Midterm 1

CSC 236 H1Y (LEC5101)

19 June 2024

Question 1. [10 marks] Multiple choice, true/false and short answer

Multiple choice and true/false. For multiple choice questions check *all* correct answer(s). You must pick *exactly* the correct answer(s) to get the point (you may leave it blank for no marks). For true/false, circle one of True or False to select it (you may leave it blank for no marks). Each question in this subsection will be worth 1 mark for a correct answer and -1 *mark* for an incorrect answer. You will get no more than five and no fewer than zero marks. *Do not guess.*

Part (a) [1 mark]

Complete induction is stronger than the well-ordering principle, i.e., there exist predicates that can be proved using complete induction which *cannot* be proved using the well-ordering principle.

True / **False**

Solution. False, they are of the same strength (this was a quiz question).

Part (b) [1 mark]

Let $P(z)$ be a predicate on integers $z \in \mathbb{Z}$. Suppose we have the following:

- $P(0)$ holds.
- If $P(i)$ holds then $P(i+1)$ holds for all $i \in \mathbb{N}$.

This constitute a valid proof of $\forall z \in \mathbb{Z}, P(z)$?

True / **False**

Solution. False, $P(z)$ for $z < 0$ would not be accounted for.

Part (c) [1 mark]

Consider the following proof of the claim that for every nonnegative integer n, $5n = 0$.

- ☐ Base Case. When $n = 0$, $5n = 0$ so the claim is true.
- ☐ Ind. Step. Let $k \geq 1$ and suppose that $P(0) \wedge \cdots \wedge P(k)$ is true. We show that $P(k+1)$ is true.
- ☐ Write $k+1 = i+j$ where i and j are integers satsifying $1 \leq i, j \leq k$.
- ☐ By IH, we can write $5i = 0$ and $5j = 0$ so that $5(k+1) = 5i + 5j = 0$.

Thus, by the property of mathematical induction $5n = 0$ for all $n \in \mathbb{N}$.

This argument is clearly wrong. **Put a checkmark next to *every* line which is incorrect by itself *(i.e. assuming all other lines are correct)***. For example, if the last line follows logically from the previous lines, regardless of the correctness of the previous lines, *do not* check it.

Solution. The only issue is line (b). Note that the base case is $k = 0$ but we are letting $k \geq 1$.

Part (d) [1 mark]

Consider the graph G shown in Figure 1. The sequence of edges $(a,d), (d,b), (b,c), (c,a)$ forms a cycle.

True / **False** True.

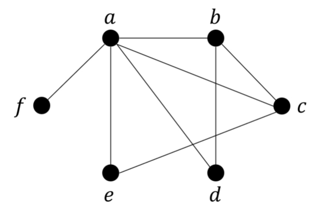

Figure 1: Graph G

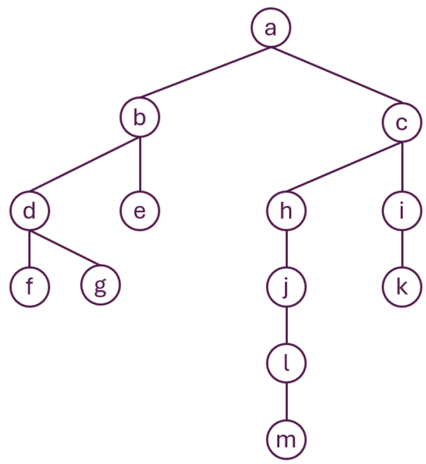

Figure 2: Graph H

Part (e) [1 MARK]

Consider the rooted tree H shown in Figure 2 with root a. The number of children of b equals its height.

True / **False** *Solution.* True.

Short Answer. Each question below is worth one to three points. Write the final answer in closed form (put answer in simplist form; no summations). No justification required.

Part (f) [1 MARK]

How many ways can k different books be distributed among 5 different people?

____

Solution. 5^k.

Part (g) [1 MARK]

How may non-negative integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = k$?

Solution. $\binom{k+4}{4}$.

Part (h) [3 marks]

How many non-negative integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = k$, with the restriction that $x_1 \leq 7$ (you can assume $k \geq 5$)?

Solution. $\binom{k+4}{4} - \binom{k-4}{4}$ where the first term is the count with no restriction (same as above) and the second term counts the number of solutions where $x_1 \geq 8$ (these are the bad cases).

Question 2. [6 marks]

Prove by induction that the following identity is true for all $n \in \mathbb{N}$ where $n \geq 1$

$$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2.$$

Solution. We induct on n.

Base Case. In the base case $n = 1$ and we have $1 \cdot 2^1 = (0) \cdot 2^2 + 2 = 2$.

Induct. Suppose that for $k \geq 1$, $P(k)$ is true. We want to show that $P(k+1)$ is true. Note that

$$\begin{aligned}
\sum_{i=1}^{k+1} i2^i &= \left(\sum_{i=1}^{k} i2^i\right) + (k+1)2^{k+1} \\
&= (k-1)2^{k+1} + 2 + (k+1)2^{k+1} && \text{(IH)} \\
&= k2^{k+2} + 2.
\end{aligned}$$

By the principle of mathematical induction $P(n)$ is true for all $n \in \mathbb{N}$ where $n \geq 1$.

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Question 3. [6 marks] Trees

You will prove the claim that every non-rooted tree with at least two nodes has *at least two* leaves.

Part (a) [2 marks]

Define a predicate to prove this claim which depends on n, the number of nodes in the tree.

Part (b) [4 marks]

Prove your predication by inducting on n.

Solution for all parts. In A2 Q1 (a) it was shown that every tree has at least one leaf.

Predicate. Let $P(n)$ be the predicate that for all graphs $G = (V, E)$ where $|V| = n$, if G is a tree then G has at least two leaves. We prove $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq 2$ by induction on n.

Base Case. When $n = 2$, G must have at least one edge between the two nodes since trees are connected. Further G can have at most one edge between the two nodes since trees are acyclic. It follows that both nodes are leaves.

Ind. Step. Fix $k \geq 2$ and suppose $P(2) \wedge \cdots \wedge P(k)$ is true. We want to show that $P(k+1)$ is true. By the A2 Q1 (a), we know that every tree has at least one leaf. Suppose v is this leaf. Let G' be the graph G with v removed. Note that G' is still a tree but has one fewer node (still at least two nodes since $k \geq 2$). It follows by IH, that G' must have two leaves. When we add v back, G can at most lose one leaf (v's neighbour) and will gain a leaf v.

Thus, by the principle of mathematical induction, $P(n)$ is true for all $n \in \mathbb{N}$, when $n \geq 2$.

Question 4. [12 MARKS] Fibonnaci Numbers

Recall the sequence of Fibonnaci numbers $f_0, f_1, f_2, f_3, f_4, f_5, f_6, \ldots$ we have seen frequently in this class. The first few terms are $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, $f_5 = 5$, $f_6 = 8$ and so on. Generally, $f_{n+2} = f_{n+1} + f_n$.

Part (a) [4 MARKS]

Prove that for every $n \in \mathbb{N}$, f_n is even if and only if n is divisible by three.

Solution.

Predicate: Let $P(n)$ be statement: n is divisible by three if and only if f_n is even. To show that $P(n)$ is true for all $n \in \mathbb{N}$, induct on n.

Base Case: When $n = 0$, f_0 is even. When $n = 1$, f_1 is odd.

Induct: Suppose for $k \geq 1$, and $P(0) \wedge \cdots \wedge P(k)$ is true. We want to show $P(k+1)$ is true.

Either $k+1$ is divisible by three or not. If $k+1$ is divisible by three, then k and $k-1$ cannot be divisible by three. By IH, f_k and f_{k-1} must be odd. Since $f_{k+1} = f_k + f_{k-1}$ is the sum of two odd numbers, it must be even. If $k+1$ is not divisible by three then one of k or $k-1$ must be divisible by three and the other not. By IH, one of f_k or f_{k-1} must be even and the other odd. Since $f_{k+1} = f_k + f_{k-1}$, f_{k+1} must be odd.

Thus, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Part (b) [3 MARKS]

We want to prove that every natural number can be written as the sum of distinct, non-consecutive Fibonacci numbers. That is, if the Fibonacci number f_i appears in the sum, it appears exactly once, and its neighbors f_{i-1} and f_{i+1} do not appear at all. For example $7 = f_3 + f_5$ and $8 = f_6$. Note that $8 = f_5 + f_6$ would *not* be a valid sum as f_5 and f_6 are consecutive.

Write 17, 42, and 54 as the sum of distinct, non-consecutive Fibonacci numbers.

Solution.

$$\begin{aligned} 17 &= f_7 + f_4 + f_2 \\ 42 &= f_9 + f_6 \\ 54 &= f_9 + f_7 + f_5 + f_3 \end{aligned}$$

Part (c) [5 MARKS]

Prove that every natural number can be written as the sum of distinct, non-consecutive Fibonacci numbers.

Solution. This is just *one solution*, others are possible.

Predicate: $P(n)$ be the statement: n can be written as the sum of distinct, non-consecutive Fibonacci numbers. We want to show that $P(n)$ is true for all $n \in \mathbb{N}$.

Base Case: $n = 0$ and $n = 1$ can both be written by a single term in the Fibonacci sequence, namely f_0 and f_1 respectively.

Induct: Suppose for $k \geq 1$, and $P(0) \wedge \cdots \wedge P(k)$ is true. We want to show $P(k+1)$ is true.

Consider the number $k+1$. Either $k+1$ is a Fibonacci number, in which case we are done, or $k+1$ is not a Fibonacci number. In the second case let f_i be the largest Fibonacci number *less than* $k+1$ i.e. $f_i < k+1$ and $f_{i+1} > k+1$. Let $r = k+1-f_i$. By IH, we have that r can be written as a sum of distinct, non-consecutive Fibonacci numbers, say $r = f_{j_1} + \cdots + f_{j_s}$ where $j_1 < \cdots < f_s$. We claim that $j_s \neq i-1$. If this is true then $k+1 = f_{j_1} + \cdots + f_{j_s} + f_i$ would be a sum of distinct, non-consecutive Fibonacci numbers for $k+1$. Suppose for a contradiction that $j_s = i-1$ then $f_{j_s} + f_i = f_{i+1}$ by the definition of Fibonacci number. Note further that $k+1 = f_{i+1} + f_{j_{s-1}} + \cdots + f_{j_1}$ so f_i was not the largest Fibonacci number less than $k+1$ (f_{i+1} clearly also works). This is a contradiction so it must be the case that $j_s \neq i-1$.

Thus, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

Question 5. [12 marks] Structural Induction

Binary strings are sequences of zeros and ones. An *even* binary string is one which contains an even number of ones. Let $\mathcal{E}$ be the set of all even binary strings. An *odd* binary string is one which contains an odd number of ones. Let $\mathcal{D}$ be the set of all odd binary strings.

Examples of strings in $\mathcal{E}$ are $0011, 10000001, 1111$, etc. Examples of strings in $\mathcal{D}$ are $0010, 1, 10101$, etc.

Part (a) [1 mark]

Which set does the empty string ϵ belong to?

Part (b) [5 marks]

Give a recursive definition of $\mathcal{E}$ and a recursive definition for $\mathcal{D}$.

Part (c) [6 marks]

Prove by structural induction that your definitions are correct (e.g. you *must show* that every element constructed by your definitions belongs to $\mathcal{E}$ and $\mathcal{D}$ and every element of $\mathcal{E}$ and $\mathcal{D}$ can be constructed using your definitions).

Solution. This is just *one* possible solution, there are other valid definitions which does not require the two sets to be related to one another.

Proof. The alphabet for both of these sets is $\{0,1\}$. We claim that $\mathcal{E}$ is defined by

- $\epsilon \in \mathcal{E}$,
- For $s \in \mathcal{E}$, $s0 \in \mathcal{E}$.
- For $t \in \mathcal{D}$, $t1 \in \mathcal{E}$.

We claim that $\mathcal{D}$ is defined by

- For $s \in \mathcal{E}$, $s1 \in \mathcal{D}$.
- For $t \in \mathcal{D}$, $t0 \in \mathcal{D}$.

Let $\mathcal{E}'$ be all the elements constructed by our first definition and $\mathcal{D}'$ be the same for the second. We want to show that $\mathcal{E}' = \mathcal{E}$ and $\mathcal{D}' = \mathcal{D}$.

1. [$\mathcal{E}' \subseteq \mathcal{E}$ and $\mathcal{D}' \subseteq \mathcal{D}$] Proof by structural induction with predicate $P(s)$ stated: if $s \in \mathcal{E}'$ then $s \in \mathcal{E}$ and if $s \in \mathcal{D}'$ then $s \in \mathcal{D}$. In the base case we have that $\epsilon \in \mathcal{E}'$ and $\epsilon \in \mathcal{E}$ (since there is an even — none — number of ones). Apply IH to $s \in \mathcal{E}'$ to get that s has an even number of ones. It follows that $s0$ also has an even number of ones (the number of ones did not change) so $s0 \in \mathcal{E}$ and $s1$ has an odd number of ones so $s1 \in \mathcal{D}$. Apply IH to $t \in \mathcal{D}'$ to get that t has an odd number of ones. It follows that $t1$ has an even number of ones as well (adding one to the number of ones) so $t1 \in \mathcal{E}$ and $t0$ has an odd number of ones so $t0 \in \mathcal{D}$.

   Thus, by Structural Induction.

2. [$\mathcal{E} \subseteq \mathcal{E}'$ and $\mathcal{D} \subseteq \mathcal{D}'$] The idea is to use something like the well-ordering principle. First you put the binary string in bijection with the natural numbers by adding a "1" to the beginning and reading the number off in binary e.g. 001 becomes 1001 which is equal to 9 while 1 become 11 which is equal to three. It follows that the binary strings are a subset of the natural numbers. Let $\mathcal{S}$ be the set of $s \in \mathcal{E}$ which is not in $\mathcal{E}'$ and $t \in \mathcal{D}$ which is not in $\mathcal{D}'$. For a contradiction we can assume that $\mathcal{S}$ is non-empty. Then $\mathcal{S}$ has a least

element, w. Note that $w \neq \epsilon$ (which is equivalent to the empty set) since $\epsilon \in \mathcal{E}$. Suppose $w \in \mathcal{E}$ (if $w \in \mathcal{D}$ is very similar). Either $w = s1$ or $w = s0$ for binary string s of length one shorter. In the first case s must have an odd number of ones and so $s \in \mathcal{D}'$. In the second case s must have an even number of ones and so $s \in \mathcal{E}'$. In both cases, the construction of $\mathcal{E}'$ would have contained w.

This is a contradiction so $\mathcal{S}$ must be empty and $\mathcal{E}' = \mathcal{E}$ as well as $\mathcal{D} = \mathcal{D}'$.

$\square$

Bonus. [2 MARKS] Well-ordering Principle

Use the well-ordering principle to prove for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ with $x \neq 1$.

$$1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1} \tag{1}$$

You should only try this problem if you have solved the five previous problems and checked that your solution for them are correct. That would be a better use of your time.

Proof. Let $P(n)$ be the predicate that Equation 1 holds for n and all $x \in \mathbb{R}$ with $x \neq 1$. Further, let $S \subseteq \mathbb{N}$ be the set of natural number for which Equation 1 is *not* true i.e. $\neg P(s)$ for all $s \in \mathcal{S}$. For a contradiction we assume that S is non-empty. By the well-ordering principle, S has a smallest element which we call m. Note that $m \neq 0$ since $1 = \frac{x^{0+1}-1}{x-1}$. Further for all $m' < m$, $P(m')$ must hold since m is the smallest element in S (no smaller element can be in S). It follows that $1 + x + x^2 + \cdots + x^{m-1} = \frac{x^{(m-1)+1}-1}{x-1}$

$$\begin{aligned} 1 + x + x^2 + \cdots + x^{m-1} + x^m &= \left(1 + x + x^2 + \cdots + x^{m-1}\right) + x^m \\ &= \frac{x^{(m-1)+1} - 1}{x - 1} + x^m \\ &= \frac{x^m - 1 + x^m(x-1)}{x - 1} \\ &= \frac{x^{m+1} - 1}{x - 1} \end{aligned}$$

This is a contradiction as $\neg P(m)$ since $m \in S$ but Equation 1 as we shown above. Thus S is empty and $P(n)$ must be true for all $n \in \mathbb{N}$. $\square$