CSC 411: Lecture 03: Linear Classification

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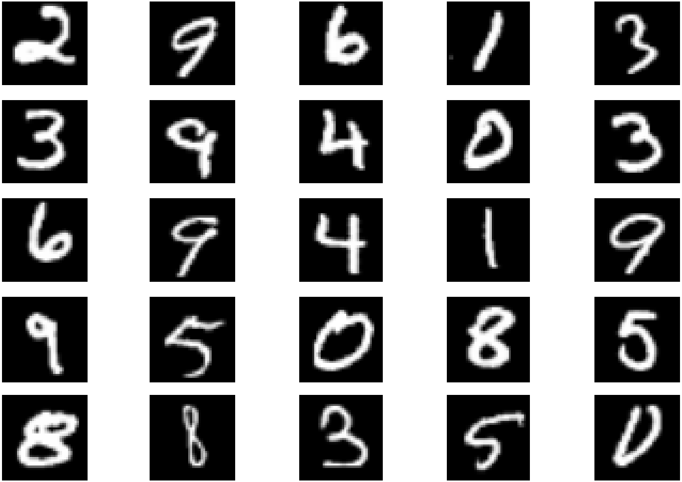
Today

- Linear Classification
- Key Concepts:
  - Classification as regression
  - Decision boundary
  - Loss functions
  - Metrics to evaluate classification

Classification vs Regression

- We are interested in mapping the input $\mathbf{x} \in \mathcal{X}$ to a *label* $t \in \mathcal{Y}$
- In regression typically $\mathcal{Y} = \Re$
- Now its a category
- Examples?

Examples of Classification



What digit is this?

How can I predict this? What are my input features?

Classification as Regression

- Can we do this task using what we have learned in previous lectures?
- Simple hack: Ignore that the input is categorical!
- Suppose we have a binary problem, $t \in \{-1, 1\}$
- Assuming the standard model used for regression

$$y = f(\mathbf{x}, \mathbf{w}) = \mathbf{w}^T \mathbf{x}$$

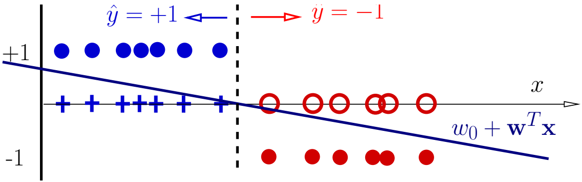
- How can we obtain $\mathbf{w}$?
- Use least squares, $\mathbf{w} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{t}$. How is $\mathbf{X}$ computed? and $\mathbf{t}$?
- Which loss are we minimizing? Does it make sense?

$$\ell_{square}(\mathbf{w}, t) = \frac{1}{N} \sum_{n=1}^{N} (t^{(n)} - \mathbf{w}^T \mathbf{x}^{(n)})^2$$

- How do I compute a label for a new example? Let's see an example

Classification as Regression

Decision Rules



- Our classifier has the form

$$f(\mathbf{x}, \mathbf{w}) = w_o + \mathbf{w}^T\mathbf{x}$$

- A reasonable decision rule is

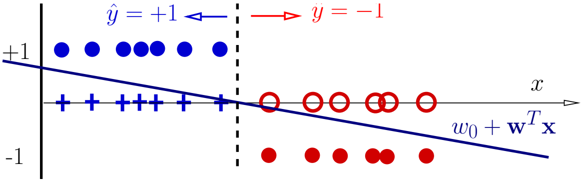
$$y = \begin{cases} 1 & \text{if } f(\mathbf{x}, \mathbf{w}) \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

- How can I mathematically write this rule?

$$y = \text{sign}(w_0 + \mathbf{w}^T\mathbf{x})$$

- How does this function look like?

Decision Rules



- How can I mathematically write this rule?

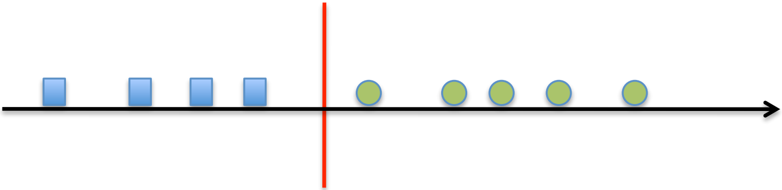
$$y = \text{sign}(w_0 + \mathbf{w}^T\mathbf{x})$$

- This specifies a linear classifier: it has a linear boundary (hyperplane)

$$w_0 + \mathbf{w}^T\mathbf{x} = 0$$

which separates the space into two "half-spaces"

Example in 1D



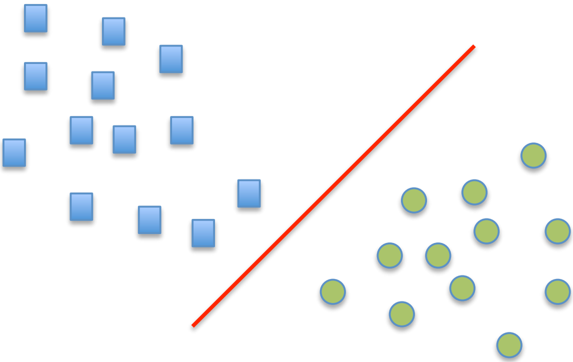
- The linear classifier has a linear boundary (hyperplane)

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

  which separates the space into two "half-spaces"

- In 1D this is simply a threshold

Example in 2D



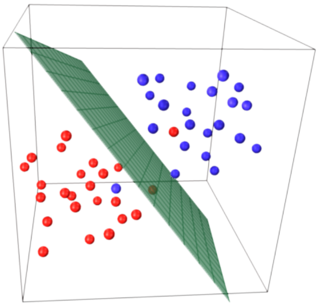
- The linear classifier has a linear boundary (hyperplane)

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

  which separates the space into two "half-spaces"

- In 2D this is a line

Example in 3D



- The linear classifier has a linear boundary (hyperplane)

$$w_0 + \mathbf{w}^T \mathbf{x} = 0$$

  which separates the space into two "half-spaces"

- In 3D this is a plane
- What about higher-dimensional spaces?

Geometry

$\mathbf{w}^T\mathbf{x} = 0$ a line passing though the origin and orthogonal to $\mathbf{w}$
$\mathbf{w}^T\mathbf{x} + w_0 = 0$ shifts it by w_0

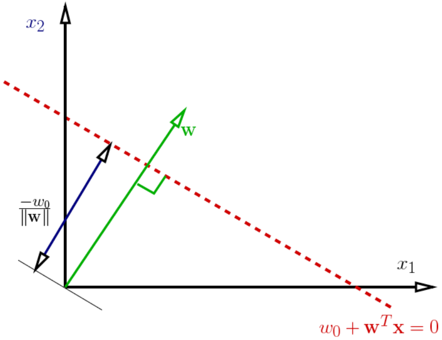
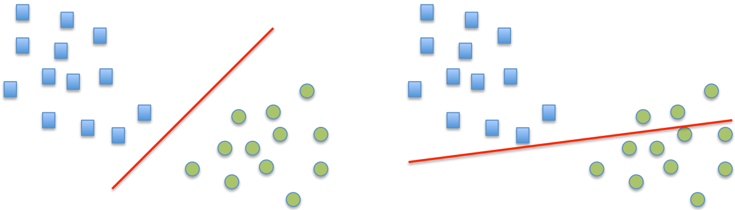


Figure from G. Shakhnarovich

Learning Linear Classifiers

- Learning consists in estimating a "good" decision boundary
- We need to find $\mathbf{w}$ (direction) and w_0 (location) of the boundary
- What does "good" mean?
- Is this boundary good?



- We need a criteria that tell us how to select the parameters
- Do you know any?

Loss functions

- Classifying using a linear decision boundary reduces the data dimension to 1

$$y(\mathbf{x}) = \text{sign}(w_0 + \mathbf{w}^T \mathbf{x})$$

- What is the cost of being wrong?
- Loss function: $L(y, t)$ is the loss incurred for predicting y when correct answer is t
- For medical diagnosis: For a diabetes screening test is it better to have false positives or false negatives?
- For movie ratings: The "truth" is that Alice thinks E.T. is worthy of a 4. How bad is it to predict a 5? How about a 2?

Loss functions

- A possible loss to minimize is the zero/one loss

$$L(y(\mathbf{x}), t) = \begin{cases} 0 & \text{if } y(\mathbf{x}) = t \\ 1 & \text{if } y(\mathbf{x}) \neq t \end{cases}$$

- Is this minimization easy to do? why?

Other Loss functions

- Zero/one loss for a classifier

$$L_{0-1}(y(\mathbf{x}), t) = \begin{cases} 0 & \text{if } y(\mathbf{x}) = t \\ 1 & \text{if } y(\mathbf{x}) \neq t \end{cases}$$

- Asymmetric Binary Loss

$$L_{ABL}(y(\mathbf{x}), t) = \begin{cases} \alpha & \text{if } y(\mathbf{x}) = 1 \wedge t = 0 \\ \beta & \text{if } y(\mathbf{x}) = 0 \wedge t = 1 \\ 0 & \text{if } y(\mathbf{x}) = t \end{cases}$$

- Squared (quadratic) loss

$$L_{squared}(y(\mathbf{x}), t) = (t - y(\mathbf{x}))^2$$

- Absolute Error

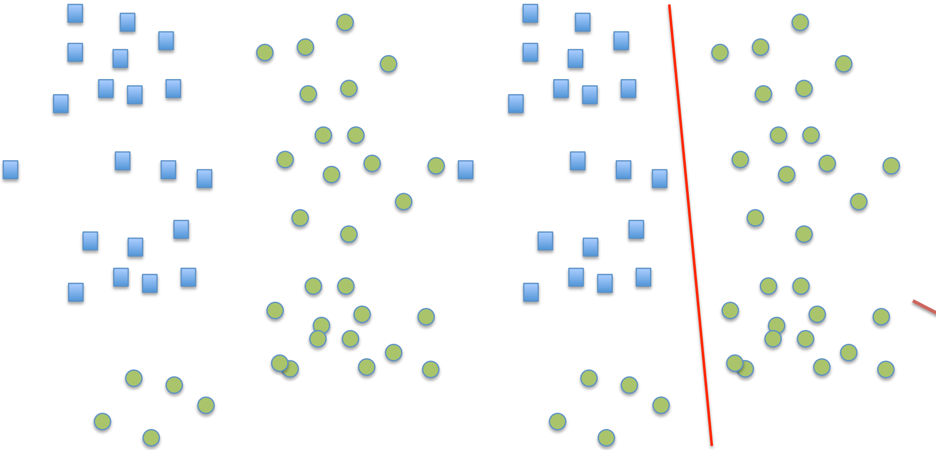
$$L_{quadratic}(y(\mathbf{x}), t) = |t - y(\mathbf{x})|$$

More complex Loss Functions

- What if the movie predictions are used for rankings? Now the predicted ratings don't matter, just the order that they imply.
- In what order does Alice prefer E.T., Amelie and Titanic?
- Possibilities:
  - 0-1 loss on the winner
  - Permutation distance
  - Accuracy of top K movies.

Can we always separate the classes?

- If we can separate the classes, the problem is linearly separable



Can we always separate the classes?

Causes of non perfect separation:

- Model is too simple
- Noise in the inputs (i.e., data attributes)
- Simple features that do not account for all variations
- Errors in data targets (miss labelings)

Should we make the model complex enough to have perfect separation in the training data?

Metrics

How to evaluate how good my classifier is?

- Precision: is the fraction of retrieved instances that are relevant

$$P = \frac{TP}{TP + FP}$$

- Recall: is the fraction of relevant instances that are retrieved

$$R = \frac{TP}{TP + FN} = \frac{TP}{P}$$

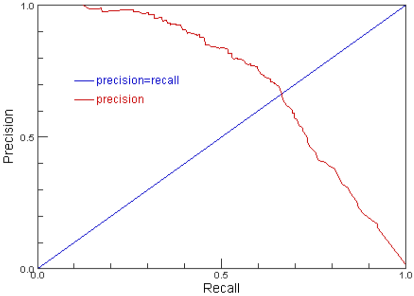
- F1 score: harmonic mean of precision and recall

$$F1 = 2\frac{P \cdot R}{P + R}$$

More on Metrics

How to evaluate how good my classifier is?

- Precision: is the fraction of retrieved instances that are relevant
- Recall: is the fraction of relevant instances that are retrieved
- Precision Recall Curve



- Average Precision (AP): mean under the curve