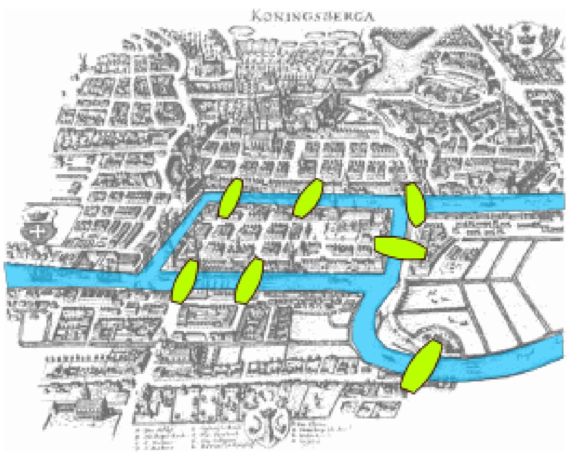
CSC236H1Y **Summer 2024**

Lecture 3: Eulerian Circuits 1

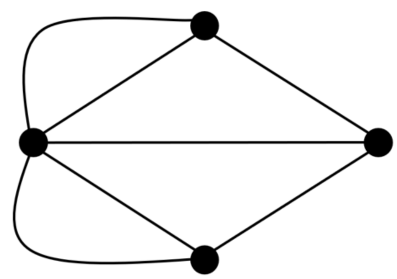
Date: May 29, 2024

3.1 Recall

In lecture three we learned about Euler and *Seven Bridges of Königsberg*. The configuration of the bridges are shown in Figure 3.1a. In-class we saw how the bridges correlate to the graph G shown in Figure 3.1b. Euler considered whether or not there exists a way to start and end on the same island while traversing all the bridges exactly once. Recall that the name of this graph structure is a *trail*. In honor of Euler, a trail which uses every edge in the graph *exactly once* is called an *Eulerian Trail*. A closed Eulerian Trail is called an *Eulerian Circuit*.



(a) Seven bridges of Königsberg



(b) Graph G representing the bridges

Figure 3.1: Picture and representation of the seven bridges of Königsburg

In today's tutorial we will show the following theorem.

Theorem 3.1. *A connected graph G has an Eulerian Circuit if and only if the degree of every node is even.*

Using Theorem 3.1, we can prove that it is *impossible* to start at one location, visit every bridge exactly once, and return to the same location: all four vertices have *odd* degree.

Ask what is required to prove an "if and only if" statement. They need to show (1) the necessity condition *where a connected graph G has an Eulerian Circuit implies the degree of every node is even (2) the* sufficiency condition *where if the degree of every node is even this implies the graph G has an Eulerian Circuit. Give the students a few minutes to consider the proof of necessity.*

Proof of Necessity. Suppose that G has an Eulerian circuit C. Each passage of C through a vertex uses two incident edges. Further the start vertex, which is same as the end vertex must have an even number of edges in addition to the first edge and the last edge. Hence every vertex has even degree. □

3.2 Proof of Sufficiency

In order to prove that "the degree of every node in G is even implies that G an Eulerian Circut", we will prove the following lemma. *State and write Lemma 3.2 on the board. Remember: cycle is a circuit with no repeated vertices. Ask the students to prove Lemma 3.2 and use it to prove sufficiency. Give students 20 minutes to do this.*

Lemma 3.2. *If every vertex of a graph G has degree at least two, then G contains a cycle.*

Proof. Let P be a maximal path[1] in G, and u be an endpoint of P. Since P cannot be extended, every neighbour of u must already be a vertex of P. Since u has degree at least two, it has a neighbour v in $V(P)$ — the vertices on P — via an edge not in P. The edge (u, v) completes a cycle with the portion of P from v to u. *Drawing a picture on the board makes the explanation much clearer.* $\square$

Proof of Sufficiency. Assume that every node in the connected graph G has even degree. We will show that G has an Eulerian Circuit by induction on the number of edges, m.

In the base case $m = 0$. Since the graph is connected, it must be the case that G contains a single vertex. This is trivially an Eulerian Circuit.

Suppose $k > 0$ and that all graphs with $0, 1, ..., k$ edges, and whose vertices all have degree at least two, contain an Eulerian Circuit. We will show that all such graphs with $k + 1$ edges contain an Eulerian circuit. Since the graph is connected and the degree of every node is even, every node must have degree at least two. By Lemma 3.2, there exists a cycle in G, denoted C. Let $E(C)$ be the edges of C and G' be the graph obtained from G by deleting the edges $E(C)$.

Note that G' *may* consist of disconnected components. Since C has zero or two edges at each vertex (zero if vertex is not on C and two if it is), each component of G' is also a graph where every vertex has even degree. By the induction hypothesis, we know that each component has a Eulerian Circuit (they have k or fewer edges). Combined with C, this forms an Eulerian circuit of G. $\square$

If there is still time, you can give them the following puzzle. Consider the classic Five-rooms Puzzle shown in Figure 3.2. The goal of the puzzle is to cross each "wall" of the diagram with a continuous line only once. Is this possible?

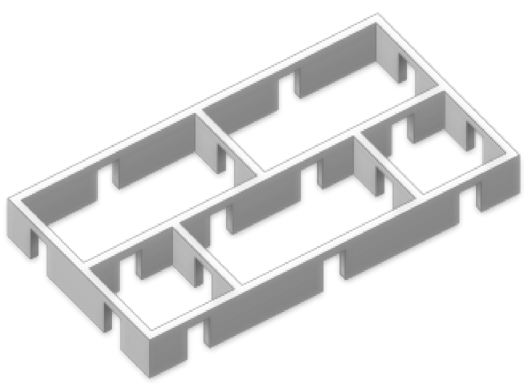


Figure 3.2: Five-rooms Puzzle

This is in Section 1.2 of Introduction to Graph Theory 2nd ed. by Douglas B. West.

[1] A path is a trail with no repeated vertices.