# Assignment 1

## Unmarked Questions

*Remember to comment out this section when submitting your assignment.*

Here are a few simple warm-up problems. Make sure you are able to do them before proceeding to the marked questions. The solutions are provided in the comment blocks (simply move them outside of the comment blocks to read).

**Q1.** Prove that $\sum_{i=0}^{n} i^2 = \frac{(2n+1)n(n+1)}{6}$ for all $n \in \mathbb{N}$.

1. **Predicate.** Let $P(n) \coloneqq \sum_{i=0}^{n} i^2 = \frac{(2n+1)n(n+1)}{6}$. To show that $P(n)$ is true for all $n \in \mathbb{N}$, induct on $n$.
2. **Base Case.** When $n = 0$,
$$0 = \sum_{i=0}^{0} i^2 = \frac{(2 \cdot 0 + 1)0(0+1)}{6} = 0$$
3. **Inductive Step.** Suppose $P(n)$ is true when $n = k - 1$. Show $P(n)$ is true when $n = k$.
$$\begin{aligned}
\sum_{i=0}^{k} i^2 &= \left(\sum_{i=0}^{k-1} i^2\right) + k^2 \\
&= \frac{(2k-1)(k-1)k}{6} + k^2 && \text{(IH)} \\
&= \frac{2k^3 - 3k^2 + k + 6k^2}{6} \\
&= \frac{2k^3 + 3k^2 + k}{6} \\
&= \frac{k(2k^2 + 3k + 1)}{6} \\
&= \frac{k(2k+1)(k+1)}{6}
\end{aligned}$$
Thus, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

**Q2.** Suppose you have a large round pizza. What's the *maximum* number of slices you can make if you use a pizza slicer to make $n$ straight cuts?

1. **Predicate.** Let $P(n)$ be the statement: the maximum number of slices you can make with $n$ straight cuts is $\frac{n(n+1)}{2} + 1$. To show that $P(n)$ is true for all $n \in \mathbb{N}$, induct on $n$.
2. **Base Case.** When $n = 0$, we made no cuts so the entire pizza is one slice. This is equal to the value $\frac{0(0+1)}{2} + 1 = 1$.

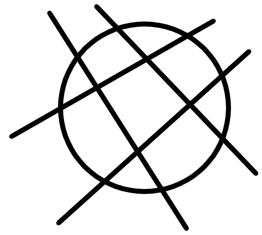

Figure 1: In this configuration of four cuts, we get nine slices.

3. **Induction Step.** Suppose $P(n)$ is true when $n = k$. Show $P(n)$ is true when $n = k + 1$.

   Suppose we have a configuration of $k$ cuts which produce the maximum number of slices. By the induction hypothesis (IH), we know that there are exactly $\frac{k(k+1)}{2} + 1$ slices so far. A line can only intersect another line in at most one place (if they are parallel, then there is no intersection). Add one more cut which intersects all the previous cuts. Assume all intersections fall inside the pizza. Otherwise, if any of the intersections fall outside the pizza, we can scale up the pizza enclose them. Then, in the original pizza, we will apply a scaled down version of the cuts to produce the same number of slices. Notice that each ray or line segment of the new cut produces a new slice. Thus $k + 1$ new slices are added. In total, the number of slices is

$$\frac{k(k+1)}{2} + 1 + k + 1 = (k+1)\left(\frac{k}{2} + 1\right) + 1 = \frac{(k+1)(k+2)}{2} + 1.$$

   Thus, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$.

## Marked Questions

### Q1. [10 Points] Fibonacci Numbers *(maximum 2 pages)*

1. Let $f_n$ be the $n^{\text{th}}$ Fibonacci Number where $f_0 = 0$ and $f_1 = 1$. Prove that $\sum_{k=0}^{n} f_k = f_{n+2} - 1$ for all $n \in \mathbb{N}$.

   *Proof.* □

2. Prove that $f_n \leq \left(\frac{7}{4}\right)^{n-1}$ for all $n \in \mathbb{N}$ when $n \geq 1$ from first principles (*do not used the equation for $f_n$ shown in class*).

   *Proof.* □

### Q2. [10 Points] Marshmallow Game *(maximum 2 pages)*

Alice and Bob's friend made a large pan of marshmallows which are cut into $m \times n$ squares but still stuck together. Alice and Bob decides to play a game with the marshmallow to pull it apart. On her turn, Alice pulls the marshmallow apart along a vertical cut while, on his turn, Bob pulls the marshmallow apart along a horizontal cut. The player pulling the marshmallow then choose one of the pieces to be the *active* and the game continues on the active piece. The first player unable to make a move loses. Alice goes first. Sees a sample game in Figure 2.

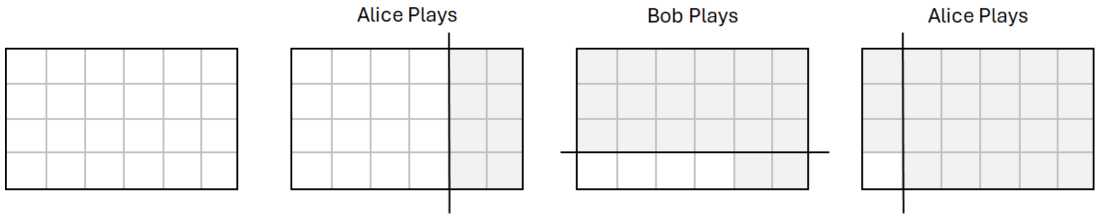


Figure 2: Initially the full grid is active. Alice pulls off two columns of marshmallows. The remaining $4 \times 4$ grid is active (the gray portion is inactive). Bob pulls off three rows leaving a $1 \times 4$ active grid. Alice pulls off three columns leaving one square. Since Bob cannot move, Alice wins!

1. If $m = 1$, who is the winner? What about when $n = 1$?
2. For $m, n \in \mathbb{N}$ with $m, n \geq 1$ who wins? Prove your claim.

   **Claim 1.**

   *Proof.* □

## Additional Questions

*Remember to comment out this section when submitting your assignment.*

If you would like more exercises consider trying the following problems from your primary and supplementary textbooks. *We will not be providing solutions to these questions* though you are free to find the solution online and discuss them with your peers.

1. David Liu's notes Chapter 2: Exercises 4, 6, 12
2. *Concrete Mathematics* Chapter 1: Exercises 1, 5, 6