# Assignment 1 [SOLUTIONS]

## Marked Questions

### Q1. [10 Points] Fibonacci Numbers *(maximum 2 pages)*

1. Let $f_n$ be the $n^{\text{th}}$ Fibonacci Number where $f_0 = 0$ and $f_1 = 1$. Prove that $\sum_{k=0}^{n} f_k = f_{n+2} - 1$ for all $n \in \mathbb{N}$.

   *Proof.* Define predicate $P(n) \coloneqq \sum_{k=0}^{n} f_k = f_{n+2} - 1$. Prove $P(n)$ is true $\forall n \in \mathbb{N}$ by induction on $n$.

   (a) Base case. Note that $f_0 = 0$ and $f_1 = 1$ (given) as well as $f_2 = 1$ and $f_3 = 2$ (deduced from the definition $f_{n+2} = f_{n+1} + f_n$. Consider the cases where $n = 0$ and $n = 1$. In the former case $\sum_{k=0}^{0} f_k = f_0 = 0 = 1 - 1 = f_2 - 1$. In the latter case, $\sum_{k=0}^{1} f_k = f_0 + f_1 = 1 = 2 - 1 = f_3 - 1$.

   (b) Inductive step. For a fixed $m$ suppose that $P(0), ..., P(m)$ are all true, we want to show that $P(m+1)$ is true. Note that

$$\begin{aligned}
\sum_{k=0}^{m+1} f_k &= \left(\sum_{k=0}^{m} f_k\right) + f_{m+1} \\
&= f_{m+2} - 1 + f_{m+1} && \text{(IH)} \\
&= f_{m+3} - 1 && \text{(Definition of } f_{m+3}\text{)}
\end{aligned}$$

   It follow by the Principle of Mathematical Induction that $P(n)$ is true $\forall n \in \mathbb{N}$. □

   *Total: 5 points. [1 Point] Predicate/tell me what you are inducting on. [2 Points] Base case must have at least two cases (one point per case). [2 Points] Inductive step* must *use the inductive hypothesis in order to get any points. Minor and major arithmetic errors are (-0.5) and (-1) respectively.*

2. Prove that $f_n \leq \left(\frac{7}{4}\right)^{n-1}$ for all $n \in \mathbb{N}$ when $n \geq 1$ from first principles (*do not used the equation for $f_n$ shown in class*).

   *Proof.* Define $P(n) \coloneqq f_n \leq (7/4)^{n-1}$. Prove $P(n)$ is true $\forall n \in \mathbb{N}$, $n \geq 1$ by induction on $n$.

   (a) Base case. Consider the cases where $n = 1$ and $n = 2$. In the former, we have $f_1 = 1 \leq (7/4)^0 = 1$. In the latter case, we have $f_2 = 1 \leq (7/4)^1$.

   (b) Inductive step. Fix some $k \geq 2$ and suppose that $P(1), P(2), ..., P(k)$ are true. Show that $P(k+1)$ is true. Note that

$$\begin{aligned}
f_{k+1} &= f_k + f_{k-1} && \text{(Definition of } f_{k+1}\text{)}\\
&\leq \left(\frac{7}{4}\right)^{k-1} + \left(\frac{7}{4}\right)^{k-2} && \text{(IH)}\\
&\leq \left(\frac{7}{4}\right)^{k-2}\left(\frac{7}{4}+1\right) = \left(\frac{7}{4}\right)^{k-2}\left(\frac{44}{16}\right)\\
&\leq \left(\frac{7}{4}\right)^{k-2}\left(\frac{49}{16}\right) = \left(\frac{7}{4}\right)^{k}.
\end{aligned}$$

It follow by the Principle of Mathematical Induction that $P(n)$ is true for all $n \in \mathbb{N}$, $n \geq 1$. $\square$

*Total: 5 points. [1 Point] Predicate/tell me what you are inducting on. [1 Point] Base case must have at least two cases (0.5 points per case) as this was pretty much already shown in Lecture 1. [3 Points] Inductive step* must *use the inductive hypothesis in order to get any points. Minor and major arithmetic errors are (-0.5) and (-1) respectively.*

**Q2. [10 Points] Marshmallow Game** *(maximum 2 pages)*

Alice and Bob's friend made a large pan of marshmallows which are cut into $m \times n$ squares but still stuck together. Alice and Bob decides to play a game with the marshmallow to pull it apart. On her turn, Alice pulls the marshmallow apart along a vertical cut while, on his turn, Bob pulls the marshmallow apart along a horizontal cut. The player pulling the marshmallow then choose one of the pieces to be the *active* and the game continues on the active piece. The first player unable to make a move loses. Alice goes first. Sees a sample game in Figure 1.

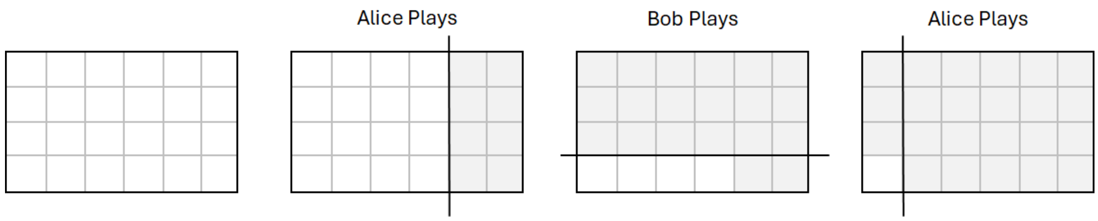


Figure 1: Initially the full grid is active. Alice pulls off two columns of marshmallows. The remaining $4 \times 4$ grid is active (the gray portion is inactive). Bob pulls off three rows leaving a $1 \times 4$ active grid. Alice pulls off three columns leaving one square. Since Bob cannot move, Alice wins!

1. If $m = 1$, who is the winner? What about when $n = 1$?

   When $m \geq 1$ and $n = 1$, Bob is the winner. In particular if $m = n = 1$, then on the first turn Alice has no valid moves and loses (Bob wins). Generally when $n = 1$, Alice has no valid moves and loses (Bob wins). Conversely, if $m = 1$ and $n > 1$, then Alice is the winner. On her first turn, Alice can pull apart the marshmallow into a single square and another rectangle which has dimension $1 \times (n-1)$. If she designates the single square to be active, then on Bob's turn he will have no valid moves and loses (Alice wins).

   *Total: 3 points. [1 Point] Consider case $m = 1$ and $n = 1$. [1 Point] Consider case $m > 1$ and $n = 1$. [1 Point] Consider case $m = 1$ and $n > 1$.*

2. For $m, n \in \mathbb{N}$ with $m, n \geq 1$ who wins? Prove your claim.

   **Claim 1.** *When $m \geq n$, Bob win. Otherwise Alice wins.*

   *Proof.* We prove the following predicate $P(m, n)$ for $m, n \in \mathbb{N}$ with $m, n \geq 1$.

Case 1. If Alice plays first and $m \geq n$ then Bob wins, otherwise Alice wins.

Case 2. If Bob plays first and $n \geq m$ then Alice wins, otherwise Bob wins.

If we show $P(m, n)$ is true $\forall m, n \in \mathbb{N}$ with $m, n \geq 1$, then the claim is true; The first case of the predicate is *exactly* our game (Alice plays first). Apply induction on the pair $(m, n)$.

(a) Base case:

Alice first. This is *exactly* what was considered in the first part of Q2 (above).

Bob first. This is *very similar* to the first part of Q2 (above). For completeness: when $n \geq 1$ and $m = 1$, Alice wins. In particular if $m = n = 1$, then on the first turn Bob has no valid moves and loses. Generally, when $m = 1$, Bob has no valid moves and loses.

Conversely, if $n = 1$ and $m > 1$, then Bob wins. On his first turn, Bob can pull apart the marshmallow into a single square and another rectangle with dimension $(m-1) \times 1$. If he designates the single square to be active, then on Alice's turn she will have no valid moves and loses.

(b) Inductive step: Fix some $k > 1$ and $\ell > 1$. Suppose $\forall k' : 1 \leq k' < k$ and $\forall \ell' :\leq \ell' < \ell$, $P(k', \ell)$, $P(k, \ell')$, and $P(k', \ell')$ are true. We want to show that $P(k, \ell)$ is true.

Alice first. Either $\ell > k$ or $\ell \leq k$.
When $\ell > k$, Alice can pull away $\ell - k$ columns to obtain a $k \times k$ square which she will designate as active. Then, Bob will be playing first on a $k \times k$ square. By IH applied to $P(k, k)$, when Bob plays first Alice wins.
When $\ell \leq k$, regardless of how Alice pulls apart the marshmallow, the two resulting rectangles will have dimension $k \times \ell_1$ and $k \times \ell_2$ where $\ell_1 + \ell_2 = \ell$. Note $\ell_1, \ell_2 < \ell \leq k$. By IH applied to $P(k, \ell_1)$ and $P(k, \ell_2)$, when Bob plays first on either of these rectangles, he will win.

Bob first. *Very similar to previous case.* The following is for completeness only. Either $k > \ell$ or $k \leq \ell$.
When $k > \ell$, Bob can pull away $k - \ell$ rows to obtain a $\ell \times \ell$ square which he will designate as active. Then, Alice will be playing first on a $\ell \times \ell$ square. By IH applied to $P(\ell, \ell)$, when Alice plays first Bob wins.
When $k \leq \ell$, regardless of how Bob pulls apart the marshmallow, the two resulting rectangles will have dimension $k_1 \times \ell$ and $k_2 \times \ell$ where $k_1 + k_2 = k$. Note $k_1, k_2 < k \leq \ell$. By IH applied to $P(k_1, \ell)$ and $P(k_2, \ell)$, when Alice plays first on either of these rectangles, she will win.

Thus, by complete induction, $P(m, n)$ is true for all $m, n \in \mathbb{N}$ where $m, n \geq 1$. $\square$

*Total: 7 points. [3 Points] Predicate. [1 Points] Base case. [3 Points] Inductive step* must *use IH to get points. Minor and major logical errors get a reduction of $-0.5$ and $-1$ respectively. If the predicate has significant errors then this part can get* at most *two points for the predicate and nothing else. Be careful when marking this, there may be multiple valid solutions e.g. with different number of variables in the predicate, using Well-ordering principle (though induction was encouraged), etc.*