGTH=20
    FAILED: codib\theorems\output\codib_tcont_theorems_5.m4.out IN 1.94 ON 2012-10-23 21:39:21
codib\theorems\output\codib_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_theorems.p9.out IN 1.94 ON 2012-10-23 21:41:06
  SUCCESS: codib\theorems\output\codib_theorems.m4.out IN 0.02 ON 2012-10-23 21:41:04, MODEL SIZE=2
codib\theorems\output\codib_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_theorems_1.p9.out IN 0.07 ON 2012-10-23 21:41:13, PROOF LENGTH=29
    FAILED: codib\theorems\output\codib_theorems_1.m4.out IN 1.96 ON 2012-10-23 21:41:15
codib\theorems\output\codib_tp_ip_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_tp_ip_theorems.p9.out IN 1.92 ON 2012-10-23 21:46:19
  SUCCESS: codib\theorems\output\codib_tp_ip_theorems.m4.out IN 0.01 ON 2012-10-23 21:46:17, MODEL SIZE=2
codib\theorems\output\codib_tp_ip_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tp_ip_theorems_1.p9.out IN 0.03 ON 2012-10-23 21:46:26, PROOF LENGTH=13
    FAILED: codib\theorems\output\codib_tp_ip_theorems_1.m4.out IN 1.93 ON 2012-10-23 21:46:28
codib\theorems\output\codib_tp_ip_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tp_ip_theorems_2.p9.out IN 63.96 ON 2012-10-23 21:47:37, PROOF LENGTH=62
    FAILED: codib\theorems\output\codib_tp_ip_theorems_2.m4.out IN 65.10 ON 2012-10-23 21:47:39
codib\theorems\output\codib_tp_ip_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tp_ip_theorems_3.p9.out IN 0.06 ON 2012-10-23 21:47:43, PROOF LENGTH=33
    FAILED: codib\theorems\output\codib_tp_ip_theorems_3.m4.out IN 1.92 ON 2012-10-23 21:47:45
codib\theorems\output\codib_tp_ip_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tp_ip_theorems_4.p9.out IN 0.05 ON 2012-10-23 21:47:49, PROOF LENGTH=16
    FAILED: codib\theorems\output\codib_tp_ip_theorems_4.m4.out IN 1.91 ON 2012-10-23 21:47:51
codib\theorems\output\codib_tp_ip_theorems_5 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tp_ip_theorems_5.p9.out IN 0.02 ON 2012-10-23 21:47:55, PROOF LENGTH=18
    FAILED: codib\theorems\output\codib_tp_ip_theorems_5.m4.out IN 1.91 ON 2012-10-23 21:47:57
codib\theorems\output\codib_tp_ip_theorems_6 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codib_tp_ip_theorems_6.p9.out IN 1.37 ON 2012-10-23 21:48:02, PROOF LENGTH=23
    FAILED: codib\theorems\output\codib_tp_ip_theorems_6.m4.out IN 1.92 ON 2012-10-23 21:48:03
codib\theorems\output\codib_updown_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codib_updown_theorems.p9.out IN 1.91 ON 2012-7-10 14:17:40
  SUCCESS: codib\theorems\output\codib_updown_theorems.m4.out IN 0.03 ON 2012-7-10 14:17:38, MODEL SIZE=2
codib\theorems\output\codib_updown_theorems_1 tested with: Mace4, Prover9
  UNKNOWN: codib\theorems\output\codib_updown_theorems_1
    FAILED: codib\theorems\output\codib_updown_theorems_1.p9.out IN 600.00 ON 2012-7-10 14:27:53
    FAILED: codib\theorems\output\codib_updown_theorems_1.m4.out IN 600.17 ON 2012-7-10 14:27:51
codib\theorems\output\codi_bcont_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codi_bcont_theorems.p9.out IN 1.94 ON 2012-10-23 21:37:50
  SUCCESS: codib\theorems\output\codi_bcont_theorems.m4.out IN 0.03 ON 2012-10-23 21:37:48, MODEL SIZE=2
codib\theorems\output\codi_bcont_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codi_bcont_theorems_1.p9.out IN 0.02 ON 2012-10-23 21:37:57, PROOF LENGTH=6
    FAILED: codib\theorems\output\codi_bcont_theorems_1.m4.out IN 1.96 ON 2012-10-23 21:37:59
codib\theorems\output\codi_bcont_theorems_2 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codi_bcont_theorems_2.p9.out IN 0.03 ON 2012-10-23 21:38:03, PROOF LENGTH=17
    FAILED: codib\theorems\output\codi_bcont_theorems_2.m4.out IN 1.93 ON 2012-10-23 21:38:05
codib\theorems\output\codi_bcont_theorems_3 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codi_bcont_theorems_3.p9.out IN 0.02 ON 2012-10-23 21:38:09, PROOF LENGTH=7
    FAILED: codib\theorems\output\codi_bcont_theorems_3.m4.out IN 1.94 ON 2012-10-23 21:38:11
codib\theorems\output\codi_bcont_theorems_4 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codi_bcont_theorems_4.p9.out IN 0.02 ON 2012-10-23 21:38:15, PROOF LENGTH=18
    FAILED: codib\theorems\output\codi_bcont_theorems_4.m4.out IN 1.92 ON 2012-10-23 21:38:17
codib\theorems\output\codi_bcont_u-a1_theorems tested with: Mace4, Prover9
    FAILED: codib\theorems\output\codi_bcont_u-a1_theorems.p9.out IN 1.94 ON 2012-10-23 21:43:23
  SUCCESS: codib\theorems\output\codi_bcont_u-a1_theorems.m4.out IN 0.01 ON 2012-10-23 21:43:21, MODEL SIZE=2
codib\theorems\output\codi_bcont_u-a1_theorems_1 tested with: Mace4, Prover9
  SUCCESS: codib\theorems\output\codi_bcont_u-a1_theorems_1.p9.out IN 9.95 ON 2012-10-23 21:43:40, PROOF LENGTH=58
    FAILED: codib\theorems\output\codi_bcont_u-a1_theorems_1.m4.out IN 11.91 ON 2012-10-23 21:43:41
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ig\consistency\output\ig_2d_lin_nontrivial tested with: Paradox3
  SUCCESS: ig\consistency\output\ig_2d_lin_nontrivial.tptp.out, MODEL SIZE=3
ig\consistency\output\ig_2d_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: ig\consistency\output\ig_2d_nontrivial.p9.out IN 0.01 ON 2012-10-24 21:27:02
  SUCCESS: ig\consistency\output\ig_2d_nontrivial.m4.out IN 0.00 ON 2012-10-24 21:27:03, MODEL SIZE=3
  SUCCESS: ig\consistency\output\ig_2d_nontrivial.tptp.out, MODEL SIZE=3
ig\consistency\output\ig_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: ig\consistency\output\ig_nontrivial.p9.out IN 7.95 ON 2012-11-4 10:01:52
  SUCCESS: ig\consistency\output\ig_nontrivial.m4.out IN 6.50 ON 2012-11-4 10:01:50, MODEL SIZE=7
  SUCCESS: ig\consistency\output\ig_nontrivial.tptp.out, MODEL SIZE=7
ig\theorems\output\ig_2d_theorems tested with: Mace4, Prover9
    FAILED: ig\theorems\output\ig_2d_theorems.p9.out IN 0.01 ON 2012-10-24 22:08:05
  SUCCESS: ig\theorems\output\ig_2d_theorems.m4.out IN 0.00 ON 2012-10-24 22:08:05, MODEL SIZE=2
ig\theorems\output\ig_2d_theorems_1 tested with: Mace4, Prover9
  SUCCESS: ig\theorems\output\ig_2d_theorems_1.p9.out IN 0.00 ON 2012-10-24 22:08:09, PROOF LENGTH=10
    FAILED: ig\theorems\output\ig_2d_theorems_1.m4.out IN 0.02 ON 2012-10-24 22:08:09
ig\theorems\output\ig_2d_theorems_2 tested with: Mace4, Prover9
  SUCCESS: ig\theorems\output\ig_2d_theorems_2.p9.out IN 0.00 ON 2012-10-24 22:08:13, PROOF LENGTH=10
    FAILED: ig\theorems\output\ig_2d_theorems_2.m4.out IN 0.02 ON 2012-10-24 22:08:13
ig\theorems\output\ig_theorems tested with: Mace4, Prover9
    FAILED: ig\theorems\output\ig_theorems.p9.out IN 0.02 ON 2012-10-24 22:07:36
  SUCCESS: ig\theorems\output\ig_theorems.m4.out IN 0.01 ON 2012-10-24 22:07:36, MODEL SIZE=2
ig\theorems\output\ig_theorems_1 tested with: Mace4, Prover9
  SUCCESS: ig\theorems\output\ig_theorems_1.p9.out IN 0.01 ON 2012-10-24 22:07:41, PROOF LENGTH=6
    FAILED: ig\theorems\output\ig_theorems_1.m4.out IN 1.91 ON 2012-10-24 22:07:43
ig\theorems\output\ig_theorems_2 tested with: Mace4, Prover9
  SUCCESS: ig\theorems\output\ig_theorems_2.p9.out IN 0.01 ON 2012-10-24 22:07:47, PROOF LENGTH=6
    FAILED: ig\theorems\output\ig_theorems_2.m4.out IN 1.90 ON 2012-10-24 22:07:49

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oig\consistency\output\woig_2d_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: oig\consistency\output\woig_2d_nontrivial.p9.out IN 1.97 ON 2012-10-24 22:37:31
  SUCCESS: oig\consistency\output\woig_2d_nontrivial.m4.out IN 0.04 ON 2012-10-24 22:37:29, MODEL SIZE=4
  SUCCESS: oig\consistency\output\woig_2d_nontrivial.tptp.out, MODEL SIZE=4
oig\consistency\output\woig_3d_nontrivial tested with: Mace4, Prover9
  UNKNOWN: oig\consistency\output\woig_3d_nontrivial
    FAILED: oig\consistency\output\woig_3d_nontrivial.p9.out IN 600.00 ON 2012-8-27 18:27:26
    FAILED: oig\consistency\output\woig_3d_nontrivial.m4.out IN 120.07 ON 2012-8-27 18:19:19
oig\consistency\output\woig_nontrivial tested with: Paradox3
  SUCCESS: oig\consistency\output\woig_nontrivial.tptp.out, MODEL SIZE=9
oig\output\woig_2d tested with: Mace4, Prover9
    FAILED: oig\output\woig_2d.p9.out IN 0.01 ON 2012-10-24 22:34:59
  SUCCESS: oig\output\woig_2d.m4.out IN 0.01 ON 2012-10-24 22:34:59, MODEL SIZE=2
oig\theorems\output\b_2d_theorems tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_2d_theorems.p9.out IN 0.00 ON 2012-8-27 16:17:57
  SUCCESS: oig\theorems\output\b_2d_theorems.m4.out IN 0.01 ON 2012-8-27 16:17:57, MODEL SIZE=2
oig\theorems\output\b_2d_theorems_1 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_2d_theorems_1.p9.out IN 1.05 ON 2012-8-27 16:18:02
  SUCCESS: oig\theorems\output\b_2d_theorems_1.m4.out IN 0.47 ON 2012-8-27 16:18:02, MODEL SIZE=7
oig\theorems\output\b_2d_theorems_2 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\b_2d_theorems_2.p9.out IN 0.02 ON 2012-8-27 16:18:05, PROOF LENGTH=10
    FAILED: oig\theorems\output\b_2d_theorems_2.m4.out IN 1.91 ON 2012-8-27 16:18:07
oig\theorems\output\b_2d_theorems_3 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_2d_theorems_3.p9.out IN 0.52 ON 2012-8-27 16:18:12
  SUCCESS: oig\theorems\output\b_2d_theorems_3.m4.out IN 0.17 ON 2012-8-27 16:18:11, MODEL SIZE=6
oig\theorems\output\b_2d_theorems_4 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_2d_theorems_4.p9.out IN 0.52 ON 2012-8-27 16:18:16
  SUCCESS: oig\theorems\output\b_2d_theorems_4.m4.out IN 0.18 ON 2012-8-27 16:18:15, MODEL SIZE=6
oig\theorems\output\b_2d_theorems_5 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_2d_theorems_5.p9.out IN 0.54 ON 2012-8-27 16:18:20
  SUCCESS: oig\theorems\output\b_2d_theorems_5.m4.out IN 0.18 ON 2012-8-27 16:18:19, MODEL SIZE=6
oig\theorems\output\b_2d_theorems_6 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_2d_theorems_6.p9.out IN 0.55 ON 2012-8-27 16:18:24
  SUCCESS: oig\theorems\output\b_2d_theorems_6.m4.out IN 0.18 ON 2012-8-27 16:18:23, MODEL SIZE=6
oig\theorems\output\b_3d_theorems tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_3d_theorems.p9.out IN 0.03 ON 2012-8-27 16:19:38
  SUCCESS: oig\theorems\output\b_3d_theorems.m4.out IN 0.01 ON 2012-8-27 16:19:38, MODEL SIZE=2
oig\theorems\output\b_3d_theorems_1 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_3d_theorems_1.p9.out IN 1.98 ON 2012-8-27 16:19:45
  SUCCESS: oig\theorems\output\b_3d_theorems_1.m4.out IN 0.03 ON 2012-8-27 16:19:43, MODEL SIZE=4
oig\theorems\output\b_3d_theorems_2 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\b_3d_theorems_2.p9.out IN 0.05 ON 2012-8-27 16:19:49, PROOF LENGTH=10
    FAILED: oig\theorems\output\b_3d_theorems_2.m4.out IN 1.95 ON 2012-8-27 16:19:51
oig\theorems\output\b_3d_theorems_3 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_3d_theorems_3.p9.out IN 1.87 ON 2012-8-27 16:19:57
  SUCCESS: oig\theorems\output\b_3d_theorems_3.m4.out IN 0.03 ON 2012-8-27 16:19:55, MODEL SIZE=4
oig\theorems\output\b_3d_theorems_4 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_3d_theorems_4.p9.out IN 1.84 ON 2012-8-27 16:20:01
  SUCCESS: oig\theorems\output\b_3d_theorems_4.m4.out IN 0.04 ON 2012-8-27 16:19:59, MODEL SIZE=4
oig\theorems\output\b_3d_theorems_5 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_3d_theorems_5.p9.out IN 1.68 ON 2012-8-27 16:20:05
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  SUCCESS: oig\theorems\output\b_3d_theorems_5.m4.out IN 0.02 ON 2012-8-27 16:20:03, MODEL SIZE=4
oig\theorems\output\b_3d_theorems_6 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\b_3d_theorems_6.p9.out IN 1.70 ON 2012-8-27 16:20:09
  SUCCESS: oig\theorems\output\b_3d_theorems_6.m4.out IN 0.02 ON 2012-8-27 16:20:07, MODEL SIZE=4
oig\theorems\output\woig_2d_theorems tested with: Mace4, Prover9
    FAILED: oig\theorems\output\woig_2d_theorems.p9.out IN 0.01 ON 2012-10-24 23:01:10
  SUCCESS: oig\theorems\output\woig_2d_theorems.m4.out IN 0.01 ON 2012-10-24 23:01:10, MODEL SIZE=2
oig\theorems\output\woig_2d_theorems_1 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_2d_theorems_1.p9.out IN 0.01 ON 2012-10-24 23:01:15, PROOF LENGTH=10
    FAILED: oig\theorems\output\woig_2d_theorems_1.m4.out IN 1.90 ON 2012-10-24 23:01:17
oig\theorems\output\woig_2d_theorems_2 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_2d_theorems_2.p9.out IN 0.01 ON 2012-10-24 23:01:21, PROOF LENGTH=10
    FAILED: oig\theorems\output\woig_2d_theorems_2.m4.out IN 1.94 ON 2012-10-24 23:01:23
oig\theorems\output\woig_2d_theorems_3 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_2d_theorems_3.p9.out IN 0.01 ON 2012-10-24 23:01:27, PROOF LENGTH=10
    FAILED: oig\theorems\output\woig_2d_theorems_3.m4.out IN 1.95 ON 2012-10-24 23:01:29
oig\theorems\output\woig_2d_theorems_4 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\woig_2d_theorems_4.p9.out IN 148.79 ON 2012-10-24 23:04:03
  SUCCESS: oig\theorems\output\woig_2d_theorems_4.m4.out IN 148.12 ON 2012-10-24 23:04:02, MODEL SIZE=10
oig\theorems\output\woig_2d_theorems_5 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_2d_theorems_5.p9.out IN 0.01 ON 2012-10-24 23:04:08, PROOF LENGTH=10
    FAILED: oig\theorems\output\woig_2d_theorems_5.m4.out IN 1.95 ON 2012-10-24 23:04:10
oig\theorems\output\woig_2d_theorems_6 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\woig_2d_theorems_6.p9.out IN 184.66 ON 2012-10-24 23:07:20
  SUCCESS: oig\theorems\output\woig_2d_theorems_6.m4.out IN 184.06 ON 2012-10-24 23:07:18, MODEL SIZE=9
oig\theorems\output\woig_2d_theorems_7 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\woig_2d_theorems_7.p9.out IN 184.72 ON 2012-10-24 23:10:30
  SUCCESS: oig\theorems\output\woig_2d_theorems_7.m4.out IN 184.57 ON 2012-10-24 23:10:29, MODEL SIZE=9
oig\theorems\output\woig_2d_theorems_8 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\woig_2d_theorems_8.p9.out IN 87.22 ON 2012-10-24 23:12:02
  SUCCESS: oig\theorems\output\woig_2d_theorems_8.m4.out IN 86.65 ON 2012-10-24 23:12:01, MODEL SIZE=9
oig\theorems\output\woig_2d_theorems_9 tested with: Mace4, Prover9
    FAILED: oig\theorems\output\woig_2d_theorems_9.p9.out IN 87.21 ON 2012-10-24 23:13:34
  SUCCESS: oig\theorems\output\woig_2d_theorems_9.m4.out IN 86.28 ON 2012-10-24 23:13:32, MODEL SIZE=9
oig\theorems\output\woig_theorems tested with: Mace4, Prover9
    FAILED: oig\theorems\output\woig_theorems.p9.out IN 0.04 ON 2012-10-24 22:18:12
  SUCCESS: oig\theorems\output\woig_theorems.m4.out IN 0.02 ON 2012-10-24 22:18:12, MODEL SIZE=2
oig\theorems\output\woig_theorems_1 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_theorems_1.p9.out IN 6.85 ON 2012-10-24 22:18:23, PROOF LENGTH=26
    FAILED: oig\theorems\output\woig_theorems_1.m4.out IN 7.89 ON 2012-10-24 22:18:25
oig\theorems\output\woig_theorems_2 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_theorems_2.p9.out IN 3.47 ON 2012-10-24 22:18:32, PROOF LENGTH=24
    FAILED: oig\theorems\output\woig_theorems_2.m4.out IN 3.91 ON 2012-10-24 22:18:33
oig\theorems\output\woig_theorems_3 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_theorems_3.p9.out IN 23.03 ON 2012-10-24 22:19:00, PROOF LENGTH=36
    FAILED: oig\theorems\output\woig_theorems_3.m4.out IN 23.75 ON 2012-10-24 22:19:01
oig\theorems\output\woig_theorems_4 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_theorems_4.p9.out IN 0.13 ON 2012-10-24 22:19:05, PROOF LENGTH=10
    FAILED: oig\theorems\output\woig_theorems_4.m4.out IN 1.95 ON 2012-10-24 22:19:07
oig\theorems\output\woig_theorems_5 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_theorems_5.p9.out IN 7.28 ON 2012-10-24 22:19:18, PROOF LENGTH=32
    FAILED: oig\theorems\output\woig_theorems_5.m4.out IN 7.88 ON 2012-10-24 22:19:19
oig\theorems\output\woig_theorems_6 tested with: Mace4, Prover9
  SUCCESS: oig\theorems\output\woig_theorems_6.p9.out IN 7.26 ON 2012-10-24 22:19:30, PROOF LENGTH=32
    FAILED: oig\theorems\output\woig_theorems_6.m4.out IN 7.87 ON 2012-10-24 22:19:31
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btw

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btw\theorems\output\btw_basic_theorems tested with: Mace4, Prover9
    FAILED: btw\theorems\output\btw_basic_theorems.p9.out IN 1.97 ON 2012-8-27 16:05:03
  SUCCESS: btw\theorems\output\btw_basic_theorems.m4.out IN 0.00 ON 2012-8-27 16:05:01, MODEL SIZE=2
btw\theorems\output\btw_basic_theorems_1 tested with: Mace4, Prover9
  SUCCESS: btw\theorems\output\btw_basic_theorems_1.p9.out IN 0.01 ON 2012-8-27 16:05:07, PROOF LENGTH=9
    FAILED: btw\theorems\output\btw_basic_theorems_1.m4.out IN 1.95 ON 2012-8-27 16:05:09
btw\theorems\output\btw_basic_theorems_2 tested with: Mace4, Prover9
  SUCCESS: btw\theorems\output\btw_basic_theorems_2.p9.out IN 0.01 ON 2012-8-27 16:05:13, PROOF LENGTH=15
    FAILED: btw\theorems\output\btw_basic_theorems_2.m4.out IN 1.93 ON 2012-8-27 16:05:15
btw\theorems\output\btw_basic_theorems_3 tested with: Mace4, Prover9
    FAILED: btw\theorems\output\btw_basic_theorems_3.p9.out IN 1.96 ON 2012-8-27 16:05:21
  SUCCESS: btw\theorems\output\btw_basic_theorems_3.m4.out IN 0.01 ON 2012-8-27 16:05:19, MODEL SIZE=4
btw\theorems\output\btw_orderable_theorems tested with: Mace4, Prover9
    FAILED: btw\theorems\output\btw_orderable_theorems.p9.out IN 1.97 ON 2012-8-27 16:05:04
  SUCCESS: btw\theorems\output\btw_orderable_theorems.m4.out IN 0.00 ON 2012-8-27 16:05:02, MODEL SIZE=2
btw\theorems\output\btw_orderable_theorems_1 tested with: Mace4, Prover9
  SUCCESS: btw\theorems\output\btw_orderable_theorems_1.p9.out IN 0.01 ON 2012-8-27 16:05:09, PROOF LENGTH=28
    FAILED: btw\theorems\output\btw_orderable_theorems_1.m4.out IN 1.96 ON 2012-8-27 16:05:11
btw\theorems\output\btw_orderable_theorems_2 tested with: Mace4, Prover9
  SUCCESS: btw\theorems\output\btw_orderable_theorems_2.p9.out IN 0.01 ON 2012-8-27 16:05:15, PROOF LENGTH=13
    FAILED: btw\theorems\output\btw_orderable_theorems_2.m4.out IN 1.95 ON 2012-8-27 16:05:17
btw\theorems\output\btw_orderable_theorems_3 tested with: Mace4, Prover9
  SUCCESS: btw\theorems\output\btw_orderable_theorems_3.p9.out IN 0.00 ON 2012-8-27 16:05:21, PROOF LENGTH=20
    FAILED: btw\theorems\output\btw_orderable_theorems_3.m4.out IN 1.92 ON 2012-8-27 16:05:23
btw\theorems\output\btw_orderable_theorems_4 tested with: Mace4, Prover9
  SUCCESS: btw\theorems\output\btw_orderable_theorems_4.p9.out IN 0.01 ON 2012-8-27 16:05:27, PROOF LENGTH=22
    FAILED: btw\theorems\output\btw_orderable_theorems_4.m4.out IN 1.96 ON 2012-8-27 16:05:29
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omt

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omt\consistency\output\omt_3d_lin_nontrivial tested with: Paradox3
  SUCCESS: omt\consistency\output\omt_3d_lin_nontrivial.tptp.out, MODEL SIZE=10
omt\consistency\output\omt_down_plp_lin_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: omt\consistency\output\omt_down_plp_lin_nontrivial.p9.out IN 600.01 ON 2012-8-27 17:08:16
    FAILED: omt\consistency\output\omt_down_plp_lin_nontrivial.m4.out IN 312.54 ON 2012-8-27 17:03:26
  SUCCESS: omt\consistency\output\omt_down_plp_lin_nontrivial.tptp.out, MODEL SIZE=10
omt\consistency\output\omt_down_plp_lin_nontrivial_btw tested with: Paradox3
  SUCCESS: omt\consistency\output\omt_down_plp_lin_nontrivial_btw.tptp.out, MODEL SIZE=10
omt\output\omt_down tested with: Mace4, Prover9
    FAILED: omt\output\omt_down.p9.out IN 1.94 ON 2012-8-27 16:37:14
  SUCCESS: omt\output\omt_down.m4.out IN 0.01 ON 2012-8-27 16:37:12, MODEL SIZE=2
omt\theorems\output\omt_down_theorems tested with: Mace4, Prover9
    FAILED: omt\theorems\output\omt_down_theorems.p9.out IN 1.87 ON 2012-8-27 18:54:47
  SUCCESS: omt\theorems\output\omt_down_theorems.m4.out IN 0.02 ON 2012-8-27 18:54:45, MODEL SIZE=2
omt\theorems\output\omt_down_theorems_1 tested with: Mace4, Prover9
  SUCCESS: omt\theorems\output\omt_down_theorems_1.p9.out IN 0.03 ON 2012-8-27 18:54:54, PROOF LENGTH=9
    FAILED: omt\theorems\output\omt_down_theorems_1.m4.out IN 1.94 ON 2012-8-27 18:54:56
omt\theorems\output\omt_down_theorems_2 tested with: Mace4, Prover9
  SUCCESS: omt\theorems\output\omt_down_theorems_2.p9.out IN 0.04 ON 2012-8-27 18:55:00, PROOF LENGTH=9
    FAILED: omt\theorems\output\omt_down_theorems_2.m4.out IN 1.93 ON 2012-8-27 18:55:02
omt\theorems\output\omt_down_theorems_3 tested with: Mace4, Prover9
  SUCCESS: omt\theorems\output\omt_down_theorems_3.p9.out IN 0.03 ON 2012-8-27 18:55:06, PROOF LENGTH=9
    FAILED: omt\theorems\output\omt_down_theorems_3.m4.out IN 1.94 ON 2012-8-27 18:55:08
omt\theorems\output\omt_down_theorems_4 tested with: Mace4, Prover9
  SUCCESS: omt\theorems\output\omt_down_theorems_4.p9.out IN 0.04 ON 2012-8-27 18:55:12, PROOF LENGTH=9
    FAILED: omt\theorems\output\omt_down_theorems_4.m4.out IN 1.94 ON 2012-8-27 18:55:14
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omtb

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omtb\consistency\output\omtb_down_plp_lin_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: omtb\consistency\output\omtb_down_plp_lin_nontrivial.p9.out IN 600.02 ON 2012-11-4 12:58:23
    FAILED: omtb\consistency\output\omtb_down_plp_lin_nontrivial.m4.out IN 98.97 ON 2012-11-4 12:49:58
  SUCCESS: omtb\consistency\output\omtb_down_plp_lin_nontrivial.tptp.out, MODEL SIZE=10
omtb\output\omtb_down tested with: Mace4, Prover9
    FAILED: omtb\output\omtb_down.p9.out IN 1.86 ON 2012-8-27 20:00:36
  SUCCESS: omtb\output\omtb_down.m4.out IN 0.02 ON 2012-8-27 20:00:34, MODEL SIZE=2
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space

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space\consistency\output\space_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: space\consistency\output\space_nontrivial.p9.out IN 600.04 ON 2012-9-3 14:23:39
    FAILED: space\consistency\output\space_nontrivial.m4.out IN 600.74 ON 2012-9-3 14:23:38
  SUCCESS: space\consistency\output\space_nontrivial.tptp.out, MODEL SIZE=9
space\consistency\output\spch_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: space\consistency\output\spch_nontrivial.p9.out IN 600.00 ON 2012-11-2 17:17:27
    FAILED: space\consistency\output\spch_nontrivial.m4.out IN 505.54 ON 2012-11-2 17:15:48
  SUCCESS: space\consistency\output\spch_nontrivial.tptp.out, MODEL SIZE=8
space\consistency\output\voids_nontrivial tested with: Mace4, Prover9, Paradox3
    FAILED: space\consistency\output\voids_nontrivial.p9.out IN 600.01 ON 2012-11-2 17:30:04
    FAILED: space\consistency\output\voids_nontrivial.m4.out IN 367.27 ON 2012-11-2 17:26:08
  SUCCESS: space\consistency\output\voids_nontrivial.tptp.out, MODEL SIZE=9

space\output\space tested with: Mace4, Prover9
    FAILED: space\output\space.p9.out IN 1.91 ON 2012-9-3 14:12:27
  SUCCESS: space\output\space.m4.out IN 0.02 ON 2012-9-3 14:12:25, MODEL SIZE=2
space\output\spch tested with: Mace4, Prover9
  UNKNOWN: space\output\spch
    FAILED: space\output\spch.p9.out IN 600.00 ON 2012-9-3 23:21:37
    FAILED: space\output\spch.m4.out IN 600.61 ON 2012-9-3 23:21:34
space\output\voids tested with: Mace4, Prover9
  UNKNOWN: space\output\voids
    FAILED: space\output\voids.p9.out IN 600.00 ON 2012-9-4 16:06:16
    FAILED: space\output\voids.m4.out
space\output\voids_extended tested with: Mace4, Prover9
  UNKNOWN: space\output\voids_extended
    FAILED: space\output\voids_extended.p9.out IN 600.01 ON 2012-11-2 16:53:46
    FAILED: space\output\voids_extended.m4.out IN 600.20 ON 2012-11-2 16:53:44
space\theorems\output\ped_theorems tested with: Mace4, Prover9
    FAILED: space\theorems\output\ped_theorems.p9.out IN 0.01 ON 2012-10-23 18:43:46
  SUCCESS: space\theorems\output\ped_theorems.m4.out IN 0.01 ON 2012-10-23 18:43:46, MODEL SIZE=2
space\theorems\output\ped_theorems_1 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\ped_theorems_1.p9.out IN 0.01 ON 2012-10-23 18:43:51, PROOF LENGTH=16
    FAILED: space\theorems\output\ped_theorems_1.m4.out IN 0.07 ON 2012-10-23 18:43:51
space\theorems\output\ped_theorems_2 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\ped_theorems_2.p9.out IN 0.01 ON 2012-10-23 18:43:55, PROOF LENGTH=17
    FAILED: space\theorems\output\ped_theorems_2.m4.out IN 0.06 ON 2012-10-23 18:43:55
space\theorems\output\space_theorems tested with: Mace4, Prover9
    FAILED: space\theorems\output\space_theorems.p9.out IN 1.92 ON 2012-9-4 01:08:13
  SUCCESS: space\theorems\output\space_theorems.m4.out IN 0.02 ON 2012-9-4 01:08:11, MODEL SIZE=2
space\theorems\output\space_theorems_1 tested with: Mace4, Prover9
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  SUCCESS: space\theorems\output\space_theorems_1.p9.out IN 0.19 ON 2012−9−4 01:08:20, PROOF LENGTH=8
    FAILED: space\theorems\output\space_theorems_1.m4.out IN 1.79 ON 2012−9−4 01:08:21
space\theorems\output\space_theorems_2 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\space_theorems_2.p9.out IN 0.31 ON 2012−9−4 01:08:26, PROOF LENGTH=46
    FAILED: space\theorems\output\space_theorems_2.m4.out IN 1.95 ON 2012−9−4 01:08:27
space\theorems\output\spch_theorems tested with: Mace4, Prover9
  UNKNOWN: space\theorems\output\spch_theorems
    FAILED: space\theorems\output\spch_theorems.p9.out IN 600.00 ON 2012−11−2 17:16:30
    FAILED: space\theorems\output\spch_theorems.m4.out IN 600.26 ON 2012−11−2 17:16:28
space\theorems\output\spch_theorems_1 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\spch_theorems_1.p9.out IN 0.06 ON 2012−11−2 17:16:34, PROOF LENGTH=5
    FAILED: space\theorems\output\spch_theorems_1.m4.out IN 1.95 ON 2012−11−2 17:16:36
space\theorems\output\spch_theorems_2 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\spch_theorems_2.p9.out IN 7.62 ON 2012−11−2 17:16:48, PROOF LENGTH=73
    FAILED: space\theorems\output\spch_theorems_2.m4.out IN 7.77 ON 2012−11−2 17:16:48
space\theorems\output\spch_theorems_3 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\spch_theorems_3.p9.out IN 0.13 ON 2012−11−2 17:16:52, PROOF LENGTH=6
    FAILED: space\theorems\output\spch_theorems_3.m4.out IN 1.97 ON 2012−11−2 17:16:54
space\theorems\output\spch_theorems_4 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\spch_theorems_4.p9.out IN 0.07 ON 2012−11−2 17:16:58, PROOF LENGTH=5
    FAILED: space\theorems\output\spch_theorems_4.m4.out IN 1.96 ON 2012−11−2 17:17:00
space\theorems\output\spch_theorems_5 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\spch_theorems_5.p9.out IN 0.40 ON 2012−11−2 17:17:04, PROOF LENGTH=32
    FAILED: space\theorems\output\spch_theorems_5.m4.out IN 1.94 ON 2012−11−2 17:17:06
space\theorems\output\spch_theorems_6 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\spch_theorems_6.p9.out IN 2.28 ON 2012−11−2 17:17:12, PROOF LENGTH=55
    FAILED: space\theorems\output\spch_theorems_6.m4.out IN 3.90 ON 2012−11−2 17:17:14
space\theorems\output\voids_extended_theorems tested with: Mace4, Prover9
  UNKNOWN: space\theorems\output\voids_extended_theorems
    FAILED: space\theorems\output\voids_extended_theorems.p9.out IN 600.00 ON 2012−11−2 18:08:16
    FAILED: space\theorems\output\voids_extended_theorems.m4.out IN 600.21 ON 2012−11−2 18:08:15
space\theorems\output\voids_extended_theorems_1 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_1.p9.out IN 3.15 ON 2012−11−2 18:08:25, PROOF LENGTH=34
    FAILED: space\theorems\output\voids_extended_theorems_1.m4.out IN 3.81 ON 2012−11−2 18:08:26
space\theorems\output\voids_extended_theorems_2 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_2.p9.out IN 0.31 ON 2012−11−2 18:08:30, PROOF LENGTH=20
    FAILED: space\theorems\output\voids_extended_theorems_2.m4.out IN 1.91 ON 2012−11−2 18:08:32
space\theorems\output\voids_extended_theorems_3 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_3.p9.out IN 0.31 ON 2012−11−2 18:08:36, PROOF LENGTH=20
    FAILED: space\theorems\output\voids_extended_theorems_3.m4.out IN 1.95 ON 2012−11−2 18:08:38
space\theorems\output\voids_extended_theorems_4 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_4.p9.out IN 0.25 ON 2012−11−2 18:08:42, PROOF LENGTH=10
    FAILED: space\theorems\output\voids_extended_theorems_4.m4.out IN 1.91 ON 2012−11−2 18:08:44
space\theorems\output\voids_extended_theorems_5 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_5.p9.out IN 0.30 ON 2012−11−2 18:08:48, PROOF LENGTH=10
    FAILED: space\theorems\output\voids_extended_theorems_5.m4.out IN 1.94 ON 2012−11−2 18:08:50
space\theorems\output\voids_extended_theorems_6 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_6.p9.out IN 0.22 ON 2012−11−2 18:08:54, PROOF LENGTH=10
    FAILED: space\theorems\output\voids_extended_theorems_6.m4.out IN 1.94 ON 2012−11−2 18:08:56
space\theorems\output\voids_extended_theorems_7 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_7.p9.out IN 0.51 ON 2012−11−2 18:09:01, PROOF LENGTH=14
    FAILED: space\theorems\output\voids_extended_theorems_7.m4.out IN 1.91 ON 2012−11−2 18:09:02
space\theorems\output\voids_extended_theorems_8 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_8.p9.out IN 0.51 ON 2012−11−2 18:09:07, PROOF LENGTH=30
    FAILED: space\theorems\output\voids_extended_theorems_8.m4.out IN 1.93 ON 2012−11−2 18:09:08
space\theorems\output\voids_extended_theorems_9 tested with: Mace4, Prover9
  SUCCESS: space\theorems\output\voids_extended_theorems_9.p9.out IN 0.33 ON 2012−11−2 18:09:12, PROOF LENGTH=16
    FAILED: space\theorems\output\voids_extended_theorems_9.m4.out IN 1.94 ON 2012−11−2 18:09:14
space\theorems\output\voids_theorems tested with: Mace4, Prover9, Vampire
  UNKNOWN: space\theorems\output\voids_theorems
    FAILED: space\theorems\output\voids_theorems.p9.out IN 600.00 ON 2012−11−2 16:54:28
    FAILED: space\theorems\output\voids_theorems.vam.out IN 599.624
    FAILED: space\theorems\output\voids_theorems.m4.out IN 600.26 ON 2012−11−2 16:54:25
space\theorems\output\voids_theorems_1 tested with: Mace4, Prover9, Paradox3, Vampire
    FAILED: space\theorems\output\voids_theorems_1.p9.out IN 600.03 ON 2012−11−2 17:04:40
  SUCCESS: space\theorems\output\voids_theorems_1.vam.out IN 6.571
    FAILED: space\theorems\output\voids_theorems_1.m4.out IN 600.26 ON 2012−11−2 17:04:40
    FAILED: space\theorems\output\voids_theorems_1.tptp.out, ATTEMPTED UP TO MODEL SIZE=24
```

Appendix F

Algebraic translations of the axioms from Chapter 3 for automated theorem proving

F.1 Axioms for automated proofs

Axioms as used for the automated proofs in Prover9. We use $\cdot$ and $+$ denote the lattice operations of meet and join. Universal closure is assumed throughout.

The theories $OCA = \{$L2$^\vee$ – L6$^\vee$, L2$^\wedge$ – L4$^\wedge$, C0 – C3, O1$'$, O2$'$, O3$'\}$ and $SPOCA = OCA \cup \{$PC1, PC2$'$, PC2$''$, S$\}$ axiomatize OCAs and SPOCAs.

Lattice: Standard axioms for commutativity, associativity, and absorption

(L2$^\wedge$) $x \cdot y = y \cdot x$

(L2$^\vee$) $x + y = y + x$

(L3$^\wedge$) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

(L3$^\vee$) $(x + y) + z = x + (y + z)$

(L4$^\wedge$) $x + (x \cdot y) = x$

(L4$^\vee$) $x \cdot (x + y) = x$

Boundedness: Existence of a null and one (universal) element

(L5$^\vee$) $0 + x = x$

(L6$^\vee$) $1 + x = 1$

Orthocomplementation and pseudocomplementation

(O1$'$) $x^{\perp\perp} = x$

(PC1$'$) $x \cdot (x \cdot y)^* = x \cdot y^*$

(O2′) $x + x^{\perp} = 1$

(PC2′) $0^* = 1$

(O3′) $x \cdot y = (x^{\perp} + y^{\perp})^{\perp}$

(PC2″) $1^* = 0$

The Stone identity

(S) $(x + y)^{**} = x^{**} + y^{**}$

Contact: Basic axioms of a weak contact algebra

(C0) $0 \neg C x$

(C1) $x \neq 0 \to x\mathbf{C}x$

(C2) $x\mathbf{C}y \to y\mathbf{C}x$

(C3) $x \cdot y = x \wedge z\mathbf{C}x \to z\mathbf{C}y$

Mereological closures as defined in Section 4.4.1

(M-I) $x \cdot y \neq 0 \to (z \cdot (x \cdot y) = z \leftrightarrow (z \cdot x = z \wedge z \cdot y = z))$

(M-S) $z \cdot (x + y) \neq 0 \leftrightarrow (x \cdot z \neq 0 \vee y \cdot z \neq 0)$

(M-C) $z \cdot x^{\perp} = 0 \leftrightarrow z \cdot x = z$

(O-Ext) $\forall z(x \cdot z = 0 \leftrightarrow y \cdot z = 0) \leftrightarrow x = y$

Topological closures as defined in Section 4.4.2

(T-I) $x \cdot y \neq 0 \to (z\mathbf{C}(x \cdot y) \to (z\mathbf{C}x \wedge w\mathbf{C}y))$

(T-S) $x\mathbf{C}(y + z) \leftrightarrow x\mathbf{C}y \vee x\mathbf{C}z$

(T-S$^{\leftarrow}$) $x\mathbf{C}(y + z) \leftarrow x\mathbf{C}y \vee x\mathbf{C}z$

(C4) $x\mathbf{C}(y + z) \to x\mathbf{C}y \vee x\mathbf{C}z$

(C5) $z \cdot x^{\perp} = z \leftrightarrow z \neg C x$ (≅ T-C)

(C5′) $(x \neq 0 \vee z \neq 1) \wedge (x \neq 1 \vee z \neq 0) \to (z\mathbf{C}x \leftrightarrow (z = x^{\perp} \vee z \cdot x^{\perp} \neq z))$ (≅ T-C′)

(C-Ext) $\forall z(x\mathbf{C}z \leftrightarrow y\mathbf{C}z) \leftrightarrow x = y$

Other axioms of interest

(Atom) $\exists a(a \neq 0 \wedge \forall x(x = 0 \vee x = a \vee x \cdot a \neq x))$

(Con) $x = 0 \vee x = 1 \vee x\mathbf{C}x^{\perp}$

(¬Con) $x \neg C x^{\perp}$

(Dis) $x \neq 1 \to \exists y(y \neq 0 \wedge x \neg C y)$

(Int) $x \neg C y \to \exists z(x \neg C z \wedge y \neg C z^{\perp})$

(¬Triv) $\exists y[y \neq 1 \wedge y \neq 0]$

(Uni) $(x \cdot y = 0 \wedge x + y = 1 \wedge x \cdot z = 0 \wedge x + z = 1) \rightarrow y = z$

F.2 Equivalence of algebraic axioms

Here we show that the axioms from Section 4.4 in a UCMT are equivalent to the algebraic versions thereof, i.e., the axioms used for the automated proofs as shown in Appendix A. We can mainly rely on Theorem 4.2 but have to show additionally that all cases involving the introduced null element 0 are properly covered. We first show the equivalence for the mereological axioms and then for the topological axioms.

F.2.1 Mereological axioms

(M-I$_{\text{UCMT}}$) $\forall w[P(w, x \odot y) \leftrightarrow (P(w,x) \cdot P(w,y))]$ (intersection)

(M-S$_{\text{UCMT}}$) $\forall w[O(w, x \oplus y) \leftrightarrow (O(w,x) + O(w,y))]$ (sum)

(M-C$_{\text{UCMT}}$) $\forall w[O(w, \ominus x) \leftrightarrow \neg P(w,x)]$ (complement)

(M-I) $x \cdot y \neq 0 \rightarrow [\forall z[z \cdot (x \cdot y) = z \leftrightarrow (z \cdot x = z \wedge z \cdot y = z)]]$

(M-S) $z \cdot (x + y) \neq 0 \leftrightarrow (x \cdot z \neq 0 \vee y \cdot z \neq 0)$

(M-C) $z \cdot x^{\perp} = 0 \leftrightarrow z \cdot x = z$

Lemma F.1. *Let UCMT be the theory of* ***UCMT*** *that satisfies P.1 – P.3, C.1 – C.3, UCMT.1 – UCMT.7 with the definitions O-D, U-D, PP-D. Let OCA be the theory of orthocomplemented contact algebras as constructed in Theorem 4.2. Then:*

1. *UCMT* $\models$ *M-I*$_{UCMT}$ *iff OCA* $\models$ *M-I;*
2. *UCMT* $\models$ *M-S*$_{UCMT}$ *iff OCA* $\cup$ *O-Ext*$\models$ *M-S;*
3. *UCMT* $\models$ *M-C*$_{UCMT}$ *iff OCA* $\cup$ *O-Ext*$\models$ *M-C.*

Proof. Let us define $z = g(w)$ throughout.

1. Assume M-I$_{\text{UCMT}}$.
   By definition $P(a,b) \Leftrightarrow a \leq b$ and because of $a \leq b \leftrightarrow a \cdot b = a$ we obtain $z \cdot (x \cdot y) = z$ iff $z \cdot x = z$ and $z \cdot y = z$ for all $x, y, z \neq 0$. If $x = 0$ or $y = 0$, then $x \cdot y = 0$ and M-I holds.
   If $z = 0$, then $z \cdot (x \cdot y) = 0 = z$ and $z \cdot x = 0 = z$ and $z \cdot y = 0 = z$ and thus M-I also holds.
   If M-I then for all $x, y, z \neq 0$ M-I$_{\text{UCMT}}$ follows from $P(a,b) \Leftrightarrow a \cdot b = a$.

2. Assume M-S$_{\text{UCMT}}$.
   Then if O is extensional by O-Ext and by the definition of O we obtain $O(w, x \oplus y) \Leftrightarrow z \cdot (x+y) \neq 0$ and thus also M-S for all $x, y, z \neq 0$. If $z = 0$, then $z \cdot (x + y) = 0$ and $x \cdot z = 0$ and $y \cdot z = 0$ (the same is true if both $x = 0$ and $y = 0$). If only $x = 0$ then $z \cdot (x + y) = z \cdot y$ iff $y \cdot z = 0$ since $x \cdot z = 0$ (the same is true for $y = 0$).
   Reversely, if M-S then for all $x, y, z \neq 0$ M-S$_{\text{UCMT}}$ directly follows.

3. Note that M-C is equivalent to $z \cdot x^{\perp} \neq 0 \leftrightarrow z \cdot x \neq z$.

   Assume M-C$_{\text{UCMT}}$.
   Since we already established that the complementation operator $\ominus$ must at least satisfy the properties of an orthocomplementation, we have $O(w, \ominus x) \Rightarrow z \cdot x^{\perp} \neq 0$ in the presence of O-Ext and $\neg P(w,x) \Rightarrow z \cdot x \neq z$ which covers all cases of M-C in which $x, z \notin \{0,1\}$. The remaining cases of M-C are:

   (i) If $z = 0$, then $z \cdot x^{\perp} = 0$ and $z \cdot x = 0 = z$.

   (ii) If $z = 1$ then $1 \cdot x^{\perp} = x^{\perp} \neq 0$ unless $x = 1$ and $z \cdot x = x \neq 1$ unless $x = 1$. The case when $x = 1$ is covered by (4).

   (iii) If $x = 0$ then $z \cdot 0^{\perp} = z \neq 0$ unless $z = 0$ and $z \cdot x = 0 \neq 0$ unless $z = 0$. The case $z = 0$ has already been covered by (1).

   (iv) If $x = 1$ then $z \cdot 1^{\perp} = 0$ and $z \cdot 1 = z$.

   Reversely, if M-C then for all $x, z \neq 0$ and $x \neq 1$ M-C$_{\text{UCMT}}$ directly follows.

□

F.2.2 Topological axioms

(T-I$_{\text{UCMT}}$) $\forall w[C(w, x \odot y) \rightarrow (C(w,x) \cdot C(w,y))]$ (intersection)

(T-S$_{\text{UCMT}}$) $\forall w[C(w, x \oplus y) \leftrightarrow (C(w,x) + C(w,y))]$ (sum)

(T-C$_{\text{UCMT}}$) $\forall w[P(w, \ominus x) \leftrightarrow \neg C(w,x)]$ (complement)

(T-C$'_{\text{UCMT}}$) $\forall w[PP(w, \ominus x) \leftrightarrow \neg C(w,x)]$ (alternative complement)

(T-I) $x \cdot y \neq 0 \rightarrow (z\mathbf{C}(x \cdot y) \rightarrow (z\mathbf{C}x \wedge w\mathbf{C}y))$

(T-S) $z\mathbf{C}(x+y) \leftrightarrow z\mathbf{C}x \vee z\mathbf{C}y$

(C5) $z \cdot x^{\perp} = z \leftrightarrow \neg z\mathbf{C}x$

(C5$'$) $(x \neq 0 \vee z \neq 1) \wedge (x \neq 1 \vee z \neq 0) \rightarrow (z\mathbf{C}x \leftrightarrow (z = x^{\perp} \vee z \cdot x^{\perp} \neq z))$

Lemma F.2. *Let UCMT be the theory of* ***UCMT*** *that satisfies P.1 – P.3, C.1 – C.3, UCMT.1 – UCMT.7 with the definitions O-D, U-D, PP-D. Let OCA be the theory of orthocomplemented contact algebras as constructed in Theorem 4.2 Then:*

1. *UCMT $\models$ T-I$_{UCMT}$ iff OCA $\models$ T-I;*

2. *UCMT $\models$ T-S$_{UCMT}$ iff OCA $\models$ T-S;*

3. *UCMT $\models$ T-C$_{UCMT}$ iff OCA $\models$ C5;*

4. *UCMT $\models$ T-C$'_{UCMT}$ iff OCA $\models$ C5$'$.*

Proof. Notice that we again define $z = g(w)$ throughout.

1. Assume T-I_{UCMT}.
   Then T-I for all $x, y, z \neq 0$. If $z = 0$, then $\forall v[\neg C(w, v)]$ and thus T-I. Otherwise, if $x = 0$ (or $y = 0$), then $x \cdot y = 0$ and thus also T-I.
   Reversely, if T-I then for all $x, y, z \neq 0$ T-I_{UCMT} directly follows.

2. Assume T-S_{UCMT}.
   Then T-S for all $x, y, z \neq 0$. If $z = 0$ then for all v $\neg C(w, v)$ and thus T-S holds (the same is true if $x = 0$ and $y = 0$). If only $x = 0$ then $C(z, y)$ if and only if $C(z, y)$ (the same is true if $y = 0$).
   Reversely, if T-S then for all $x, y, z \neq 0$ T-S_{UCMT} directly follows.

3. Assume T-C_{UCMT}.
   Since $\ominus$ must at least satisfy the properties of an orthocomplementation $P(w, \ominus x) \Leftrightarrow z \cdot x^{\perp} = z$. The remaining cases are:

   (i) If $z = 0$ then $z \cdot x^{\perp} = 0 = z$ and $\neg C(0, x)$.

   (ii) If $z = 1$ then $1 \cdot x^{\perp} = x^{\perp} \neq z$ unless $x = 0$ and $C(w, x)$ unless $x = 0$. The case when $x = 0$ is covered by (3).

   (iii) If $x = 0$ then $z \cdot 0^{\perp} = z$ and $\neg C(w, 0)$.

   (iv) If $x = 1$ then $z \cdot 1^{\perp} = 0 \neq z$ unless $z = 0$ and $C(z, 1)$ unless $z = 0$. The case when $z = 0$ is covered by (1).

   Reversely, if T-C then for all $x, z \neq 0$ and $x \neq 1$ T-C_{UCMT} directly follows.

4. We can rewrite T-C'_{UCMT} as $\forall w[\neg PP(w, \ominus x) \leftrightarrow C(w, x)]$.
   Assume T-C'_{UCMT}.
   Then $\neg PP(w, \ominus x) \Leftrightarrow z = x^{\perp}$ or $z \cdot x^{\perp} \neq z$ since $\ominus$ must at least satisfy the properties of an orthocomplementation. Then for all $x, z \notin \{0, 1\}$, C5$'$ holds. Trivially, C5$'$ holds if $x = 0$ or $z = 1$ or $x = 1$ and $z = 0$. The remaining cases are:

   (i) If $z = 0$ then $\neg C(0, x)$ and $0 \neq x^{\perp}$ unless $x = 1$ and $0 \cdot x^{\perp} = 0 = z$. The case $x = 1$ is covered by the precondition of C5$'$.

   (ii) If $x = 1$ then $C(z, 1)$ unless $z = 0$ and $z \neq 1^{\perp}$ unless $z = 0$ and $z \cdot 1^{\perp} = 0 \neq z$ unless $z = 0$. The case $z = 0$ is covered by the precondition of C5$'$.

   Reversely, if T-C$'$ then for all $x, z \neq 0$ and $x \neq 1$ T-C'_{UCMT} directly follows.

□

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