CSC236H1Y **Summer 2024**

Lecture 2: Generating Functions

Date: May 23, 2024

2.1 Recall

This tutorial is a bit more interactive than the last. Go through the material and pause to let the students think before moving on.

Recall the Fibonacci numbers f_n we saw in the first lecture. $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$. The first few terms of the sequence are $0, 1, 1, 3, 5, 8, 13, 21, \ldots$. We also saw the closed-form expression

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

In this tutorial we will see how this equation comes about. We will need:

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n \tag{2.1}$$

for $|x| < 1$. *Ask the students if they are familiar with this and explain it if they are not.*

2.2 Warming Up

Before we deal with a three term sequence, let's first consider a two term sequence $a_0, a_1, \ldots$ where $a_{n+1} = 2a_n + 1$ for $n \geq 0$ and $a_0 = 0$. The first few terms of this sequence is $0, 1, 3, 7, 15, 31\ldots$. *Question: is there a clear pattern? Answer: seems to be $a_n = 2^n - 1$.* Let's go about this more systematically. In the method of generating function we want to find a function $A(x)$ such that $A(x) = \sum_{n\geq 0} a_n x^n$. $A(x)$ is the *generating function* for the sequence a_n. *$A(x)$ is the "clotheline" upon which we hang the sequence $a_0, a_1, \ldots$.* Once we know $A(x)$ we can read off an explicit formula for a_n by looking at the coefficient of x^n in $A(x)$.

There's how we find $A(x)$. We take the equation $a_{n+1} = 2a_n + 1$ and multiply both sides by x^n to get $a_{n+1}x^n = x^n(2a_n + 1)$. Summing over all terms for $n \geq 0$, we have $\sum_{n\geq 0} a_{n+1}x^n = \sum_{n\geq 0} x^n(2a_n + 1)$. Now, the LHS of the equation is:

$$\begin{aligned}
\sum_{n\geq 0} a_{n+1}x^n &= a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \cdots \\
&= \frac{\big((a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) - a_0\big)}{x} \\
&= \frac{A(x)}{x}
\end{aligned}$$

since $a_0 = 0$. *Go slowly here, this might be the first time the students have seen this kind of algebraic* *manipulation.* While the RHS of the equation is:

$$\begin{aligned}\sum_{n\geq 0}(2a_n+1)x^n &= 2A(x)+\sum_{n\geq 0}x^n \\ &= 2A(x)+\frac{1}{1-x}\end{aligned}$$

Comparing these two equations we have:

$$\frac{A(x)}{x} = 2A(x)+\frac{1}{1-x} \implies A(x) = \frac{x}{(1-x)(1-2x)}.$$

This is the closed form expression for $A(x)$, but we need a way to easily extract the coefficients. In particular, we want to write $A(x)$ as a sum of terms like those in Equation 2.1. Find C and D such that

$$\frac{1}{(1-x)(1-2x)} = \frac{C}{1-x}+\frac{D}{1-2x}.$$

Give the students 10 minutes to do this. It turns out that $\frac{1}{(1-x)(1-2x)} = \frac{-1}{1-x}+\frac{2}{1-2x}$ and

$$A(x) = \frac{2x}{1-2x}-\frac{x}{1-x} = \left(\sum_{n\geq 1}(2x)^n\right)-\left(\sum_{n\geq 1}x^n\right) = \sum_{n\geq 1}\left(2^n-1\right)x^n.$$

2.3 Fibonacci Sequence

Now let's apply the same procedure to the Fibonacci Numbers. Recall that $f_{n+2} = f_{n+1}+f_n$. Find the closed form of the generating function $F(x) = \sum_{n\geq 0} f_n x^n$ for the sequence f_n. *Give the students 15 minutes to do this.* We multiply the definition of Fibonacci numbers by x^n on both sides and sum over all n to obtain

$$\left(f_2+f_3x+f_4x^2+\cdots\right) = \frac{F(x)-x}{x^2} \qquad \text{(LHS)}$$

$$\left(f_1+f_2x+f_3x^2+\cdots\right)+\left(f_0+f_1x+f_2x^2+\cdots\right) = \frac{F(x)}{x}+F(x) \qquad \text{(RHS)}$$

Solving for $F(x)$, we obtain $F(x) = \frac{x}{1-x-x^2}$. Note that $1-x-x^2 = (1-r_+x)(1-r_-x)$ with roots $\frac{1}{r_\pm}$ where $r_\pm = (1\pm\sqrt{5})/2$. We want to find C and D such that

$$\frac{x}{1-x-x^2} = \frac{C}{1-xr_+}+\frac{D}{1-xr_-}.$$

It turns out that $C = \frac{1}{r_+-r_-}$ while $D = \frac{-1}{r_+-r_-}$. Together we have

$$F(x) = \frac{1}{r_+-r_-}\left(\left(\sum_{n\geq 0}(r_+x)^n\right)-\left(\sum_{n\geq 0}(r_-x)^n\right)\right).$$

Picking out the coefficient of x^n, we get the closed form for f_n as desired.

Note that as $n\to\infty$, $r_+^n\to 0$ so we can get a great approximation to f_n by taking

$$f_n \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n. \qquad (2.2)$$

In fact, the error is *never greater than 0.5 in magnitude*, so f_n is just the nearest integer to Equation 2.2.

This is the first chapter of Generatingfunctionology *by Herbert S. Wilf.*