Week 8: Regular Languages and Finite Automaton

CSC 236:Introduction to the Theory of Computation

Summer 2024

Instructor: Lily

Announcement

- Midterm 2: 6:00~8:00pm EX 100 (July 17)
  - Bonus Question (midterm 2). Course material (up to and including W7)
  - Missed Exam? Submit request by July 22 (documents by July 25)
- Structure for finite automaton section:
  - Week 8: DFA, NFA, and regular expressions
  - Week 9: Proof of correctness of finite automatons, equivalence of DFA-NFA-regular expressions
  - Week 10: Limitations of regular languages, pumping lemma
- Tutorials this week: more examples of DFAs and NFAs

Regular Languages

General Definitions of Languages

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	English	**Boolean**	**Hexadecimal**
Alphabet			

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Language	all words	$\{0, 1\}^*$	$\{c \in \Sigma^* : \lvert c\rvert = 6\}$

Operations on Languages

- **Union**: Let I and J be languages over the alphabet Σ, then the language
$$I \cup J = \{x \in \Sigma^* : x \in I \text{ or } x \in J\}.$$
- **Concatenation**: Let I and J be as the above, then the language
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- **Kleene Star**: Let I be as above, then
$$I^{\circledast} = \{x \in \Sigma^* : x \in I^k \text{ for any } k \in \mathbb{N}\}$$

$I = \{w \in \{0,1\}^* : w = \underbrace{0 \cdots 0}_{n}, n \in \mathbb{N}\}$

$J = \{w \in \{0,1\}^* : w = \underbrace{1 \cdots 1}_{n}, n \in \mathbb{N}\}$

$I \cup J = \{w \in \{0,1\}^* : w = \underbrace{0 \cdots 0}_{n}$ OR $w = \underbrace{1 \cdots 1}_{n}, n \in \mathbb{N}\}$

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Example.

1. Σ^* (take the Kleene Star of $\{a \in \Sigma\}$).
2. Over the Boolean alphabet $\Sigma = \{0, 1\}$, the language

$$L = \{w \in \Sigma : w = \epsilon \text{ or } w = 01 \cdots 01\}.$$

Regular Expressions

Overview

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... or even more succinctly still

$$(0+1)1^*0$$

Regular Expressions

Let R and S be regular expressions. Then valid operations include:

$R + S$: formally known as **alternation**, but can be read as *or*. Analogous to the $\cup$ operation on languages.

RS: formally known as **concatenation**. Analogous to the $\circ$ operation.

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Regular expression R **represents** the language $\mathcal{L}(R)$ (in the previous example $L = \mathcal{L}((0 + 1)1^*0)$).

Finite Automaton

Deterministic Finite Automaton (DFA)

Definition

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DFA M **accepts** the language $\mathcal{L}(M)$.

DFA Example

Consider the following DFA $M = (Q, \{0, 1\}, \delta, s, F)$ where:

- $Q = \{q_0, q_1, q_2, q_3\}$.
- $s = q_0$.
- $F = \{q_3\}$.

State	0	1
q_0	q_0	q_1
q_1	q_1	q_2
q_2	q_2	q_3
q_3	q_3	q_3

Table 1: Transition table for δ.

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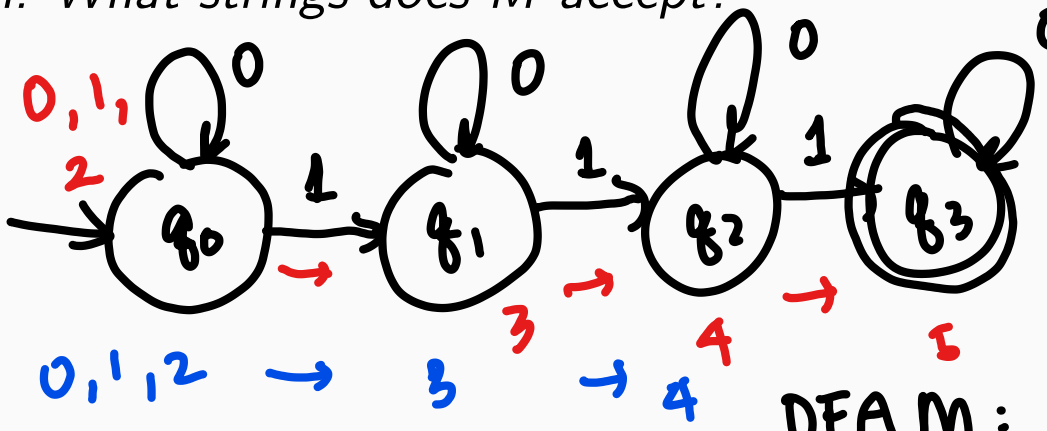
Table 1: Transition table for δ.

Question: What strings does M accept?



S = 00111 accepted by M

S_2 = 0011 NOT accepted by M

DFA M: accepts strings with ≥ 3 "1"s.

Now you try!

Design a DFA which accepts the following languages

1. $L_1 = \{w \in \{0,1\}^* : w \text{ starts with } 0 \text{ and ends with } 1\}$.
2. $L_2 = \{w \in \{0,1\}^* : w \text{ contains an odd number of } 1\}$.
3. $L_3 = \{0^n 1^m : m, n \in \mathbb{N}, m + n \text{ is even}\}$.

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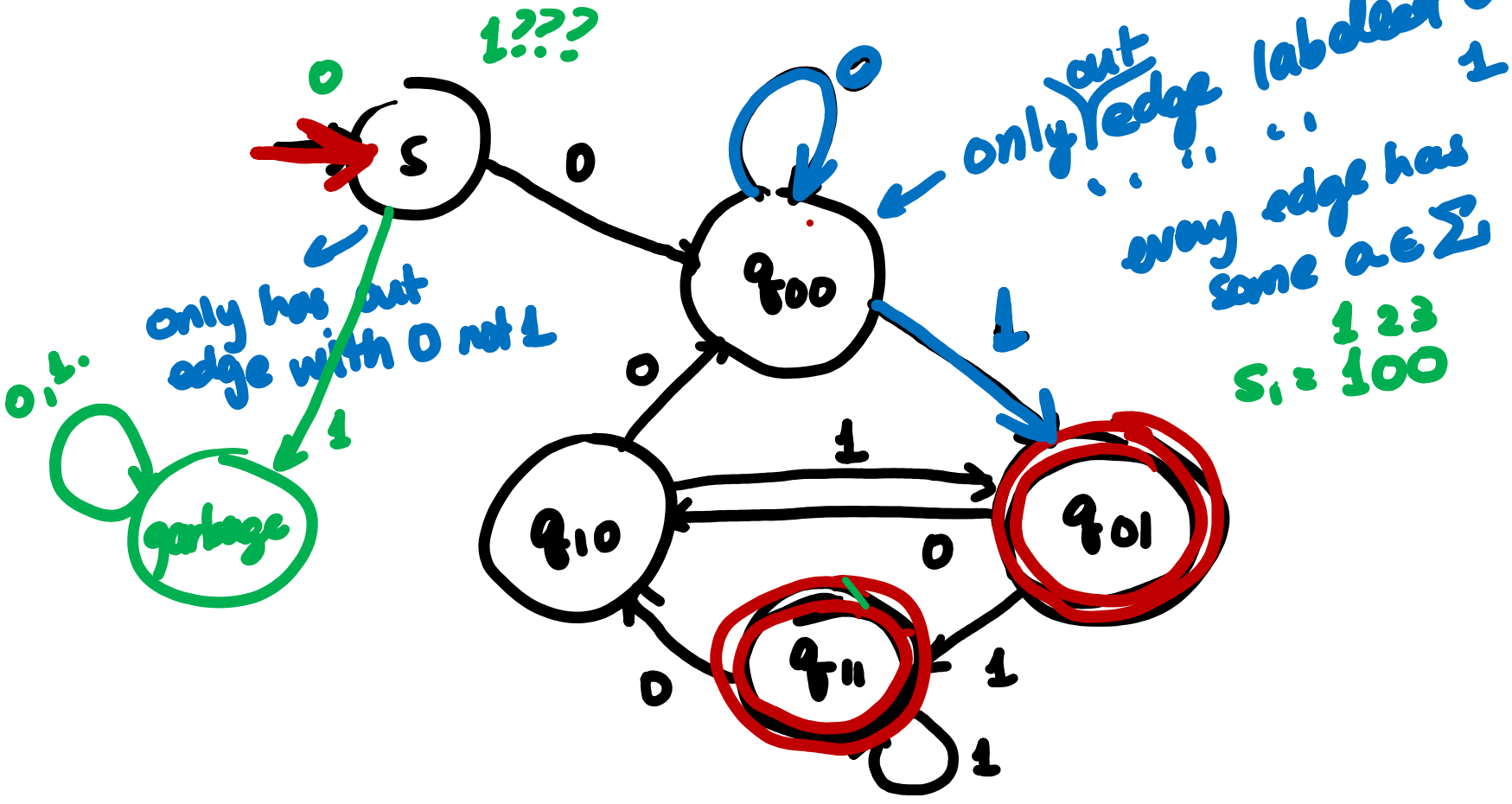

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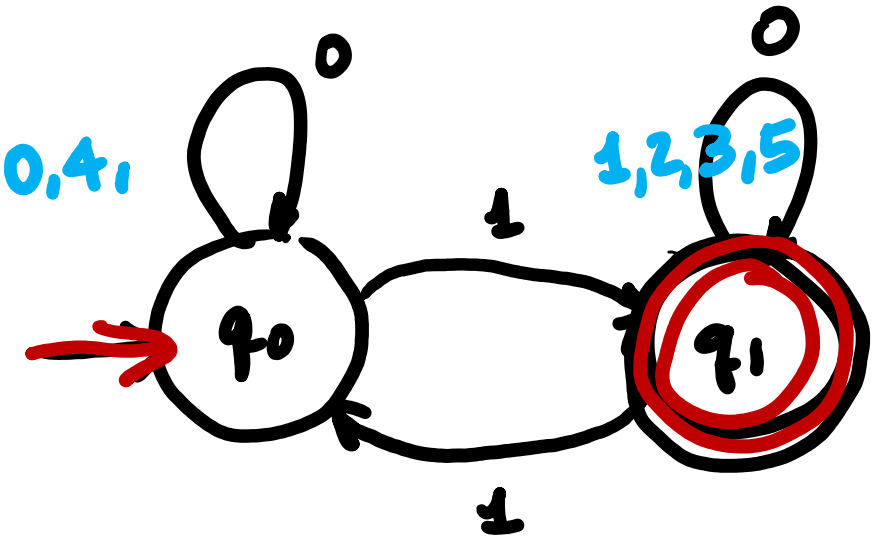
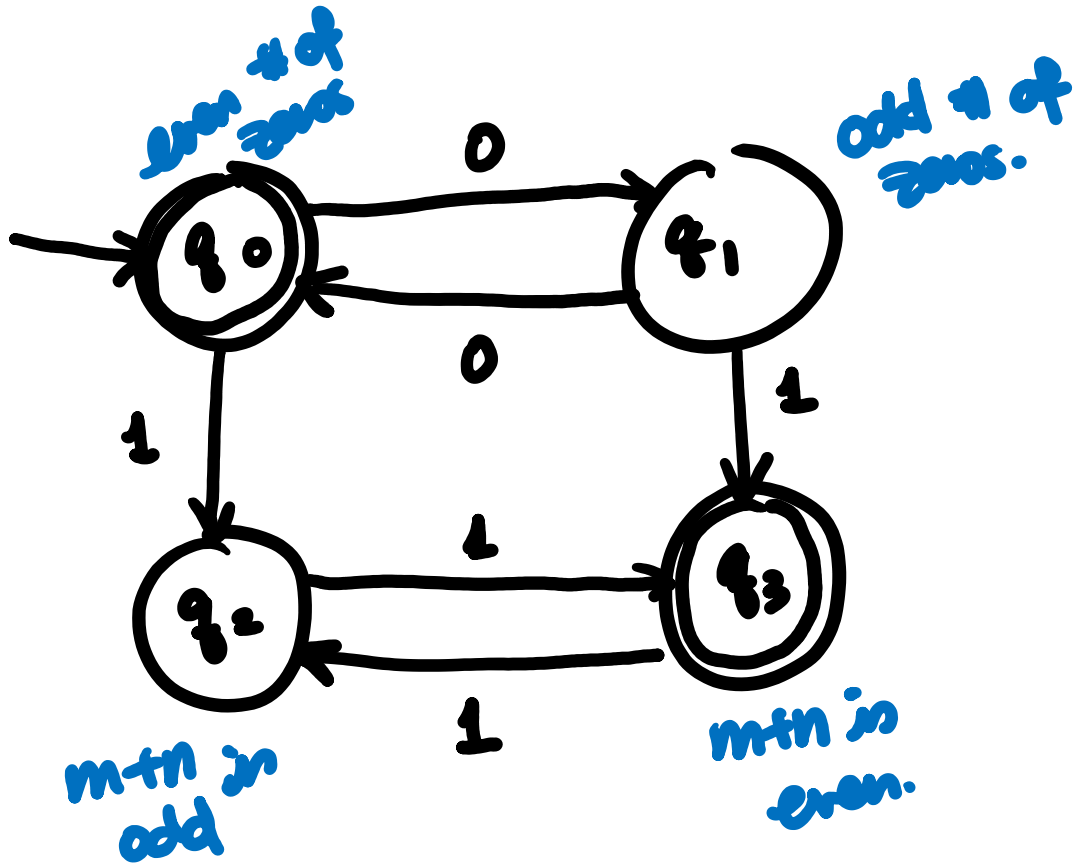


1 2 3 4 5
$S_1 =$ 1 0 0 1 1

$L_3 = \{0^n 1^m : m, n \in \mathbb{N}, m + n \text{ is even}\}.$



Intuition

First some intuition for what an automaton is: consider *Manufactoria*.

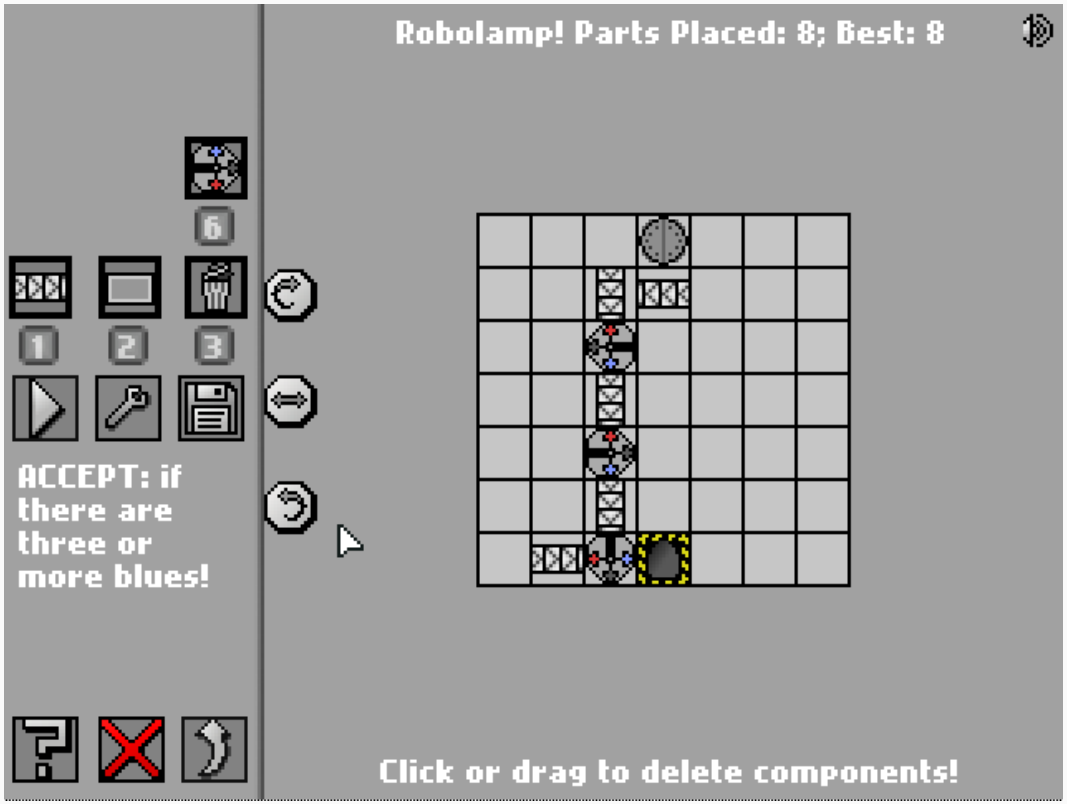


Figure 1: Putting robots in their place!

Non-Deterministic Finite Automaton (NFA)

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NFA

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- Q is a *finite set* of **states**.
- Σ is a finite **alphabet**.
- $\delta : Q \times \{\Sigma \cup \{\epsilon\}\} \to \mathcal{P}(Q)$ is the **transition function** where $\mathcal{P}(Q)$ is the *power set* of Q (the set of all subsets of Q). That is to say: δ outputs a set of states instead of a single state.
- s is *the* start or **initial** state.
- $F \subset Q$ is the *set* of **accepting states**.

NFA Example

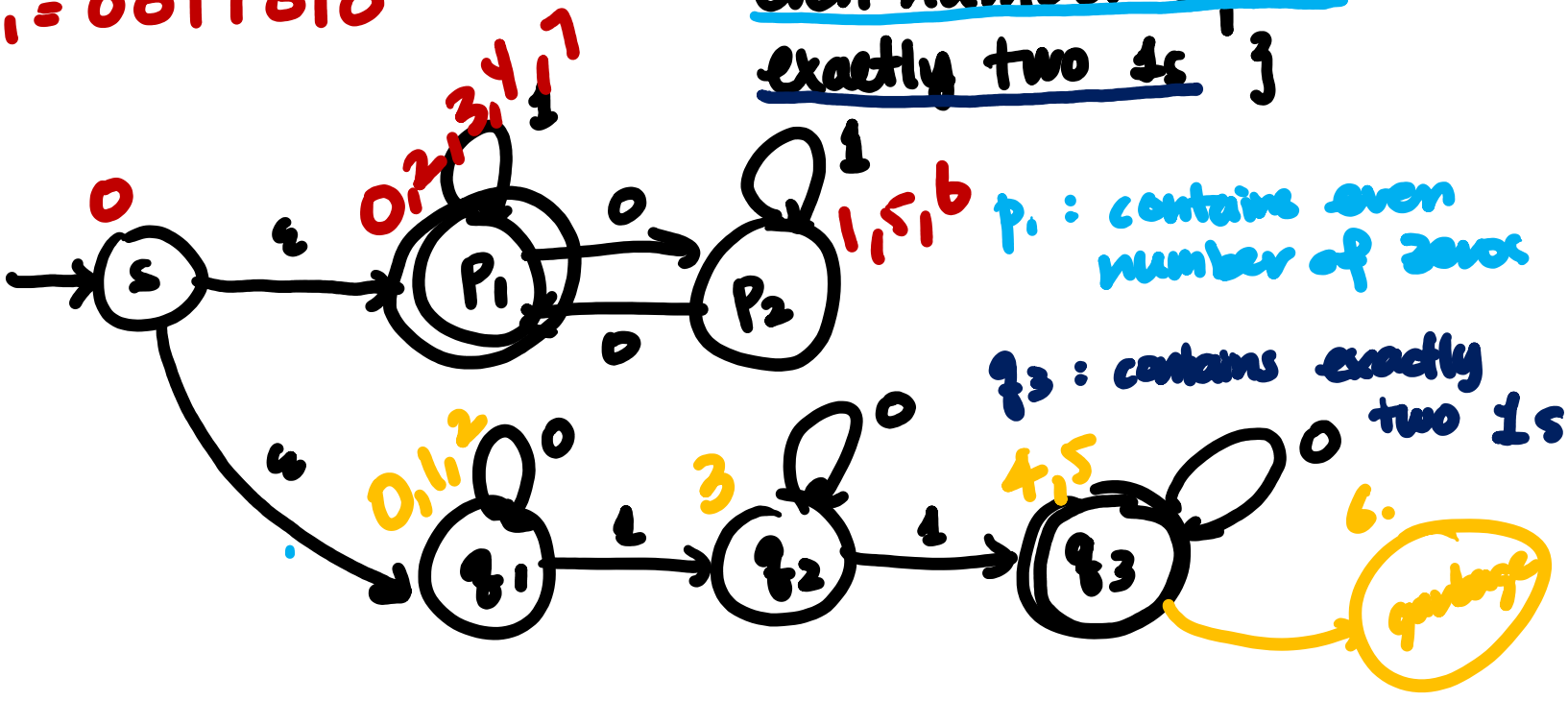
We want to construct an NDFA M which accepts language:

$L = \{w \in \{0,1\}^* : w \text{ contains an even number of 0s or exactly two 1s}\}$.

Construct NFA M which accepts:

$L = \{ w \in \{0,1\}^* : w$ contains an even number of 0s or exactly two 1s $\}$

$s_1 = 0\,0\,1\,1\,0\,1\,0$ (positions 1 2 3 4 5 6 7)



RECAP:

1. Language L : over alphabet Σ (Σ^* : all strings over alphabet)
   - strings : composed of $a \in \Sigma$
   - L is <u>set of strings</u> ($L \subseteq \Sigma^*$)
   - OPS: Union ($\cup$), concat ($\circ$), Kleene Star ($*$)

2. Regular Expressions — alternation ($+$), concat ($\cdot$), repetition ($*$) reps.

3. FINITE AUTOMATONS : DFA, NFA

DFA	NFA
each state $q \in Q$ has 1 transition for each $a \in \Sigma$	$q \in Q$ has any # of transitions on $a \in \Sigma$
ε-transition NOT allowed	ε-transition allowed