Week 2: More Induction and Counting

CSC 236:Introduction to the Theory of Computation

Summer 2024

Instructor: Lily

Announcements

- Volunteer notetaker for Accessibility Services
- Registered study groups
- Solution to unmarked Q2 updated
- New Office Hours (still in BA 2272):
  - Tuesdays: 4-5pm **(NEW! – starts May 21)**
  - Fridays: 1-2pm **(MODIFIED!)**
- Assignment Office Hours (BA 2270):
  - Thursday, 1-2pm
  - Friday 2-3pm

Announcements

- Waitlisted students: **email us!**
  - Q1 and A1 gets average of other quizzes (best 7) and assignments (best 3)

Subject [Waitlist] First Name, Last Name, Student Number, UTORID

- Bonus points: **email us!**
  - 4+ endorsed answers or corrections on piazza (send us list of links, your name, and student ID) --- processed near the end of the semester

Subject [Bonus] First Name, Last Name, Student Number, UTORID will be defined on test.

$H_1 H_2$

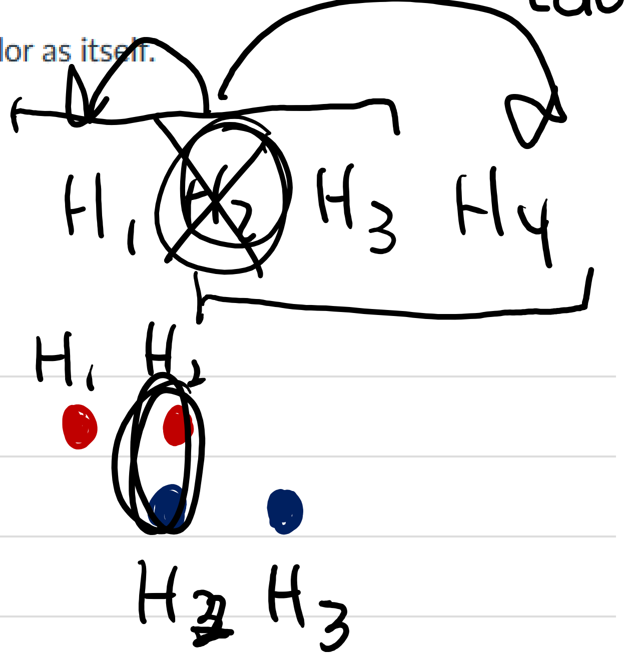
$P(n) :=$ any group of n horses have the same color.

Question 1 1 pts

Consider the following proof that all horses have the same color. We induction on n, the number of horses. Choose every line which is *incorrect*.

1. In the base case, if there are no horses then the statement is vacuously true .If there is one horse it is the same color as itself.
2. For a fixed k, assume that $P(k)$ is true. Show that $P(k+1)$ is true.
3. Take a group of $k+1$ horses and label them $1, \ldots, k+1$. By IH, horses $1, \ldots, k$ are equal.
4. Similarly, by IH, horses $2, \ldots, k+1$ are the same color.
5. Note that the middle horses $2, \ldots, k$ must all be the same color which is the color of horse 1 and horse $k+1$.
6. Thus all horses are the same color.

if $1 \sim 5 \Rightarrow 6$



- [ ] 4
- [ ] 5
- [ ] 1
- [ ] 3
- [ ] 6
- [ ] 2

TUTORIAL 5|012 maybe canceled

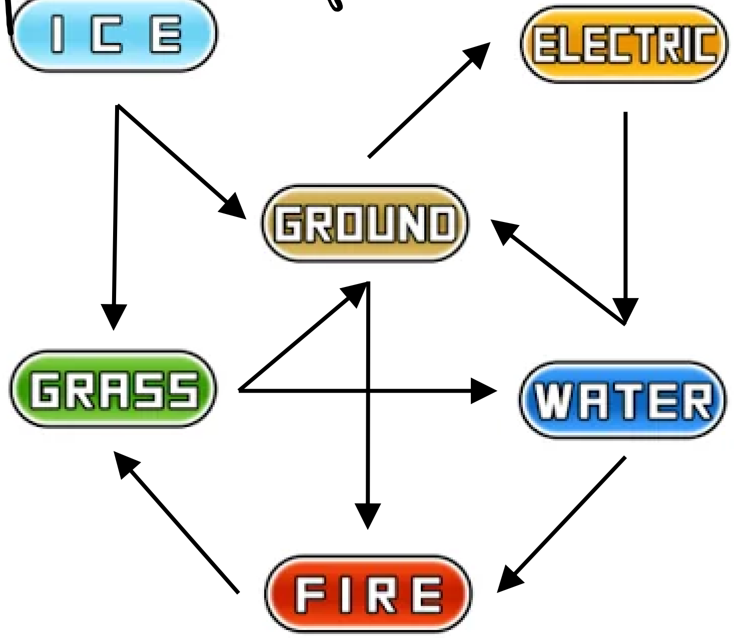
Relations

Ordering: Set S ordered pairs on S $:= \{(a,b) : a \in S, b \in S\}$

↳ tells me pair satisfies / not satisfy relation

$S =$ pokemon types

order relation = "super effective against"



$S = \mathbb{N}$

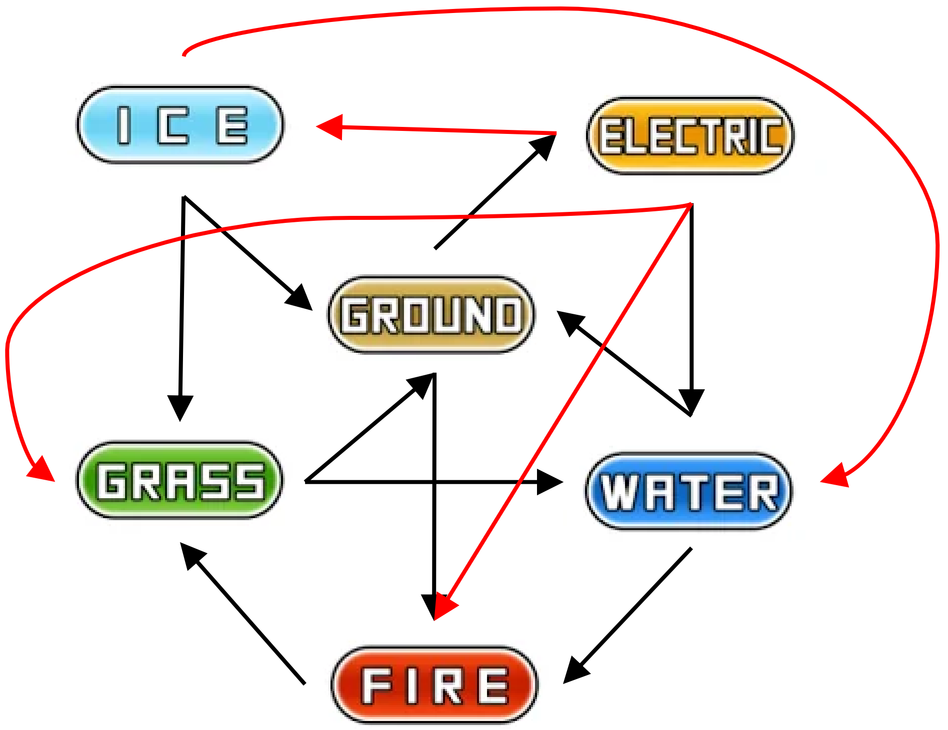
order relation = "<"

$(1, 100)$ sat

$(2, 0)$ not sat

Relations

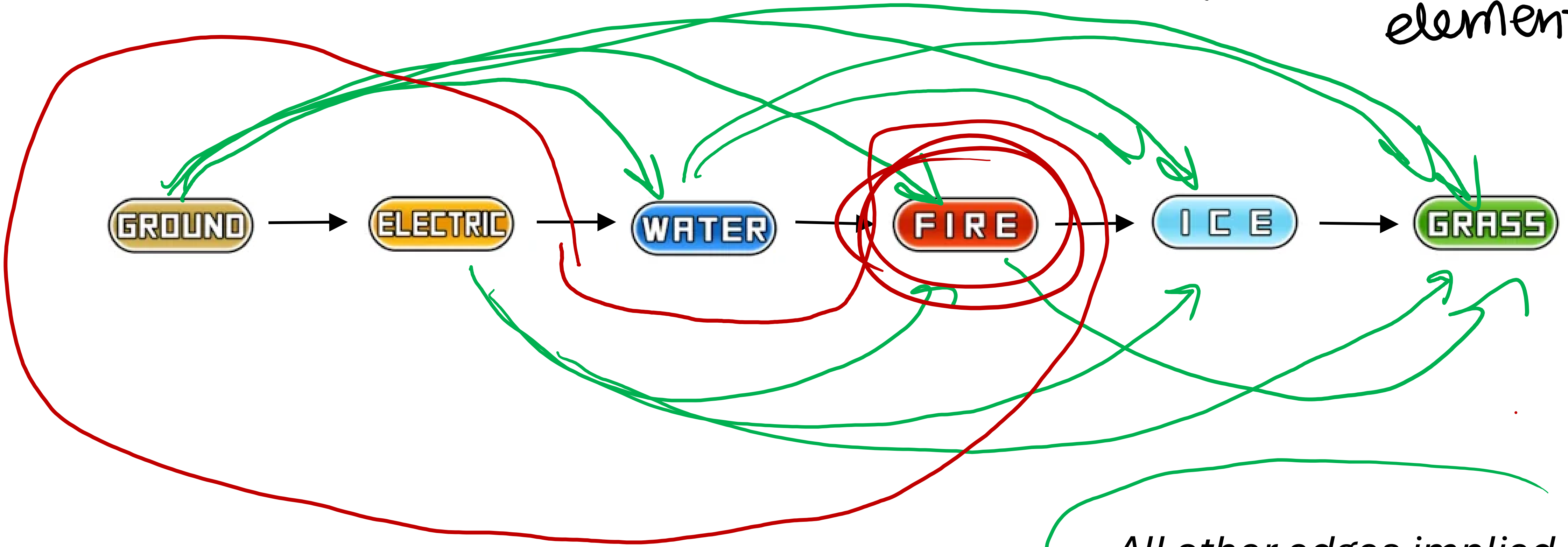
Total Ordering: all elements of the ordered pairs of S must satisfy / not satisfy relation



(as graph
types := nodes
effective against := edges)

Relations

Well-Ordering: total ordering where every non-empty subset has a least element



All other edges implied

Well-Ordering: $S = \mathbb{N}$ '$<$' $U \subseteq S$ U has minimum element

A – on a set S is a *total ordering* such that every *non-empty* subset of S has a least element

Well-Ordering Theorem: (Zermelo's Thm)

Every *nonempty* set S has a relation $<$ on S which is a well ordering.

+ Zorn's Lemma $\Rightarrow$ Axiom of Choice

Well-Ordering Principle:

Every *nonempty* subset of $\mathbb{N}$ has a *smallest* element.

If not subset of $\mathbb{N}$, $U \subseteq \mathbb{Z} \Rightarrow U = \{0, -1, -2, \ldots\}$

If empty, then no least element.

Induction implies Well-Ordering Principle

Giusepp Peano 19th century arithmetics on $\mathbb{N}$.

WOP follows from $P(n)\ \forall n$ because S is non-empty.

Well-Ordering Principle (WOP):

Every *nonempty* subset of ℕ has a *smallest* element.

Induction : $\left(P(0) \wedge \forall k, \left(P(0) \wedge \cdots \wedge P(k) \right) \Rightarrow P(k+1) \right) \Rightarrow \forall n \in \mathbb{N}, P(n)$

$P(n)$: if $n \in S$ then S has a min element.

base : $n=0$ if $0 \notin S$ $P(0)$ is trivially true (False $\Rightarrow$ true) same as

$0 \in S$ then 0 is min of S

inductive step : fix k, assume $P(0) \wedge \cdots \wedge P(k)$

if any of $0, \ldots, k \in S$, IH, S has min element.

if ___ not in S. $k+1$ in S then it min element.

if $k+1$ not in S

Well-Ordering

For all $n \in \mathbb{N}$ where $n > 1$, n has a prime factorization.

let $S \subseteq \mathbb{N}$, $S := \{ n \in \mathbb{N}, n > 1 :$ n does not have prime factorization$\}$

assume for contradiction S non-empty, WOP gives us min element w of S. Note

- w is not prime

$\Rightarrow w = a \cdot b$ with $a, b < w$ and $a, b > 1$

$\Rightarrow a, b \notin S$ (s has min element w, ab smaller)

prime factor a,b $a = p_1 \cdots p_u$ $b = q_1 \cdots q_v \Rightarrow w = p_1 \cdots p_u q_1 \cdots q_v$

BUT WAIT! w is not suppose to have a prime factorization

$\cancel{\#}$

Fundamentals of Counting

- Applications: telephone numbers, IP addresses, password security, biology (e.g. sequencing DNA)

\# TORONTO PHONE NUMBERS : 416, 647, 437 $\times$ 3 digits $\times$ 4 digits. $\Rightarrow 3 \times 10^7$

10^3 10^4

$\sim$ 30 mill.

1 call per second

$\sim 60 \times 60 \times 24 \times 365$ 1 call/year

- Putting items (n) into different slots (m)
- Factorials: $n! = n \cdot (n-1) \cdots 2 \cdot 1$
- Binomial Coefficient: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Permutation: Order Matters

How many ways can n children stand in a line for a picture?

n choices n-1 choices ... 2 choices 1 choice $\Rightarrow n!$

Combinations: Order *Does Not* Matter

How many ways can we draw a hand of m cards from a deck of n? Assume $m \leq n$.

$m=3 \quad n=52$

n choices n-1 choices ... n-m+1 choices

A♡ K♢ Q♡

$= \boxed{n! / m!(n-m)!} = \binom{n}{m}$

Q♡ = A♡ K♢

Permutation: Order Matters

How many ways can n children stand in a line for a picture?

Combinations: Order *Does Not* Matter

$$\underbrace{(1+x)^n}_{\overbrace{(1+x)\cdots(1+x)}^{n \text{ times}}} = \binom{n}{0} + \binom{n}{1}x + \cdots + \binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$$

must choose all 1s

linear tearm

choose 1s for all but one

. . . .

how many ways?

choose all x's

Examples

1. How many permutations of the letters ABCDEFG contains the string “ABC”?
2. How many possibilities are there for the first, second, and third positions in a horse race with 12 horses?
3. A group contains n men and n women. How many ways are there to arrange theses people in a row if the men and women alternate?

Now You Try!

1. How many binary strings contain 3 ones and 5 zeros?

2. Prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ for all positive integer n.

3. Suppose there are $2n$ people. How many ways can we form a committee of n people?

4. Prove that $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}$ for all positive integer n.

Q1. How many binary strings contain 3 ones and 5 zeros?

1 1 0 1 0 1 0 1

contains 0

\# of items

$\Rightarrow \binom{8}{3} = \binom{8}{5}$

\# slots for 0; order does not matter

Q2. Prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ for all positive integer n.

$$(1+x)^n = \sum_{k=0}^{n} x^k \binom{n}{k} \qquad x=-1.$$

$x \in \mathbb{R}$

Q3. Suppose there are $2n$ people. How many ways can we form a committee of n people?

$\binom{2n}{n}$

items

slots

Q4. Prove that $\underbrace{\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}}_{\text{what does this mean?}} = \binom{2n}{n}$ for all positive integer n.

n men $\quad n-k \Rightarrow \binom{n}{n-k}$ # ways choose $n-k$ men

n women $\quad k \Rightarrow \binom{n}{k}$ # of ways of choosing k women.

there could be $0 \to n$ women

so $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$ → sums all cases

Permutation with Replacement

There are n children and m chores. How many ways are there to assign the children to the chores if each child can do any number of chores?

chore 1 chore 2 chore 3

n choices n choices n choices

$$n^m$$

Combination with Replacement

general case

$$\binom{n+m-1}{n} = \binom{n+m-1}{m-1}$$

Choose scoops → n

$m-1$ → choose dividers

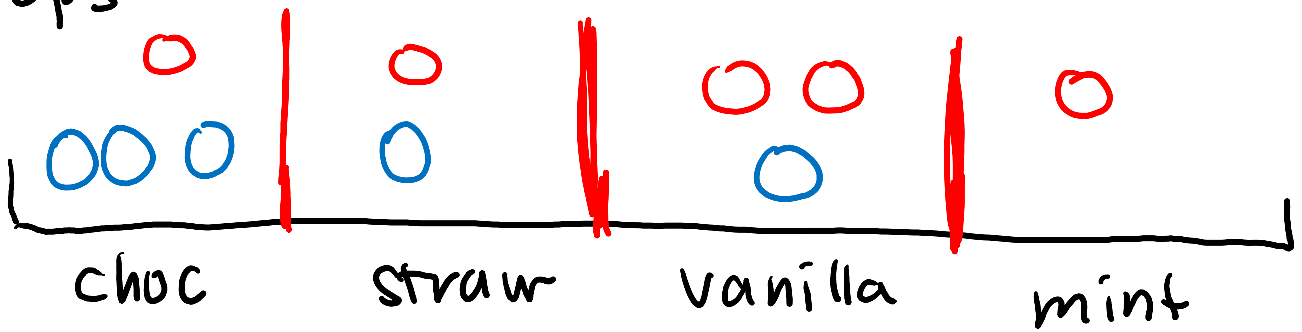
There are m flavors of ice-cream. How many ways are there to make an ice-cream cone with n balls of ice-cream (order does not matter)?

4 flavors (choc. Straw, vanilla, mint)

5 scoops

A

B



A	B
C	C
S	C
V	C
V	C
m	S
	V

A 0 1 0 1 0 0 1 0

B 0 0 0 1 0 1 0 1

3 "1"s, 5 "0"s ↳ $\binom{8}{3} = \binom{8}{5}$

Example. How many solutions are there to the equation $x_1 + x_2 + x_3 = 11$ where $x_1, x_2, x_3 \in \mathbb{N}$.

divide up "1"s

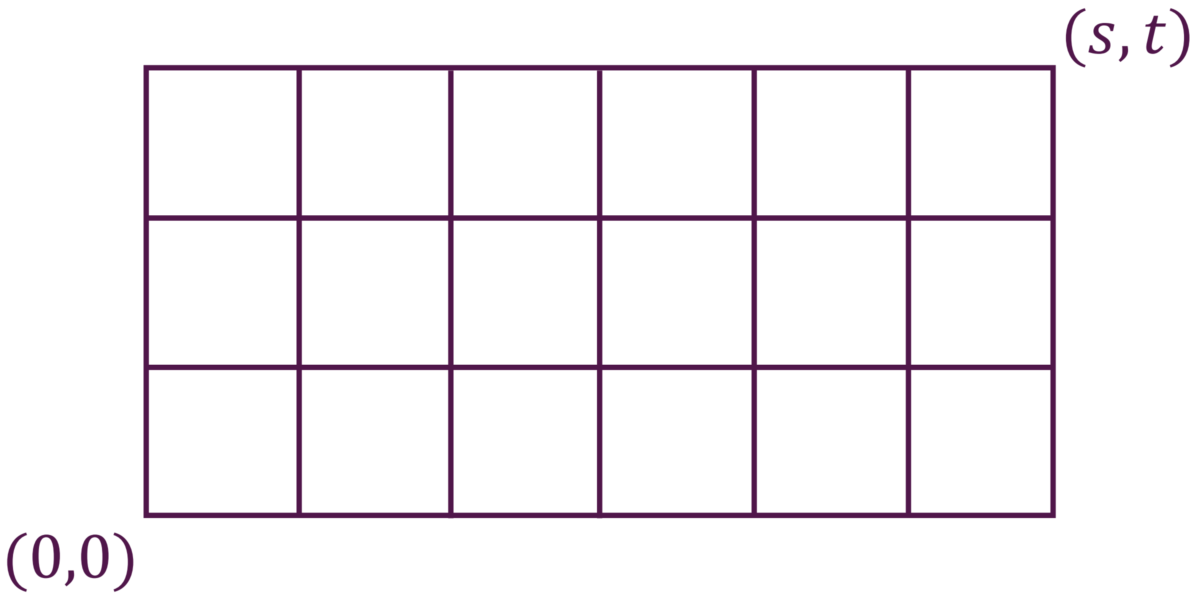
$$\underbrace{x_1 + x_2 + x_3}_{\text{different "flavors"}} = 11$$

$$= \binom{11+3-1}{11}$$

$$= \binom{13}{11}$$

	Items (n)		
Type		No Replacement	Replacement
	Permutation	Ex. pick top m dogs in dog show among n $(m \leq n)$ $n(n-1)\ldots(n-m+1) = n!/(n-m)!$	Ex. pick m numbers for lottery ticket among n n^m
	Combination	Ex. Pick group of m children from n to form a club $(m \leq n)$ $\binom{n}{m}$	Ex. Picking m scoops of icecream from among n flavors $\binom{m+n-1}{m}$

Example. Start at $(0,0)$ and end at (s,t). Valid steps: one step up or one step right. How many different paths?



Recap

- Order relations, total ordering, well-ordering
- Induction and the Well-Ordering Principle are equivalent (*we only saw induction implies well-ordering principle*)
- Permutation (with/without replacement)
- Combination (with/without replacement)
- Lots of Examples

Next time… Discrete Probability, Pigeonhole Principle and Introduction to Graphs.