Week 3: Combinatorics and Graph Theory

CSC 236:Introduction to the Theory of Computation

Summer 2024

Instructor: Lily

Announcements

- Deferred students: **email us!**
  - Gives access to syllabus and quizzes (marks don't count)
- Tutorial location changed:
  - Section 5101 + 5104: BA 1130 (Logan)
  - Section 5102 + 5103: BA 2135 (Lawrence)
- Peer review procedure

  - available on Friday (Markus)
  - 4 submissions to mark
    - rubric + sample marked solutions provided EOD Fri

Discrete Probability: Definitions

$[n] = \{1, \dots, n\}$

- Ω: probability space, $w \in \Omega$ is an event
- $\mathbb{P}: \Omega \to [0,1]$ Probability distribution
  - $\mathbb{P}[w] \geq 0$
  - $\sum_{w \in \Omega} \mathbb{P}[w] = 1$
  - For disjoint $S_1, \dots, S_n \subseteq \Omega$, $\mathbb{P}[\cup_{i=1}^{n} S_i] = \sum_{i=1}^{n} \mathbb{P}[S_i]$
- Random Variable: functions $X : \underbrace{\Omega}_{\text{(domain)}} \to \underbrace{E}_{\text{(range)}} \subseteq \mathbb{R}$
- Expectation: $\underbrace{\mathbb{E}X}_{\text{scalar value}} = \sum_{x \in E} x \cdot \mathbb{P}[X = x]$
- Variance: $Var(X) = \mathbb{E}(X - \mathbb{E}X)^2$

Examples	X	Ω	E
1.	value of roll of fair die	$\{1, \dots, 6\}$	$\{1, \dots, 6\}$
2.	sum of values when rolling 2 fair die	$\{(i,j) : i \in [6], j \in [6]\}$	$\{2, \dots, 12\}$
3.	# of "H" in 10 coin flips.	$\{H, T\}^{10}$	$\{0, \dots 10\}$

Tossing n coins and counting number of heads.

coins with bias $p \in [0,1]$

with prob. p coin comes up heads.

$$X : \{H,T\}^n \longrightarrow \{0,\dots,n\}$$

$n=3$ $\quad X(HHT)=2, \quad X(HTT)=1, \quad X(TTT)=0$

$$P_X(k) = \mathbb{P}[X=k] = \binom{n}{k} p^k \underbrace{(1-p)}_{\text{define as } q}{}^{n-k}$$

$$\underbrace{0,1,2,3}_{}\quad P_X(k) = \mathbb{P}[X=k]$$

Non-neg ✓ $\quad$ disjoint sets ✓

$$\underbrace{T\ H\ T\ T\ H\ T\ H}_{n}$$

$$\sum_{k\in\Omega} \mathbb{P}[X=k] = \sum_{k\in\Omega} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= (p + (1-p))^n = 1$$

(binomial coeff $\quad p \quad q$

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n \text{)}$$

Tossing n coins and counting number of heads.

$$\mathbb{E}X = \sum_{k=0}^{n} \underbrace{\binom{n}{k} p^k (1-p)^{n-k}}_{P[X=k]} \cdot k = \cancel{\binom{n}{0} p^0 (1-p)^n} \cdot 0 + \left(\sum_{k=1}^{n} \binom{n}{k} p^k (1-p)^{n-k} k \right)$$

$$= \sum_{k=1}^{n} n \binom{n-1}{k-1} p^k (1-p)^{n-(k-1)-1}$$

$$\binom{n}{k} \cdot k = \frac{n! \, \cancel{k}}{\cancel{k!}(n-k)!} = n \binom{n-1}{k-1}$$

$(k-1)!$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$X_1, \ldots, X_n$ Head for toss $1 \ldots n$

$\mathbb{E}X_i = p \cdot 1 + \cancel{(1-p)} \cdot 0$

$= p$

$$= np\,(p + (1-p))^{n-1} = np$$

$$\mathbb{E}X_1 + \mathbb{E}X_2 \cdots + \mathbb{E}X_n = \mathbb{E}[X_1 + X_2 + \cdots + X_n]$$

$np = n\mathbb{E}X_i = \mathbb{E}[X]$ $\quad \mathrm{Var}(X) = np(1-p)$

Poker Hands

of 5 card Hands $\Rightarrow \binom{52}{5}$

Imagine being dealt a hand of five cards from a standard 52 card deck (4 suits, 13 values)

A♡ A♧

3♤ 3♧ J♡

1. How many hands are two pairs (not full house)

$$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}$$

choose 2 values — choose suits for values — pick remaining val — pick suit for remaining val.

2. How many hands are flushes (not straight flushes)?

count : all flushes − straight flushes

fix suit ♡

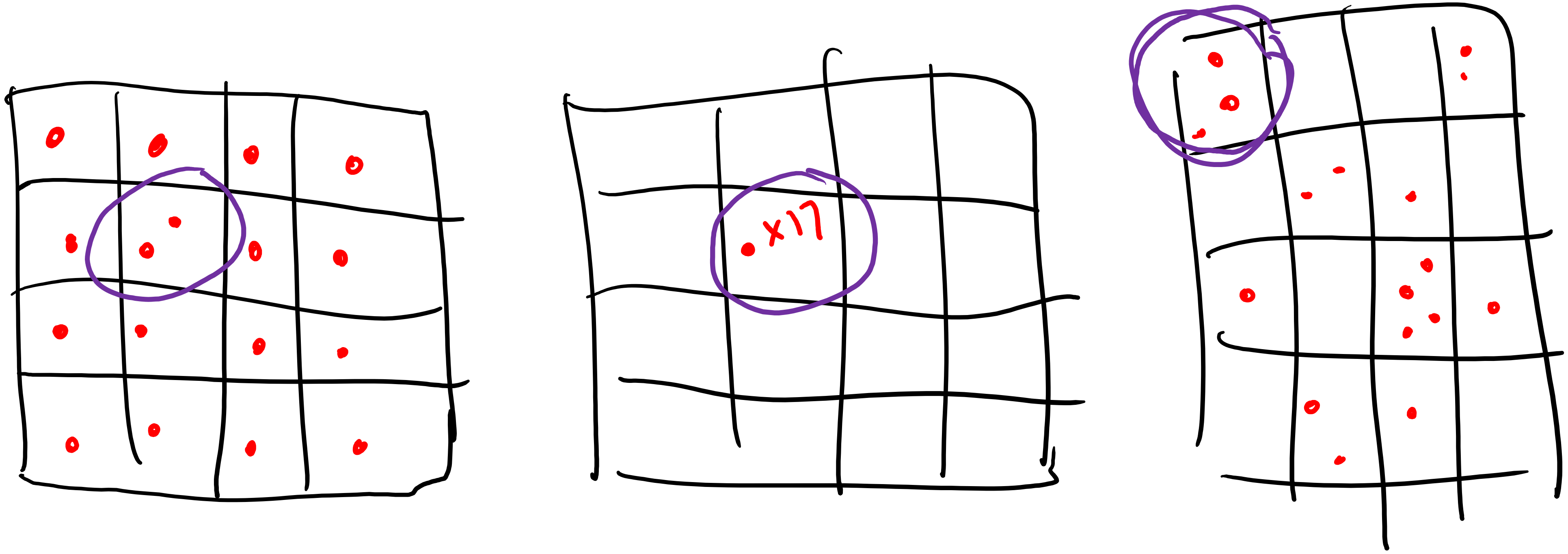
$$4\left(\binom{13}{5} - 10\right)$$

A K Q J 10 9 8 7 6 5 4 3 2 A

smallest card in straight

Pigeonhole Principle (PHP)

Flock of 17 pigeons flies into 16 roosts:



Pigeonhole Principle (PHP)

Flock of n pigeons flies into m roosts ($n > m$):

There is at least one roost with more than one pigeon.

Pigeonhole Principle (PHP):

For $m \in \mathbb{Z}_+$, if there are $m + 1$ or more objects and m boxes to put them in, then at least one box contains two or more objects.

367 people $\Rightarrow$ at least 2 will have same birthday

27 words (english) $\Rightarrow$ at least 2 start w/ the same letter

Example. Among $n+1$ positive integers $a_0, \dots, a_n$ where $a_i \le 2n$, there must be one integer which divides another.

$S = \{2, 7, 5, 3, 9, 4\}$ $n = 5$ $(2|4)$

$a_0 = 2^{k_0} q_0, \quad a_1 = 2^{k_1} q_1, \quad \dots, \quad a_n = 2^{k_n} q_n$

odd numbers

\# of odd numbers $\le 2n \rightarrow n$ ROOST

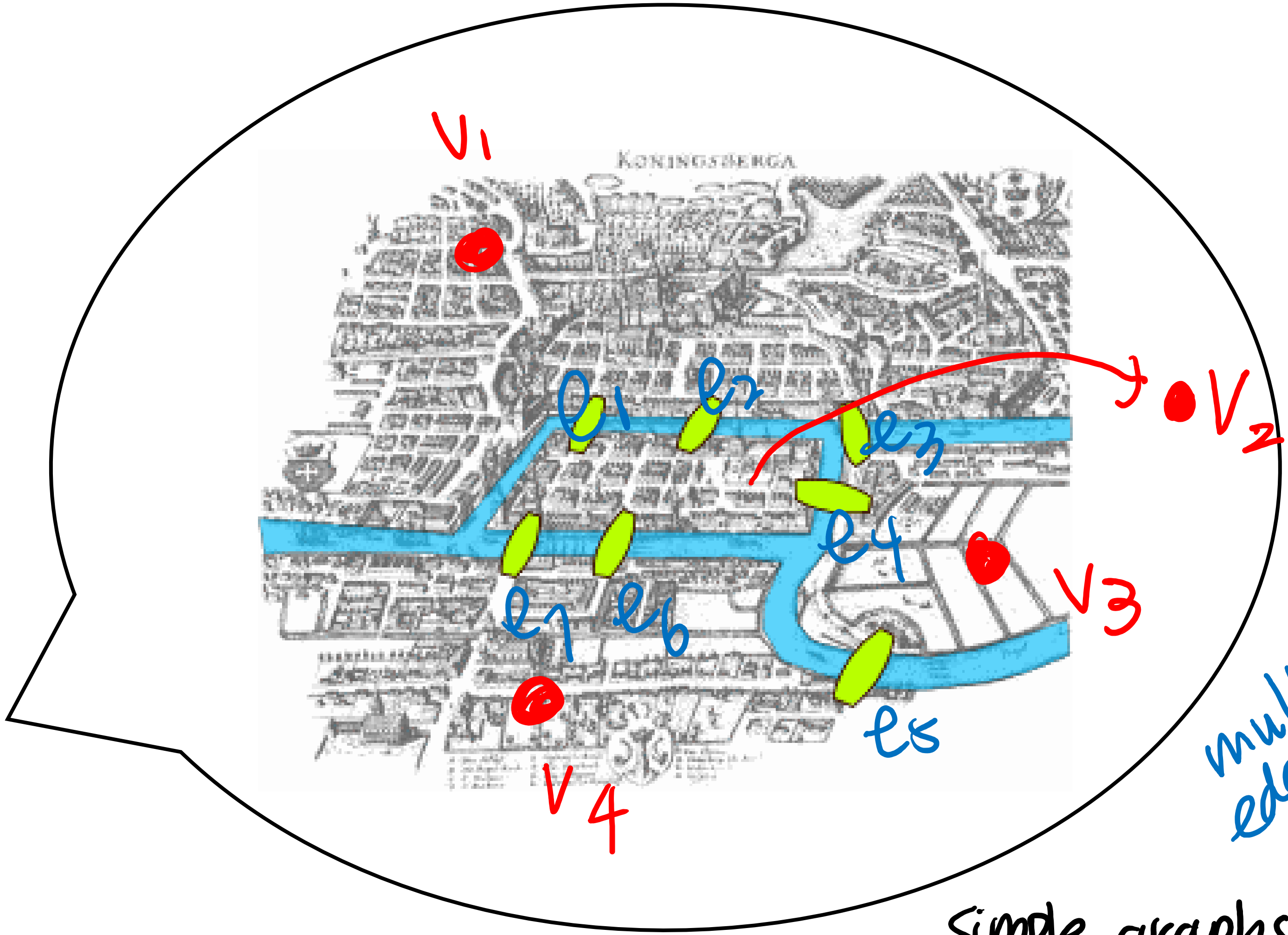
$n+1$ PIGEONS

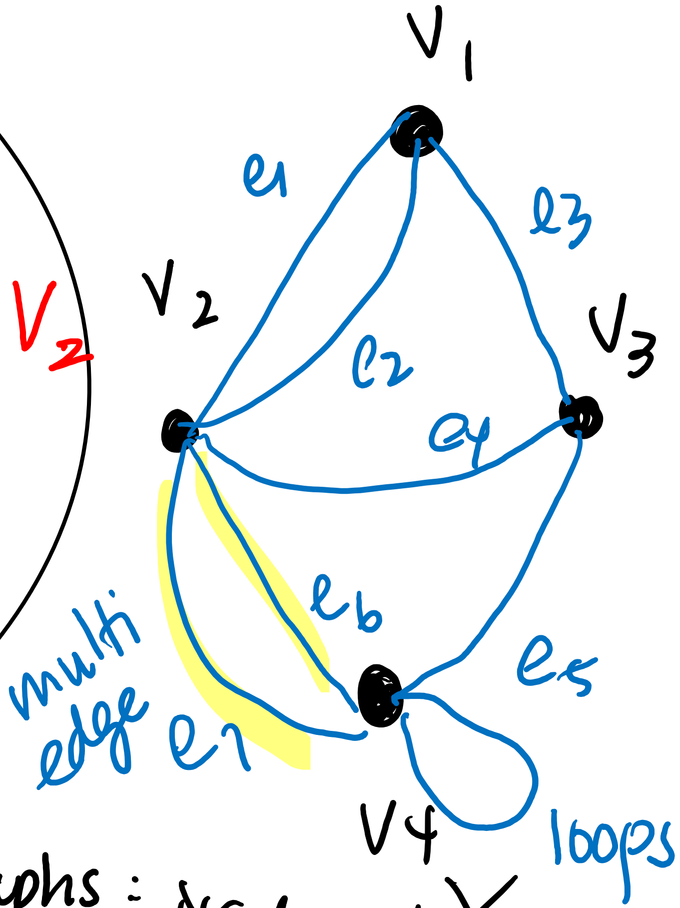
$\exists\, a_i, a_j$ where $q_i = q_j$ values $a_0, \dots, a_n$

$i \ne j$ then either $a_i | a_j$ or $a_j | a_i$ (depends on k_i vs k_j)

Graph Theory Fundamentals

graph $G = (V, E)$





simple graphs : ~~multi edge~~, ~~loops~~

https://en.wikipedia.org/wiki/Graph_theory#/media/File:Konigsberg_bridges.png

Now You Try!

1. What's the probability that a hand of five cards contains a four of a kind? What about a full house?
2. Every integer n has a multiple that has only 0s and 1s.

   10 (mult of 2), 111 (mult of 3), 100 (mult of 4)

3. **Bridges of Königsberg:** Does there exist a Euler Tour for the bridge configuration?

Q1. What's the probability that a hand of five cards contains a four of a kind? What about a full house?

a) $\binom{13}{1}\binom{4}{4}\binom{48}{1}$

pick value

all suits of value

pick another card

b) $\binom{13}{2}\binom{4}{2}\binom{4}{3}\binom{2}{1}$

AAA33

AA333

Q2. Every integer n has a multiple that has only 0s and 1s.

Pigeons

$n+1$ numbers : $a_0 = 1$, $a_1 = 11$, ..., $a_n = \underbrace{1 \cdots 1}_{n+1}$

Holes : remainder divided by n (n remainders)

suppose a_i, a_j have the same remainder when divided by n, $i < j$

$a_j - a_i = 11 \ldots 0 \ldots$

$a_j - a_i \bmod n = 0 \bmod n$.

Graph Terminology

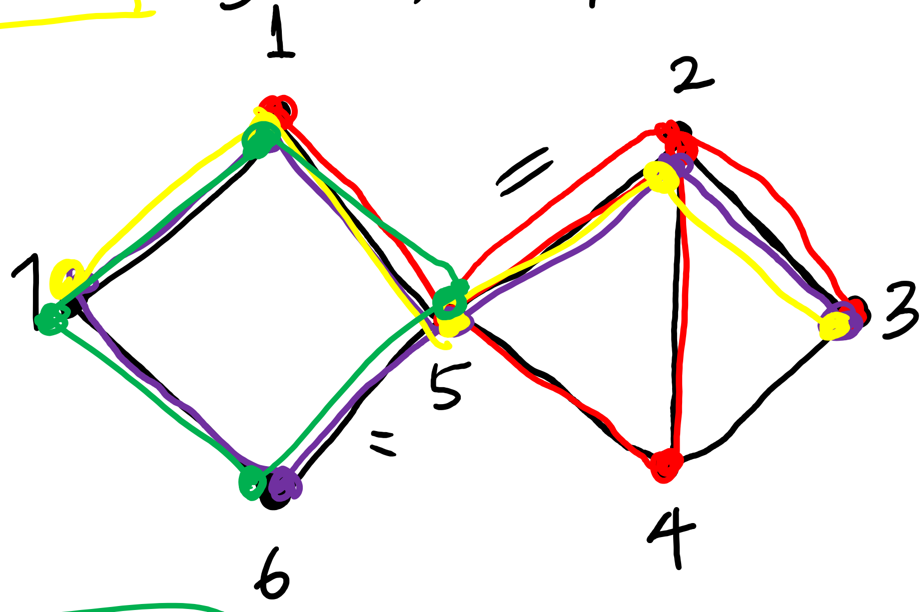
- Walk: List of vertex, edge, vertex, …, vertex
- Trail: *walk* where no edge is repeated
- Path: *trail* with no repeated *internal* vertices
  - u, v-Path: *path with* endpoints u and v
- Cycle: u, u-Path
- Length of walk/trail/path/cycle: number of edges

walk 1 (1,5) 5 (5,2) 2 (2,4) 6

4, (4,5) 5, (5,2), 2, (2,3), 3

Trail: 3, 2, 5, 1, 7, 6, 5 6

Path: 3, 2, 5, 1, 7 → 3,7-path 4



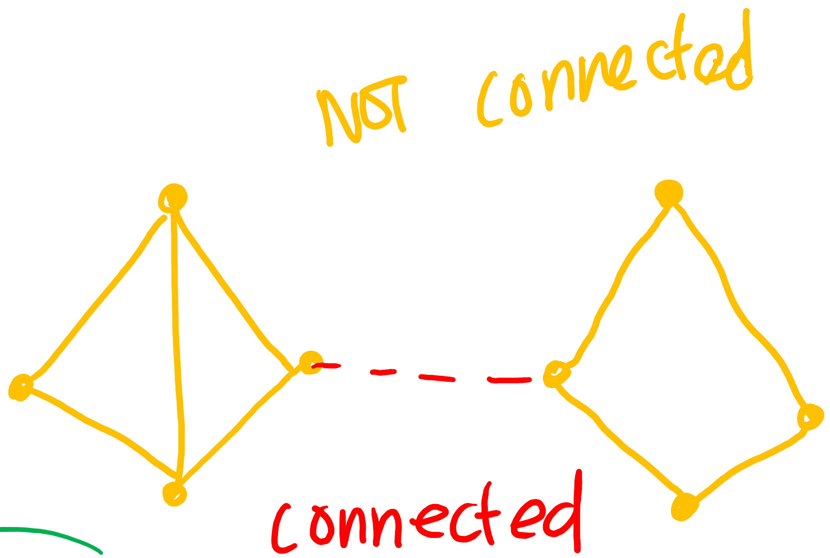
cycle: 1, 7, 6, 5 1 4

Graph Terminology

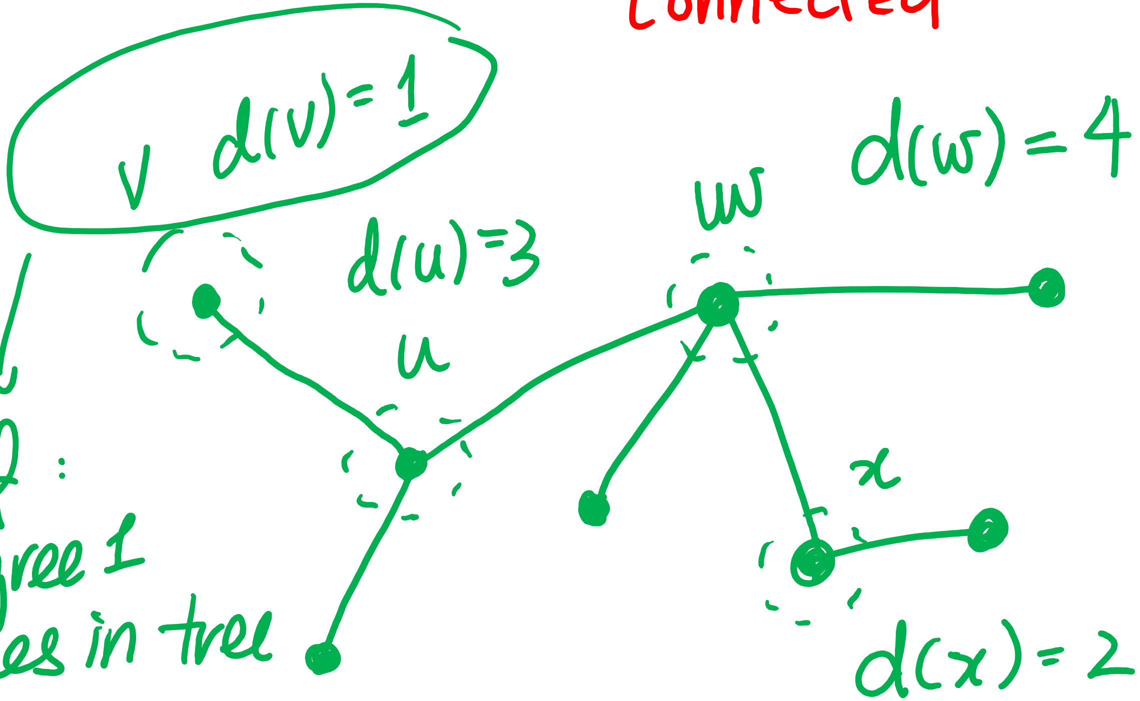
- Connected: between every pair of vertices there exists a path
- Tree: Connected but no cycles
- Degree: # of neighbours

u and w ✓

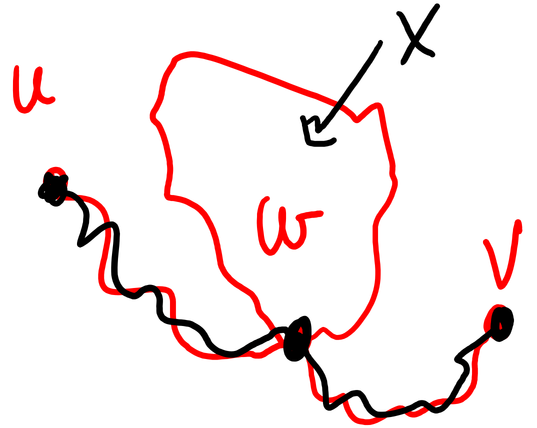
u and x ✗



leaf: degree 1 nodes in tree



Every u, v-walk contains a u, v-path.



walk - can have repeated edges

path - NO repeated vertices or edges

induct on length of u,v-walk, n

$p(n)$: u,v-walk of length n contains u,v path

Base case: $n=0$ ✓ $(u=v)$

Inductive step: fix $k \geq 0$, suppose $p(0) \wedge \cdots \wedge p(k)$ true

show $p(k+1)$ true

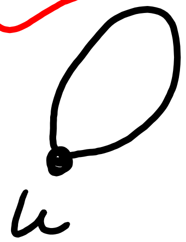
Repeat vertex in u,v-walk?

1. No → done u,v-path already
2. YES → remove closed walk on repeated vertex IH on remainder.

Every odd closed walk contains an odd cycle.

induction on odd closed walk, length n

$p(n) :=$ if closed walk has $2n+1$ edges then it has odd cycle



Base : $n=0$ closed walk has 1 edge is odd cycle. ✓

Inductive Step : fix $k \geq 0$, $p(0) \wedge \cdots \wedge p(k)$ true $\rightarrow$ show $p(k+1)$

REPEAT VERTEX ?

1. NO ✓
2. consider edged between repeated vertex $w \neq u$. one cycle odd. IH.

Recap

- Discrete Probability
  - Probability distributions
  - Expectation
  - Variance
- Pigeonhole Principle + Examples
- Graph Theory
  - Motivation: Bridges of Königsberg
  - Definitions: walk, trail, path, cycle
  - Basic properties (proof by induction)

Next time… more graph theory and structural induction.