Inapproximability of Treewidth, Graph-layout and One-shot Pebbling

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Problems we consider

- One-shot Pebbling game:
  Pebbling game on a DAG where each node can only be pebbled once.

- Graph layout problems:
  Minimum Linear Arrangement, Interval Graph Completion, etc.

- Width parameters: treewidth, pathwidth.

Our Contribution:

- Treat those problems in a unified way.
- Prove that assuming the Small Set Expansion (SSE) Conjecture, the above problems are hard to approximate within any constant factor.
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One-shot Pebbling Game

Given a DAG, pebble the sink node according to the following rules, while minimize \( \# \) pebbles used.

- A pebbled can be placed on any source nodes.
- A pebble can be placed on a vertex \( v \) if all of the immediate predecessors of \( v \) are pebbled.
- A pebble can be removed from a vertex.

Additional rule: each vertex can be pebbled only once.
Graph Layout Problems

Given a graph $G = (V, E)$ $V = \{1, \ldots, n\}$, find a permutation $\pi$ on $V$. Define $\text{length}(u, v) = |\pi(u) - \pi(v)|$.

Minimum Linear Arrangement:

$$\min \sum_{(u, v) \in E} \text{length}(u, v)$$
Graph Layout Problems

Minimum Cut Linear Arrangement:

$$Cut_i(\pi) = \{ e \in E | \pi(u) \leq i < \pi(v), e = (u, v) \}$$

Want:

$$\min_{\pi} \max_{i \in [n]} |Cut_i(\pi)|$$

8 variations:

- Undirected / directed acyclic
- Counting edges or vertices at each cut
- Aggregation by sum or max
Graph Layout Problems

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Want:

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\min_{\pi} \max_{i \in [n]} |\text{Cut}_i(\pi)|
\]

8 variations:

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**Width-parameters: treewidth, pathwidth**

\((T, \mathcal{V})\) is called a tree decomposition of \(G\) if:

1. \((T1)\) \(\mathcal{V} = \bigcup_{t \in T} V_t\);
2. \((T2)\) \(\forall e \in E, \exists t \in T\), s.t. both endpoints of \(e\) lie in \(V_t\);
3. \((T3)\) for every \(v \in \mathcal{V}\), \(\{t \in T \mid v \in V_t\}\) is a subtree of \(T\).

\[
Treewidth(G) = \min_T \max T |V_t| - 1.
\]

Cited from Wikipedia
A complete list of problems covered

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<th>Problem</th>
<th>Also known as / Equivalent with</th>
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Complexity Status

- All of the problems are \textbf{NP}-Complete (decision version).

- Approximation Algorithms known (optimization version) best known approximation ratio $\text{poly}(\log n)$.

- Very few hardness of approximation results.
Known approximation algorithms

- $O(\sqrt{\log n})$: Treewidth [Feige, Hajiaghayi and Lee 05’]
- $O(\sqrt{\log n \log \log n})$: Minimum Linear Arrangement, Interval Graph Completion, Directed MLA. [Charikar, Hajiaghayi, Karloff and Rao 10’]
- $O(\log n \sqrt{\log n})$: MCLA, Vertex Separation (Pathwidth) [Feige, Hajiaghayi and Lee 05’], Register Sufficiency.
Known hardness result

- No constant approximation for MLA assuming UGC with an additional condition (which is equivalent to SSE) [Devanur, Khot, and Saket 06’, Raghavendra, Steurer and Tulsiani 10’].

- No PTAS for MLA, unless $\mathbf{NP} \subseteq$ randomized subexponential time. [Ambuhl, Mastrolilli and Svensson 07’]

- $\mathbf{NP}$-hard to find a tree-decomposion with width $\text{Treewidth}(G) + n^\epsilon$ for some constant $0 < \epsilon < 1$. [Bodlaender, Gilbert, Hafsteinsson and Kloks 95’]
**Unique Games Conjecture**

**Definition**

$Y = (V, E, \Pi, \{R\})$ consists of a graph $G = (V, E)$, a set of labels $\{R\} = \{1, \ldots, R\}$ and a set of permutations $\pi_{v \leftarrow w} : [R] \rightarrow [R]$ for each edge $e = (w, v) \in E$.

An assignment $F : V \rightarrow [R]$ of labels to vertices is said to satisfy an edge $e = (w, v)$, if $\pi_{v \leftarrow w}(F(w)) = F(v)$. 

![Diagram](image)
Unique Games Conjecture

Problem (Unique Games \((R, 1 - \epsilon, \eta)\)).

\(Y = (V, E, \Pi = \{\pi_{v \leftarrow w} : [R] \rightarrow [R] | e = (w, v) \in E\}, [R])\),

distinguish between the following:

- **Yes** There exists a labeling assignment satisfies a fraction of \((1 - \epsilon)\) edges.
- **No** No labeling assignment satisfies a fraction of \(\eta\) edges.

Conjecture (Unique Games Conjecture (Khot 02))

*For all constants \(\epsilon, \eta > 0\), there exists large enough constant \(R\) such that Unique Games \((R, 1 - \epsilon, \eta)\) is \textbf{NP}-hard.*
Some facts about UGC

- Implies that approximating Vertex Cover within a factor of $2 - \epsilon$ is hard. [Khot and Regev 08]

- Implies that SDP approximation algorithms for several problems, such as MAX-CUT, are optimal. [Khot, Kindler, Mossel and O’Donnell 07]

- Proving / Disproving UGC is a hot topic in complexity.
Small Set Expansion (SSE) Conjecture

$G = (V, E)$: undirected, $d$-regular graph

$\Phi_G(S)$: the (normalized) edge expansion of $S$, for $S \subseteq V$

$$\Phi_G(S) = \frac{|E(S, \bar{S})|}{d|S|}$$

The SSE Problem asks if $G$ has a small set $S$ which does not expand or whether all small sets are highly expanding. [Raghavendra and Steurer 10']
The Small Set Expansion Conjecture

- $G = (V, E)$, $\text{SSE}(\eta, \delta)$ is the problem of distinguishing:
  - Yes: There is an $S \subseteq V$ with $|S| = \delta|V|$ and $\Phi_G(S) \leq \eta$.
  - No: For every $S \subseteq V$ with $|S| = \delta|V|$ it holds that $\Phi_G(S) \geq 1 - \eta$.

- Small Set Expansion Conjecture
  For every $\eta > 0$, there is a $\delta > 0$ such that $\text{SSE}(\eta, \delta)$ is \textbf{NP}-hard.
Some facts about SSE

- SSE-hardness implies UGC-hardness.

- It is equivalent to UGC with an additional condition on the expansion of the graph. [Raghavendra, Steurer and Tulsiani 10’]

- Can be solved in subexponential time (also true for Unique Games). [Arora, Barak and Steurer 11’], [Barak, Raghavendra and Steurer 11’]
Strong version of the SSE conjecture

SSE Conjecture (strong form)
For every integer $q > 0$ and $\epsilon, \gamma > 0$, it is NP-hard to distinguish between the following two cases for a given regular graph $G = (V, E)$

Yes There is a partition of $V$ into $q$ equi-sized sets $S_1, \ldots, S_q$ such that $\Phi_G(S_i) \leq 2\epsilon$ for every $1 \leq i \leq q$.

No For $S \subseteq V$, $|V|/10 \leq |S| \leq 9|V|/10$, we have $\Phi_G(S) \geq c\sqrt{\epsilon}$
Minimum Cut Linear Arrangement:

\[ \text{Cut}_i(\pi) = \{ e \in E | \pi(u) \leq i < \pi(v), e = (u, v) \} \]

Want:

\[ \min_{\pi} \max_{i \in [n]} |\text{Cut}_i(\pi)| \]

**Theorem**

*MCLA (Layout[undirected, edge, max]) is hard to approximate within any constant factor.*
MCLA is hard to approximate (1)

YES case: let \( q = 1/\epsilon \)

FACT: \( |E(S, \bar{S})| = \Phi_G(S)d|S| \)

\[
\begin{align*}
|\text{edges inside each set}| & \leq d|S_i| = dn/q = 2\epsilon |E| \\
|\text{edges that expand}| & \leq n|E(S_i, \bar{S}_i)| = n \cdot \Phi_G(S_i)d|S_i| \leq 2\epsilon |E| \\
MCLA(G) & \leq 4\epsilon |E|
\end{align*}
\]
MCLA is hard to approximate (1)

YES case: let $q = 1/\epsilon$

FACT: $|E(S, \bar{S})| = \Phi_G(S)d|S|$

- $|\#\text{edges inside each set}| \leq d|S_i| = dn/q = 2\epsilon|E|$
- $|\#\text{edges that expand}| \leq n|E(S_i, \bar{S}_i)| = n \cdot \Phi_G(S_i)d|S_i| \leq 2\epsilon|E|$

$MCLA(G) \leq 4\epsilon|E|$
MCLA is hard to approximate (2)

No case: let $S$ = the first half vertices according to a given permutation.

\[ |\text{#edges between the first half and the second half}| \geq c\sqrt{\epsilon}|E| \]

\[ \text{MCLA}(G) \geq c\sqrt{\epsilon}|E| \]
Hardness result of layout problems

- What we showed: For a given graph $G$, $MCLA(G)$ ($\text{Layout[undirected, edge, max]}$) is hard to approximate within any constant factor.

- Can also get the same hardness for MLA ($\text{Layout[undirected, edge, sum]}$) (also proved by [Devanur, Khot, and Saket 06’, Raghavendra, Steurer and Tulsiani 10’]).

- For the rest of the layout problems, we need:
  1. Reduction from undirected version to directed version.
  2. Reduction from edge version to vertex version.
Hardness result of layout problems

- **What we showed:** For a given graph $G$, $\text{MCLA}(G)$ ($\text{Layout[undirected, edge, max]}$) is hard to approximate within any constant factor.

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- For the rest of the layout problems, we need:
  1. Reduction from undirected version to directed version.
  2. Reduction from edge version to vertex version.
Reduction from undirected to directed version

Given $G = (V, E)$, produce a directed graph $D = (V', E')$.

$$V' = V \cup E$$
$$E' = \{(e, v) \mid e \in E, v \in V, v \in e\}.$$
Reduction from edge to vertex version

Given $G = (V, E)$, produce $G' = (V', E')$, where

$$V' = \{v^i | v \in V, i \in [r]\}$$

$$E' = \{(e, v^i) | e \in E, v \in V, v \in e, i \in [r]\}.$$
Inapproximability of Treewidth

Cited from Wikipedia

\( V = \bigcup_{t \in T} V_t; \)

\( \forall e \in E, \exists t \in T, \text{s.t.} \) both endpoints of \( e \) lie in \( V_t; \)

\( \forall v \in V, \) \( \{ t \in T \mid v \in V_t \} \) is a subtree of \( T. \)

**Theorem**

It’s SSE-hard to approximate \( \text{Treewidth}(G) \) within any constant factor.
Inapproximability for Treewidth

**Proof:** Use the following facts.

- $\text{Treewidth}(G) \leq \text{Pathwidth}(G)$.
- $\text{Pathwidth}(G) = \text{Layout}[\text{undirected}, \text{vertex}, \text{max}]$.

[Kinnersley 92]
Inapproximability of Treewidth

In the Yes case, we use the fact that
\[ \text{Treewidth}(G') \leq \text{Pathwidth}(G') \leq O(\epsilon)|E|. \]
In the No case, we show that \( \text{Treewidth}(G') \geq c\sqrt{\epsilon}|E| \).

\[ (u, v) \quad (u, w) \]
\[ v \quad w \]

\[ G \]

\[ a^1, \ldots, a^r \]
\[ b^1, \ldots, b^r \]
\[ c^1, \ldots, c^r \]

\[ G' \]
Conclusion and Discussion

Main Results

- SSE-hard to approximate graph layout problems within any constant factor.
- SSE-hard to approximate treewidth within any constant factor.

Open Problems

- Weaker Assumptions such as UGC?
- Would hardness of our problems imply hardness for the SSE problem?
- Approximability of the original pebbling problem?
Thank you all!