Time

$\sigma = t\,;\; x\,;\; y\,;\; ...$

state = time variable; memory variables

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t is the time at which execution starts

t' is the time at which execution ends

Time

$\sigma = t ; x ; y ; ...$

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t is the time at which execution starts

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t, t': *xnat* or §r: *xreal*· $r \geq 0$

Time

$\sigma \;=\; t\,;\; x\,;\; y\,;\; ...$

state = time variable; memory variables

t is the time at which execution starts

t' is the time at which execution ends

t, t': *xnat* or $\S r$: *xreal*$\cdot$ $r{\ge}0$

Specification S is **implementable** if and only if

$\forall\sigma\cdot\ \exists\sigma'\cdot\ S \wedge t'{\ge}t$

real time

real time

$x := e$

real time

t:= t+(the time to evaluate and store e). x:= e

real time

t:= t+(the time to evaluate and store e). x:= e

if b **then** P **else** Q **fi**

real time

$t:= t+$(the time to evaluate and store e). $x:= e$

$t:= t+$(the time to evaluate b and branch). **if** b **then** P **else** Q **fi**

real time

$t{:=}\ t+$(the time to evaluate and store e). $x{:=}\ e$

$t{:=}\ t+$(the time to evaluate b and branch). **if** b **then** P **else** Q **fi**

P

real time

$t := t+$(the time to evaluate and store e). $x := e$

$t := t+$(the time to evaluate b and branch).   **if** b **then** P **else** Q **fi**

$t := t+$(the time for the call and return). P

real time

$t := t + (\text{the time to evaluate and store } e)$. $x := e$

$t := t + (\text{the time to evaluate } b \text{ and branch})$. **if** b **then** P **else** Q **fi**

$t := t + (\text{the time for the call and return})$. P

$$t' = t + f\,\sigma$$

real time

$t := t +$(the time to evaluate and store e). $x := e$

$t := t +$(the time to evaluate b and branch). **if** b **then** P **else** Q **fi**

$t := t +$(the time for the call and return). P

$$t' = t + f\,\sigma$$

$$t' \le t + f\,\sigma$$

real time

$t := t + (\text{the time to evaluate and store } e)$. $x := e$

$t := t + (\text{the time to evaluate } b \text{ and branch})$.   **if** b **then** P **else** Q **fi**

$t := t + (\text{the time for the call and return})$. P

$$t' = t + f\,\sigma$$

$$t' \le t + f\,\sigma$$

$$t' \ge t + f\,\sigma$$

real time

$P \Leftarrow$ **if** $x=0$ **then** *ok* **else** $x:= x-1.$ P **fi**

real time

$P \Leftarrow t{:=}\ t{+}1.$ **if** $x{=}0$ **then** *ok* **else** $x{:=}\ x{-}1.$ P **fi**

real time

$P \Leftarrow t{:=}\ t+1.$ **if** $x{=}0$ **then** *ok* **else** $t{:=}\ t+1.\ \ x{:=}\ x-1.\qquad P$ **fi**

real time

$P \Leftarrow t{:=}\ t{+}1.\ \textbf{if}\ x{=}0\ \textbf{then}\ ok\ \textbf{else}\ \ t{:=}\ t{+}1.\ \ x{:=}\ x{-}1.\ t{:=}\ t{+}1.\ P\ \textbf{fi}$

real time

$P \Leftarrow t{:=}\ t+1.\ \textbf{if}\ x=0\ \textbf{then}\ ok\ \textbf{else}\ t{:=}\ t+1.\ x{:=}\ x-1.\ t{:=}\ t+1.\ P\ \textbf{fi}$

is a theorem when

$P \;=\; x'=0$

$P \;=\; \textbf{if}\ x\ge 0\ \textbf{then}\ t'=t+3\times x+1\ \textbf{else}\ t'=\infty\ \textbf{fi}$

$P \;=\; \textbf{if}\ x\ge 0\ \textbf{then}\ x'=0 \land t'=t+3\times x+1\ \textbf{else}\ t'=\infty\ \textbf{fi}$

$P \;=\; x'=0 \;\land\; \textbf{if}\ x\ge 0\ \textbf{then}\ t'=t+3\times x+1\ \textbf{else}\ t'=\infty\ \textbf{fi}$

real time

$P \Leftarrow$ $t{:=}\, t{+}1.$ **if** $x{=}0$ **then** ok **else** $t{:=}\, t{+}1.\ \ x{:=}\, x{-}1.\ t{:=}\, t{+}1.\ P$ **fi**

is a theorem when

$P \;=\; x'{=}0$ ⟵

$P \;=\;$ **if** $x{\ge}0$ **then** $t'{=}t{+}3{\times}x{+}1$ **else** $t'{=}\infty$ **fi**

$P \;=\;$ **if** $x{\ge}0$ **then** $x'{=}0 \land t'{=}t{+}3{\times}x{+}1$ **else** $t'{=}\infty$ **fi**

$P \;=\; x'{=}0 \;\land\;$ **if** $x{\ge}0$ **then** $t'{=}t{+}3{\times}x{+}1$ **else** $t'{=}\infty$ **fi**

real time

$P \Leftarrow t{:=}\ t+1.\ $ **if** $x{=}0$ **then** ok **else** $t{:=}\ t+1.\ \ x{:=}\ x-1.\ t{:=}\ t+1.\ P$ **fi**

is a theorem when

$P\ =\ x'{=}0$

$P\ =$ **if** $x{\geq}0$ **then** $t'{=}t+3{\times}x+1$ **else** $t'{=}\infty$ **fi** ⟵

$P\ =$ **if** $x{\geq}0$ **then** $x'{=}0 \wedge t'{=}t+3{\times}x+1$ **else** $t'{=}\infty$ **fi**

$P\ =\ x'{=}0\ \wedge$ **if** $x{\geq}0$ **then** $t'{=}t+3{\times}x+1$ **else** $t'{=}\infty$ **fi**

real time

$P \Leftarrow t:= t+1.$ **if** $x=0$ **then** ok **else** $t:= t+1.\ x:= x-1.\ t:= t+1.\ P$ **fi**

is a theorem when

$P = x'=0$

$P =$ **if** $x\ge 0$ **then** $t'=t+3\times x+1$ **else** $t'=\infty$ **fi**

$P =$ **if** $x\ge 0$ **then** $x'=0 \wedge t'=t+3\times x+1$ **else** $t'=\infty$ **fi** ⟵

$P = x'=0 \wedge$ **if** $x\ge 0$ **then** $t'=t+3\times x+1$ **else** $t'=\infty$ **fi**

real time

$P \Leftarrow$ $t{:=}\ t+1.$ **if** $x=0$ **then** ok **else** $t{:=}\ t+1.\ \ x{:=}\ x-1.\ t{:=}\ t+1.\ P$ **fi**

is a theorem when

$P\ =\ x'=0$

$P\ =$ **if** $x\ge 0$ **then** $t'=t+3\times x+1$ **else** $t'=\infty$ **fi**

$P\ =$ **if** $x\ge 0$ **then** $x'=0\ \land\ t'=t+3\times x+1$ **else** $t'=\infty$ **fi**

$P\ =\ x'=0\ \land$ **if** $x\ge 0$ **then** $t'=t+3\times x+1$ **else** $t'=\infty$ **fi** ⟵

recursive time

- Each recursive call costs time 1.
- All else is free.

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$P \Leftarrow$ **if** x=0 **then** *ok* **else** x:= x–1. t:= t+1. P **fi**

recursive time

- Each recursive call costs time 1.
- All else is free.

$P \Longleftarrow$ **if** $x=0$ **then** *ok* **else** $x:= x-1.\ t:= t+1.\ P$ **fi**

is a theorem when

$P = x'=0$

$P =$ **if** $x\geq 0$ **then** $t'=t+x$ **else** $t'=\infty$ **fi**

$P =$ **if** $x\geq 0$ **then** $x'=0 \land t'=t+x$ **else** $t'=\infty$ **fi**

$P = x'=0 \land$ **if** $x\geq 0$ **then** $t'=t+x$ **else** $t'=\infty$ **fi**

recursive time

- Each recursive call costs time 1.
- All else is free.

$P \;\Leftarrow\;$ **if** $x=0$ **then** ok **else** $x:= x-1.\; t:= t+1.\; P$ **fi**

is a theorem when

$P \;=\; x'=0$

$P \;=\;$ **if** $x\ge 0$ **then** $t'=t+x$ **else** $t'=\infty$ **fi**

$P \;=\;$ **if** $x\ge 0$ **then** $x'=0 \wedge t'=t+x$ **else** $t'=\infty$ **fi**

$P \;=\; x'=0 \;\wedge\;$ **if** $x\ge 0$ **then** $t'=t+x$ **else** $t'=\infty$ **fi**

Recursion can be direct or indirect.

In every loop of calls, there must be a time increment of at least one time unit.

Prove $R \Leftarrow$ **if** x=1 **then** *ok* **else** x:= *div* x 2. t:= t+1. R **fi**

Prove $R \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $R = x'=1 \land$ **if** $x\ge 1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R = x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$= x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R\ =\ x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$=\ x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts; prove:

$x'{=}1 \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x'{=}1$ **fi**

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{\ge}1 \Rightarrow t' \le t + log\ x$ **fi**

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{<}1 \Rightarrow t'{=}\infty$ **fi**

Prove $R \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $R = x'=1 \land$ **if** $x\ge 1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$= x'=1 \land (x\ge 1 \Rightarrow t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$

use Refinement by Parts and Cases; prove:

$x'=1 \Leftarrow x=1 \land ok$

$x'=1 \Leftarrow x\ne 1 \land (x:= div\ x\ 2.\ \ t:= t+1.\ \ x'=1)$

$x\ge 1 \Rightarrow t' \le t + log\ x \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ x\ge 1 \Rightarrow t' \le t + log\ x$ **fi**

$x<1 \Rightarrow t'=\infty \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ x<1 \Rightarrow t'=\infty$ **fi**

Prove $R \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $R = x'=1 \land$ **if** $x\ge 1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$= x'=1 \land (x\ge 1 \Rightarrow t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$

use Refinement by Parts and Cases; prove:

$x'=1 \Leftarrow x=1 \land ok$

$x'=1 \Leftarrow x\ne 1 \land (x:= div\ x\ 2.\ \ t:= t+1.\ \ x'=1)$

$x\ge 1 \Rightarrow t' \le t + log\ x \Leftarrow x=1 \land ok$

$x\ge 1 \Rightarrow t' \le t + log\ x \Leftarrow x\ne 1 \land (x:= div\ x\ 2.\ \ t:= t+1.\ \ x\ge 1 \Rightarrow t' \le t + log\ x)$

$x<1 \Rightarrow t'=\infty \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ x<1 \Rightarrow t'=\infty$ **fi**

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R \ = \ x'{=}1 \wedge$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$= \ x'{=}1 \wedge (x{\ge}1 \ \Rightarrow\ t' \le t + log\ x) \wedge (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \ \Leftarrow\ x{=}1 \wedge ok$

$x'{=}1 \ \Leftarrow\ x{\ne}1 \wedge (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x'{=}1)$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \ \Leftarrow\ x{=}1 \wedge ok$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \ \Leftarrow\ x{\ne}1 \wedge (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{\ge}1 \Rightarrow t' \le t + log\ x)$

$x{<}1 \ \Rightarrow\ t'{=}\infty \ \Leftarrow\ x{=}1 \wedge ok$

$x{<}1 \ \Rightarrow\ t'{=}\infty \ \Leftarrow\ x{\ne}1 \wedge (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{<}1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R = x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$= x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \Leftarrow x{=}1 \land ok$ ←

$x'{=}1 \Leftarrow x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x'{=}1)$

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{=}1 \land ok$ ←

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{\ge}1 \Rightarrow t' \le t + log\ x)$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{=}1 \land ok$ ←

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{<}1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R = x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$= x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$

$x'{=}1 \Leftarrow x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x'{=}1)$

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{\ge}1 \Rightarrow t' \le t + log\ x)$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{<}1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R \ =\ x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$=\ x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \ \Leftarrow\ x{=}1 \land x'{=}x \land t'{=}t$

$x'{=}1 \ \Leftarrow\ x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x'{=}1)$ ⟵

$x{\ge}1 \Rightarrow t' \le t + log\ x \ \Leftarrow\ x{=}1 \land x'{=}x \land t'{=}t$

$x{\ge}1 \Rightarrow t' \le t + log\ x \ \Leftarrow\ x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{\ge}1 \Rightarrow t' \le t + log\ x)$

$x{<}1 \Rightarrow t'{=}\infty \ \Leftarrow\ x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \Rightarrow t'{=}\infty \ \Leftarrow\ x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{<}1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** x=1 **then** ok **else** x:= $div\ x$ 2. t:= t+1. R **fi**

where $R = x'=1 \land$ **if** $x \ge 1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$= x'=1 \land (x\ge1 \Rightarrow t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$

use Refinement by Parts and Cases; prove:

$x'=1 \Leftarrow x=1 \land x'=x \land t'=t$

$x'=1 \Leftarrow x \ne 1 \land x'=1$

$x\ge1 \Rightarrow t' \le t + log\ x \Leftarrow x=1 \land x'=x \land t'=t$

$x\ge1 \Rightarrow t' \le t + log\ x \Leftarrow x \ne 1 \land (x:= div\ x\ 2.\ t:= t+1.\ x\ge1 \Rightarrow t' \le t + log\ x)$

$x<1 \Rightarrow t'=\infty \Leftarrow x=1 \land x'=x \land t'=t$

$x<1 \Rightarrow t'=\infty \Leftarrow x \ne 1 \land (x:= div\ x\ 2.\ t:= t+1.\ x<1 \Rightarrow t'=\infty)$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $\quad R \quad = \quad x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$\qquad\qquad = \quad x'{=}1 \land (x{\ge}1 \ \Rightarrow\ t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x'{=}1 \quad \Leftarrow \quad x{\ne}1 \land x'{=}1$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{\ge}1 \Rightarrow t' \le t + log\ x)$ $\quad \longleftarrow$

$x{<}1 \ \Rightarrow\ t'{=}\infty \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \ \Rightarrow\ t'{=}\infty \quad \Leftarrow \quad x{\ne}1 \land (x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ x{<}1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $R = x'=1 \land$ **if** $x\ge1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$= x'=1 \land (x\ge1 \ \Rightarrow\ t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$

use Refinement by Parts and Cases; prove:

$x'=1 \ \Leftarrow\ x=1 \land x'=x \land t'=t$

$x'=1 \ \Leftarrow\ x\ne1 \land x'=1$

$x\ge1 \ \Rightarrow\ t' \le t + log\ x \ \Leftarrow\ x=1 \land x'=x \land t'=t$

$x\ge1 \ \Rightarrow\ t' \le t + log\ x \ \Leftarrow\ x\ne1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))$

$x<1 \ \Rightarrow\ t'=\infty \ \Leftarrow\ x=1 \land x'=x \land t'=t$

$x<1 \ \Rightarrow\ t'=\infty \ \Leftarrow\ x\ne1 \land (x:= div\ x\ 2.\ \ t:= t+1.\ \ x<1 \Rightarrow t'=\infty)$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $\quad R \quad = \quad x'=1 \land$ **if** $x\geq 1$ **then** $t' \leq t + log\ x$ **else** $t'=\infty$ **fi**

$$= \quad x'=1 \land (x\geq 1 \ \Rightarrow\ t' \leq t + log\ x) \land (x<1 \Rightarrow t'=\infty)$$

use Refinement by Parts and Cases; prove:

$$x'=1 \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$$

$$x'=1 \quad \Leftarrow \quad x\neq 1 \land x'=1$$

$$x\geq 1 \ \Rightarrow\ t' \leq t + log\ x \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$$

$$x\geq 1 \ \Rightarrow\ t' \leq t + log\ x \quad \Leftarrow \quad x\neq 1 \land (div\ x\ 2 \geq 1 \ \Rightarrow\ t' \leq t + 1 + log\ (div\ x\ 2))$$

$$x<1 \ \Rightarrow\ t'=\infty \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$$

$$x<1 \ \Rightarrow\ t'=\infty \quad \Leftarrow \quad x\neq 1 \land (x:= div\ x\ 2.\ \ t:= t+1.\ \ x<1 \Rightarrow t'=\infty) \qquad \longleftarrow$$

Prove $R \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $R \ = \ x'=1 \land$ **if** $x\ge 1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$= \ x'=1 \land (x\ge 1 \ \Rightarrow\ t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$

use Refinement by Parts and Cases; prove:

$x'=1 \ \Leftarrow\ x=1 \land x'=x \land t'=t$

$x'=1 \ \Leftarrow\ x\ne 1 \land x'=1$

$x\ge 1 \ \Rightarrow\ t' \le t + log\ x \ \Leftarrow\ x=1 \land x'=x \land t'=t$

$x\ge 1 \ \Rightarrow\ t' \le t + log\ x \ \Leftarrow\ x\ne 1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))$

$x<1 \ \Rightarrow\ t'=\infty \ \Leftarrow\ x=1 \land x'=x \land t'=t$

$x<1 \ \Rightarrow\ t'=\infty \ \Leftarrow\ x\ne 1 \land (div\ x\ 2 < 1 \ \Rightarrow\ t'=\infty)$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R = x'{=}1 \wedge$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$= x'{=}1 \wedge (x{\ge}1 \Rightarrow t' \le t + log\ x) \wedge (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \Leftarrow x{=}1 \wedge x'{=}x \wedge t'{=}t$ ←

$x'{=}1 \Leftarrow x{\ne}1 \wedge x'{=}1$

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{=}1 \wedge x'{=}x \wedge t'{=}t$

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{\ne}1 \wedge (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{=}1 \wedge x'{=}x \wedge t'{=}t$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{\ne}1 \wedge (div\ x\ 2 < 1 \Rightarrow t'{=}\infty)$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $\quad R \quad = \quad x'=1 \land$ **if** $x\ge1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$$= \quad x'=1 \land (x\ge1 \ \Rightarrow\ t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$$

use Refinement by Parts and Cases; prove:

$$x'=1 \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$$

$$x'=1 \quad \Leftarrow \quad x\ne1 \land x'=1$$

$$x\ge1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$$

$$x\ge1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x\ne1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))$$

$$x<1 \ \Rightarrow\ t'=\infty \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$$

$$x<1 \ \Rightarrow\ t'=\infty \quad \Leftarrow \quad x\ne1 \land (div\ x\ 2 < 1 \ \Rightarrow\ t'=\infty)$$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R\ =\ x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$=\ x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$$x'{=}1 \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$$

$$x'{=}1 \Leftarrow x{\ne}1 \land x'{=}1$$

$$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$$

$$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{\ne}1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$$

$$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$$

$$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{\ne}1 \land (div\ x\ 2 < 1 \Rightarrow t'{=}\infty)$$

Prove $\quad R \quad \Leftarrow \quad \textbf{if } x{=}1 \textbf{ then } ok \textbf{ else } x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R \textbf{ fi}$

where $\quad R \quad = \quad x'{=}1 \land \textbf{if } x{\ge}1 \textbf{ then } t' \le t + log\ x \textbf{ else } t'{=}\infty \textbf{ fi}$

$\qquad\qquad\quad = \quad x'{=}1 \land (x{\ge}1 \ \Rightarrow\ t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x'{=}1 \quad \Leftarrow \quad x{\ne}1 \land x'{=}1$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x{\ne}1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))$

$x{<}1 \ \Rightarrow\ t'{=}\infty \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \ \Rightarrow\ t'{=}\infty \quad \Leftarrow \quad x{\ne}1 \land (div\ x\ 2 < 1 \ \Rightarrow\ t'{=}\infty)$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $\quad R \quad = \quad x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$$= \quad x'{=}1 \land (x{\ge}1 \ \Rightarrow\ t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$$

use Refinement by Parts and Cases; prove:

$x'{=}1 \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x'{=}1 \quad \Leftarrow \quad x{\ne}1 \land x'{=}1$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x{\ne}1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))$

$x{<}1 \ \Rightarrow\ t'{=}\infty \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \ \Rightarrow\ t'{=}\infty \quad \Leftarrow \quad x{\ne}1 \land (div\ x\ 2 < 1 \ \Rightarrow\ t'{=}\infty)$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $\quad R \quad = \quad x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$$= \quad x'{=}1 \land (x{\ge}1 \ \Rightarrow\ t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$$

use Refinement by Parts and Cases; prove:

$x'{=}1 \ \Leftarrow\ x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x'{=}1 \ \Leftarrow\ x{\ne}1 \land x'{=}1$ ←

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \ \Leftarrow\ x{=}1 \land x'{=}x \land t'{=}t$

$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \ \Leftarrow\ x{\ne}1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))$

$x{<}1 \ \Rightarrow\ t'{=}\infty \ \Leftarrow\ x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \ \Rightarrow\ t'{=}\infty \ \Leftarrow\ x{\ne}1 \land (div\ x\ 2 < 1 \ \Rightarrow\ t'{=}\infty)$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $\quad R \quad = \quad x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$$= \quad x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$$

use Refinement by Parts and Cases; prove:

$x'{=}1 \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x'{=}1 \quad \Leftarrow \quad x{\ne}1 \land x'{=}1$ ←

↑ ↑

$x{\ge}1 \Rightarrow t' \le t + log\ x \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x{\ge}1 \Rightarrow t' \le t + log\ x \quad \Leftarrow \quad x{\ne}1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$

$x{<}1 \Rightarrow t'{=}\infty \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \Rightarrow t'{=}\infty \quad \Leftarrow \quad x{\ne}1 \land (div\ x\ 2 < 1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** *ok* **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where R $=$ $x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$= \ x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x'{=}1 \Leftarrow x{\ne}1 \land x'{=}1$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{\ne}1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{\ne}1 \land (div\ x\ 2 < 1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $R = x'=1 \land$ **if** $x \ge 1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$= x'=1 \land (x \ge 1 \Rightarrow t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$

use Refinement by Parts and Cases; prove:

$x'=1 \Leftarrow x=1 \land x'=x \land t'=t$ ✔

$x'=1 \Leftarrow x \ne 1 \land x'=1$ ✔

$x \ge 1 \Rightarrow t' \le t + log\ x \Leftarrow x=1 \land x'=x \land t'=t$ ⟵

$x \ge 1 \Rightarrow t' \le t + log\ x \Leftarrow x \ne 1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$

$x<1 \Rightarrow t'=\infty \Leftarrow x=1 \land x'=x \land t'=t$

$x<1 \Rightarrow t'=\infty \Leftarrow x \ne 1 \land (div\ x\ 2 < 1 \Rightarrow t'=\infty)$

$$(x\geq 1 \;\Rightarrow\; t' \leq t + log\ x \quad\Leftarrow\quad x=1 \wedge x'=x \wedge t'=t)$$

$(x \geq 1 \ \Rightarrow \ t' \leq t + log\ x \ \ \Leftarrow \ \ x{=}1 \wedge x'{=}x \wedge t'{=}t)$ context $x{=}1$ and $t'{=}t$

$= \quad (1 \geq 1 \ \Rightarrow \ t \ \leq t + log\ 1 \ \ \Leftarrow \ \ x{=}1 \wedge x'{=}x \wedge t'{=}t)$

$(x{\ge}1 \;\Rightarrow\; t' \le t + log\ x \;\;\Leftarrow\;\; x{=}1 \wedge x'{=}x \wedge t'{=}t)$ context $x{=}1$ and $t'{=}t$

$= \quad (1{\ge}1 \;\Rightarrow\; t \le t + log\ 1 \;\;\Leftarrow\;\; x{=}1 \wedge x'{=}x \wedge t'{=}t)$ simplify

$= \quad (\top \;\;\Leftarrow\;\; x{=}1 \wedge x'{=}x \wedge t'{=}t)$

$(x{\ge}1 \;\Rightarrow\; t' \le t + log\ x \;\Leftarrow\; x{=}1 \wedge x'{=}x \wedge t'{=}t)$ context $x{=}1$ and $t'{=}t$

$= \quad (1{\ge}1 \;\Rightarrow\; t \le t + log\ 1 \;\Leftarrow\; x{=}1 \wedge x'{=}x \wedge t'{=}t)$ simplify

$= \quad (\top \;\Leftarrow\; x{=}1 \wedge x'{=}x \wedge t'{=}t)$ base law

$= \quad \top$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R = x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$= x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x'{=}1 \Leftarrow x{\ne}1 \land x'{=}1$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{\ne}1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{\ne}1 \land (div\ x\ 2 < 1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x=1$ **then** ok **else** $x:= div\ x\ 2.\ \ t:= t+1.\ \ R$ **fi**

where $R = x'=1 \land$ **if** $x\ge 1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$= x'=1 \land (x\ge 1 \Rightarrow t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$

use Refinement by Parts and Cases; prove:

$x'=1 \Leftarrow x=1 \land x'=x \land t'=t$ ✔

$x'=1 \Leftarrow x\ne 1 \land x'=1$ ✔

$x\ge 1 \Rightarrow t' \le t + log\ x \Leftarrow x=1 \land x'=x \land t'=t$ ✔

$x\ge 1 \Rightarrow t' \le t + log\ x \Leftarrow x\ne 1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$ ←

$x<1 \Rightarrow t'=\infty \Leftarrow x=1 \land x'=x \land t'=t$

$x<1 \Rightarrow t'=\infty \Leftarrow x\ne 1 \land (div\ x\ 2 < 1 \Rightarrow t'=\infty)$

$$(x \ge 1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x \neq 1 \wedge (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2)))$$

$$(x \geq 1 \;\Rightarrow\; t' \leq t + log\ x \;\;\Leftarrow\;\; x \neq 1 \wedge (div\ x\ 2 \geq 1 \;\Rightarrow\; t' \leq t + 1 + log\ (div\ x\ 2)))$$

portation

$$a \wedge b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c)$$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a})$$

portation

$$a{\wedge}b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c)$$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \quad\Leftarrow\quad \underbrace{x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a})$$

portation

$$a{\wedge}b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c)$$

$$(b \Rightarrow c) \Leftarrow a$$

$$= \qquad a{\wedge}b \Rightarrow c$$

$$(\underbrace{x\ge 1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c}) \;\Leftarrow\; \underbrace{x \ne 1 \land (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a} \qquad \text{portation}$$

$$= \qquad x \ne 1 \land (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \land x\ge 1 \;\Rightarrow\; t' \le t + log\ x$$

portation

$$a \land b \Rightarrow c \;=\; a \Rightarrow (b \Rightarrow c)$$

$$(b \Rightarrow c) \Leftarrow a$$

$$= \qquad a \land b \Rightarrow c$$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \land (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a})$$ portation

$$= \quad x{\ne}1 \land (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \land x{\ge}1 \;\Rightarrow\; t' \le t + log\ x$$

$$(\underbrace{x{\geq}1}_{b} \;\Rightarrow\; \underbrace{t' \leq t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\neq}1 \wedge (div\ x\ 2 \geq 1 \;\Rightarrow\; t' \leq t + 1 + log\ (div\ x\ 2))}_{a})$$

portation

$$= \quad x{\neq}1 \wedge (div\ x\ 2 \geq 1 \;\Rightarrow\; t' \leq t + 1 + log\ (div\ x\ 2)) \wedge x{\geq}1 \;\Rightarrow\; t' \leq t + log\ x$$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + \mathit{log}\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \wedge (\mathit{div}\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + \mathit{log}\ (\mathit{div}\ x\ 2))}_{a})$$

portation

$$= \quad x{\ne}1 \wedge (\mathit{div}\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + \mathit{log}\ (\mathit{div}\ x\ 2)) \wedge x{\ge}1 \;\Rightarrow\; t' \le t + \mathit{log}\ x$$

$$(\underset{b}{\underline{x \ge 1}} \;\Rightarrow\; \underset{c}{\underline{t' \le t + log\ x}}) \;\Leftarrow\; \underset{a}{\underline{x \ne 1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}}) \qquad \text{portation}$$

$$= \quad x \ne 1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \wedge x \ge 1 \;\Rightarrow\; t' \le t + log\ x \qquad \text{simplify}$$

$$= \quad x > 1 \wedge (x > 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \;\Rightarrow\; t' \le t + log\ x$$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a})$$ portation

$= \quad x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \wedge x{\ge}1 \;\Rightarrow\; t' \le t + log\ x$ simplify

$= \quad x{>}1 \wedge (x{>}1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \;\Rightarrow\; t' \le t + log\ x$

discharge

$$a \wedge (a \Rightarrow b) \;=\; a{\wedge}b$$

$$(\underbrace{x \ge 1}_{b} \ \Rightarrow\ \underbrace{t' \le t + log\ x}_{c} \ \Leftarrow\ \underbrace{x \ne 1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))}_{a}) \qquad \text{portation}$$

$$=\quad x \ne 1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2)) \land x \ge 1 \ \Rightarrow\ t' \le t + log\ x \qquad \text{simplify}$$

$$=\quad \underbrace{x>1}_{a} \land (\underbrace{x>1}_{a} \ \Rightarrow\ \underbrace{t' \le t + 1 + log\ (div\ x\ 2)}_{b}) \ \Rightarrow\ t' \le t + log\ x$$

discharge

$$a \land (a \Rightarrow b) \ = \ a \land b$$

$$(\underbrace{x\geq 1}_{b} \;\Rightarrow\; \underbrace{t' \leq t + \mathit{log}\; x}_{c} \;\Leftarrow\; \underbrace{x \neq 1 \land (\mathit{div}\; x\; 2 \geq 1 \;\Rightarrow\; t' \leq t + 1 + \mathit{log}\; (\mathit{div}\; x\; 2))}_{a})$$ portation

$$= \quad x \neq 1 \land (\mathit{div}\; x\; 2 \geq 1 \;\Rightarrow\; t' \leq t + 1 + \mathit{log}\; (\mathit{div}\; x\; 2)) \land x\geq 1 \;\Rightarrow\; t' \leq t + \mathit{log}\; x$$ simplify

$$= \quad \underbrace{x>1}_{a} \land (\underbrace{x>1}_{a} \;\Rightarrow\; \underbrace{t' \leq t + 1 + \mathit{log}\; (\mathit{div}\; x\; 2)}_{b}) \;\Rightarrow\; t' \leq t + \mathit{log}\; x$$ discharge

$$= \quad x>1 \;\land\; t' \leq t + 1 + \mathit{log}\; (\mathit{div}\; x\; 2) \;\Rightarrow\; t' \leq t + \mathit{log}\; x$$

discharge

$$a \land (a \Rightarrow b) \;=\; a \land b$$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \land (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a}) \qquad \text{portation}$$

$$= \quad x{\ne}1 \land (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \land x{\ge}1 \;\Rightarrow\; t' \le t + log\ x \qquad \text{simplify}$$

$$= \quad \underbrace{x{>}1}_{a} \land (\underbrace{x{>}1}_{a} \;\Rightarrow\; \underbrace{t' \le t + 1 + log\ (div\ x\ 2)}_{b}) \;\Rightarrow\; t' \le t + log\ x \qquad \text{discharge}$$

$$= \quad x{>}1 \;\land\; t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x$$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a}) \qquad \text{portation}$$

$$= \quad x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \wedge x{\ge}1 \;\Rightarrow\; t' \le t + log\ x \qquad \text{simplify}$$

$$= \quad \underbrace{x{>}1}_{a} \wedge (\underbrace{x{>}1}_{a} \;\Rightarrow\; \underbrace{t' \le t + 1 + log\ (div\ x\ 2)}_{b}) \;\Rightarrow\; t' \le t + log\ x \qquad \text{discharge}$$

$$= \quad x{>}1 \;\wedge\; t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x \qquad \text{portation}$$

$$= \quad x{>}1 \;\Rightarrow\; (t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x)$$

$$(\underbrace{x{\geq}1}_{b} \;\Rightarrow\; \underbrace{t' \leq t + log\ x}_{c} \;\;\Leftarrow\;\; \underbrace{x{\neq}1 \land (div\ x\ 2 \geq 1 \;\Rightarrow\; t' \leq t + 1 + log\ (div\ x\ 2))}_{a})$$

portation

$$= \quad x{\neq}1 \land (div\ x\ 2 \geq 1 \;\Rightarrow\; t' \leq t + 1 + log\ (div\ x\ 2)) \land x{\geq}1 \;\;\Rightarrow\;\; t' \leq t + log\ x$$

simplify

$$= \quad \underbrace{x{>}1}_{a} \land (\underbrace{x{>}1}_{a} \;\Rightarrow\; \underbrace{t' \leq t + 1 + log\ (div\ x\ 2)}_{b}) \;\;\Rightarrow\;\; t' \leq t + log\ x$$

discharge

$$= \quad x{>}1 \;\land\; t' \leq t + 1 + log\ (div\ x\ 2) \;\;\Rightarrow\;\; t' \leq t + log\ x$$

portation

$$= \quad x{>}1 \;\;\Rightarrow\;\; (t' \leq t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \leq t + log\ x)$$

Connection Law $t'{\leq}a \Rightarrow t'{\leq}b \;\;\Leftarrow\;\; a{\leq}b$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \land (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a})$$

portation

$$= \quad x{\ne}1 \land (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \land x{\ge}1 \;\Rightarrow\; t' \le t + log\ x$$

simplify

$$= \quad \underbrace{x{>}1}_{a} \land (\underbrace{x{>}1}_{a} \;\Rightarrow\; \underbrace{t' \le t + 1 + log\ (div\ x\ 2)}_{b}) \;\Rightarrow\; t' \le t + log\ x$$

discharge

$$= \quad x{>}1 \;\land\; t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x$$

portation

$$= \quad x{>}1 \;\Rightarrow\; (t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x)$$

Connection Law $t'{\le}a \Rightarrow t'{\le}b \;\Leftarrow\; a{\le}b$

$$\Leftarrow \quad x{>}1 \;\Rightarrow\; t + 1 + log\ (div\ x\ 2) \;\le\; t + log\ x$$

$$(\underset{b}{\underline{x \ge 1}} \;\Rightarrow\; \underset{c}{\underline{t' \le t + log\, x}} \;\Leftarrow\; \underset{a}{\underline{x \neq 1 \wedge (div\; x\; 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\, (div\; x\; 2))}})$$ portation

$$= \quad x \neq 1 \wedge (div\; x\; 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\, (div\; x\; 2)) \wedge x \ge 1 \;\Rightarrow\; t' \le t + log\, x$$ simplify

$$= \quad \underset{a}{\underline{x > 1}} \wedge (\underset{a}{\underline{x > 1}} \;\Rightarrow\; \underset{b}{\underline{t' \le t + 1 + log\, (div\; x\; 2))}} \;\Rightarrow\; t' \le t + log\, x$$ discharge

$$= \quad x > 1 \;\wedge\; t' \le t + 1 + log\, (div\; x\; 2) \;\Rightarrow\; t' \le t + log\, x$$ portation

$$= \quad x > 1 \;\Rightarrow\; (t' \le t + 1 + log\, (div\; x\; 2) \;\Rightarrow\; t' \le t + log\, x)$$

Connection Law $t' \le a \Rightarrow t' \le b \;\Leftarrow\; a \le b$

$$\Leftarrow \quad x > 1 \;\Rightarrow\; t + 1 + log\, (div\; x\; 2) \;\le\; t + log\, x$$ subtract $t+1$ from each side

$$= \quad x > 1 \;\Rightarrow\; log\, (div\; x\; 2) \;\le\; log\, x - 1$$

$$(\underbrace{x\ge 1}_{b} \;\Rightarrow\; \underbrace{t' \le t + \mathit{log}\ x}_{c} \;\Leftarrow\; \underbrace{x\ne 1 \land (\mathit{div}\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + \mathit{log}\ (\mathit{div}\ x\ 2))}_{a})$$

portation

$= \quad x\ne 1 \land (\mathit{div}\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + \mathit{log}\ (\mathit{div}\ x\ 2)) \land x\ge 1 \;\Rightarrow\; t' \le t + \mathit{log}\ x$ — simplify

$$= \quad \underbrace{x>1}_{a} \land (\underbrace{x>1}_{a} \;\Rightarrow\; \underbrace{t' \le t + 1 + \mathit{log}\ (\mathit{div}\ x\ 2)}_{b}) \;\Rightarrow\; t' \le t + \mathit{log}\ x$$

discharge

$= \quad x>1 \land t' \le t + 1 + \mathit{log}\ (\mathit{div}\ x\ 2) \;\Rightarrow\; t' \le t + \mathit{log}\ x$ — portation

$= \quad x>1 \;\Rightarrow\; (t' \le t + 1 + \mathit{log}\ (\mathit{div}\ x\ 2) \;\Rightarrow\; t' \le t + \mathit{log}\ x)$

Connection Law $t'\le a \Rightarrow t'\le b \;\Leftarrow\; a\le b$

$\Leftarrow \quad x>1 \;\Rightarrow\; t + 1 + \mathit{log}\ (\mathit{div}\ x\ 2) \le t + \mathit{log}\ x$ — subtract $t+1$ from each side

$= \quad x>1 \;\Rightarrow\; \mathit{log}\ (\mathit{div}\ x\ 2) \le \mathit{log}\ x - 1$ — property of log

$= \quad x>1 \;\Rightarrow\; \mathit{log}\ (\mathit{div}\ x\ 2) \le \mathit{log}\ (x/2)$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)))}_{a}$$

portation

$$=\quad x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \wedge x{\ge}1 \;\Rightarrow\; t' \le t + log\ x$$

simplify

$$=\quad \underbrace{x{>}1}_{a} \wedge (\underbrace{x{>}1}_{a} \;\Rightarrow\; \underbrace{t' \le t + 1 + log\ (div\ x\ 2))}_{b} \;\Rightarrow\; t' \le t + log\ x$$

discharge

$$=\quad x{>}1 \;\wedge\; t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x$$

portation

$$=\quad x{>}1 \;\Rightarrow\; (t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x)$$

Connection Law $t'{\le}a \Rightarrow t'{\le}b \;\Leftarrow\; a{\le}b$

$$\Leftarrow\quad x{>}1 \;\Rightarrow\; t + 1 + log\ (div\ x\ 2) \;\le\; t + log\ x$$

subtract $t{+}1$ from each side

$$=\quad x{>}1 \;\Rightarrow\; log\ (div\ x\ 2) \;\le\; log\ x - 1$$

property of log

$$=\quad x{>}1 \;\Rightarrow\; log\ (div\ x\ 2) \;\le\; log\ (x/2)$$

log is monotonic for $x{>}0$

$$\Leftarrow\quad div\ x\ 2 \le x/2$$

$$(\underbrace{x{\ge}1}_{b} \;\Rightarrow\; \underbrace{t' \le t + log\ x}_{c} \;\Leftarrow\; \underbrace{x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2))}_{a})$$ portation

$$=\quad x{\ne}1 \wedge (div\ x\ 2 \ge 1 \;\Rightarrow\; t' \le t + 1 + log\ (div\ x\ 2)) \wedge x{\ge}1 \;\Rightarrow\; t' \le t + log\ x$$ simplify

$$=\quad \underbrace{x{>}1}_{a} \wedge (\underbrace{x{>}1}_{a} \;\Rightarrow\; \underbrace{t' \le t + 1 + log\ (div\ x\ 2))}_{b} \;\Rightarrow\; t' \le t + log\ x$$ discharge

$$=\quad x{>}1 \;\wedge\; t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x$$ portation

$$=\quad x{>}1 \;\Rightarrow\; (t' \le t + 1 + log\ (div\ x\ 2) \;\Rightarrow\; t' \le t + log\ x)$$

Connection Law $t'{\le}a \Rightarrow t'{\le}b \;\Leftarrow\; a{\le}b$

$$\Leftarrow\quad x{>}1 \;\Rightarrow\; t + 1 + log\ (div\ x\ 2) \;\le\; t + log\ x$$ subtract $t{+}1$ from each side

$$=\quad x{>}1 \;\Rightarrow\; log\ (div\ x\ 2) \;\le\; log\ x - 1$$ property of log

$$=\quad x{>}1 \;\Rightarrow\; log\ (div\ x\ 2) \;\le\; log\ (x/2)$$ log is monotonic for $x{>}0$

$$\Leftarrow\quad div\ x\ 2 \le x/2$$

$$=\quad \top$$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $\quad R \quad = \quad x'{=}1 \wedge$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$$= \quad x'{=}1 \wedge (x{\ge}1 \ \Rightarrow\ t' \le t + log\ x) \wedge (x{<}1 \Rightarrow t'{=}\infty)$$

use Refinement by Parts and Cases; prove:

$$x'{=}1 \quad \Leftarrow \quad x{=}1 \wedge x'{=}x \wedge t'{=}t$$

$$x'{=}1 \quad \Leftarrow \quad x{\ne}1 \wedge x'{=}1$$

$$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x{=}1 \wedge x'{=}x \wedge t'{=}t$$

$$x{\ge}1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x{\ne}1 \wedge (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))$$

$$x{<}1 \ \Rightarrow\ t'{=}\infty \quad \Leftarrow \quad x{=}1 \wedge x'{=}x \wedge t'{=}t$$

$$x{<}1 \ \Rightarrow\ t'{=}\infty \quad \Leftarrow \quad x{\ne}1 \wedge (div\ x\ 2 < 1 \ \Rightarrow\ t'{=}\infty)$$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $\quad R \quad = \quad x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$$= \quad x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$$

use Refinement by Parts and Cases; prove:

$x'{=}1 \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x'{=}1 \quad \Leftarrow \quad x{\neq}1 \land x'{=}1$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \quad \Leftarrow \quad x{\neq}1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$ ✔

$x{<}1 \Rightarrow t'{=}\infty \quad \Leftarrow \quad x{=}1 \land x'{=}x \land t'{=}t$ ←

$x{<}1 \Rightarrow t'{=}\infty \quad \Leftarrow \quad x{\neq}1 \land (div\ x\ 2 < 1 \Rightarrow t'{=}\infty)$

$$(x<1 \Rightarrow t'=\infty \quad \Leftarrow \quad x=1 \;\land\; x'=x \;\land\; t'=t)$$

$$(x<1 \Rightarrow t'=\infty \quad \Leftarrow \quad x=1 \ \wedge \ x'=x \ \wedge \ t'=t)$$ portation

$$= \quad x<1 \wedge x=1 \ \wedge \ x'=x \ \wedge \ t'=t \quad \Rightarrow \quad t'=\infty$$

$$(x<1 \Rightarrow t'=\infty \quad \Leftarrow \quad x=1 \ \land \ x'=x \ \land \ t'=t)$$ portation

$=$ $$x<1 \land x=1 \ \land \ x'=x \ \land \ t'=t \ \Rightarrow \ t'=\infty$$ generic, base

$=$ $$\bot \ \Rightarrow \ t'=\infty$$

$(x<1 \Rightarrow t'=\infty \quad \Leftarrow \quad x=1 \;\land\; x'=x \;\land\; t'=t)$ — portation

$= \quad x<1 \land x=1 \;\land\; x'=x \;\land\; t'=t \quad \Rightarrow \quad t'=\infty$ — generic, base

$= \quad \bot \;\Rightarrow\; t'=\infty$ — base

$= \quad \top$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where $R\ =\ x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$=\ x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x'{=}1 \Leftarrow x{\ne}1 \land x'{=}1$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{\ne}1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$ ✔

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{\ne}1 \land (div\ x\ 2 < 1 \Rightarrow t'{=}\infty)$

Prove $R \Leftarrow$ **if** $x{=}1$ **then** ok **else** $x{:=}\ div\ x\ 2.\ \ t{:=}\ t{+}1.\ \ R$ **fi**

where R $=$ $x'{=}1 \land$ **if** $x{\ge}1$ **then** $t' \le t + log\ x$ **else** $t'{=}\infty$ **fi**

$=$ $x'{=}1 \land (x{\ge}1 \Rightarrow t' \le t + log\ x) \land (x{<}1 \Rightarrow t'{=}\infty)$

use Refinement by Parts and Cases; prove:

$x'{=}1 \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x'{=}1 \Leftarrow x{\ne}1 \land x'{=}1$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x{\ge}1 \Rightarrow t' \le t + log\ x \Leftarrow x{\ne}1 \land (div\ x\ 2 \ge 1 \Rightarrow t' \le t + 1 + log\ (div\ x\ 2))$ ✔

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{=}1 \land x'{=}x \land t'{=}t$ ✔

$x{<}1 \Rightarrow t'{=}\infty \Leftarrow x{\ne}1 \land (div\ x\ 2 < 1 \Rightarrow t'{=}\infty)$ ⟵

$$(x<1 \Rightarrow t'=\infty \quad \Leftarrow \quad x \neq 1 \;\land\; (div\ x\ 2 < 1 \;\Rightarrow\; t'=\infty))$$

$(x{<}1 \Rightarrow t'{=}\infty \;\Leftarrow\; x{\neq}1 \;\wedge\; (div\ x\ 2 < 1 \;\Rightarrow\; t'{=}\infty))$ portation

$= \quad x{<}1 \wedge x{\neq}1 \wedge (div\ x\ 2 < 1 \;\Rightarrow\; t'{=}\infty) \;\Rightarrow\; t'{=}\infty$

$(x<1 \Rightarrow t'=\infty \quad \Leftarrow \quad x \neq 1 \;\wedge\; (div\ x\ 2 < 1 \;\Rightarrow\; t'=\infty))$ portation

$= \quad x<1 \;\wedge\; x \neq 1 \;\wedge\; (div\ x\ 2 < 1 \;\Rightarrow\; t'=\infty) \quad\Rightarrow\quad t'=\infty$

$$(x<1 \Rightarrow t'=\infty \;\;\Leftarrow\;\; x\neq 1 \;\wedge\; (div\ x\ 2 < 1 \;\Rightarrow\; t'=\infty)) \qquad \text{portation}$$

$$= \qquad x<1 \;\wedge\; x\neq 1 \;\wedge\; (div\ x\ 2 < 1 \;\Rightarrow\; t'=\infty) \;\;\Rightarrow\;\; t'=\infty$$

$(x{<}1 \Rightarrow t'{=}\infty \quad \Leftarrow \quad x{\ne}1 \;\land\; (div\ x\ 2 < 1 \;\Rightarrow\; t'{=}\infty))$ portation

$= \quad x{<}1 \land x{\ne}1 \land (div\ x\ 2 < 1 \;\Rightarrow\; t'{=}\infty) \;\Rightarrow\; t'{=}\infty$ discharge

$= \quad x{<}1 \land t'{=}\infty \;\Rightarrow\; t'{=}\infty$

$$(x<1 \Rightarrow t'=\infty \;\; \Leftarrow \;\; x \neq 1 \;\wedge\; (div\ x\ 2 < 1 \;\Rightarrow\; t'=\infty)) \qquad \text{portation}$$

$$= \quad x<1 \wedge x \neq 1 \wedge (div\ x\ 2 < 1 \;\Rightarrow\; t'=\infty) \;\;\Rightarrow\;\; t'=\infty \qquad \text{discharge}$$

$$= \quad x<1 \wedge t'=\infty \;\;\Rightarrow\;\; t'=\infty \qquad \text{specialization}$$

$$= \quad \top$$

Prove $\quad R \quad \Leftarrow \quad$ **if** $x=1$ **then** ok **else** $x:=\ div\ x\ 2.\ \ t:=t+1.\ \ R$ **fi**

where $\quad R \quad = \quad x'=1 \land$ **if** $x \ge 1$ **then** $t' \le t + log\ x$ **else** $t'=\infty$ **fi**

$$= \quad x'=1 \land (x \ge 1 \ \Rightarrow\ t' \le t + log\ x) \land (x<1 \Rightarrow t'=\infty)$$

use Refinement by Parts and Cases; prove:

$x'=1 \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$ ✔

$x'=1 \quad \Leftarrow \quad x \neq 1 \land x'=1$ ✔

$x \ge 1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$ ✔

$x \ge 1 \ \Rightarrow\ t' \le t + log\ x \quad \Leftarrow \quad x \neq 1 \land (div\ x\ 2 \ge 1 \ \Rightarrow\ t' \le t + 1 + log\ (div\ x\ 2))$ ✔

$x<1 \ \Rightarrow\ t'=\infty \quad \Leftarrow \quad x=1 \land x'=x \land t'=t$ ✔

$x<1 \ \Rightarrow\ t'=\infty \quad \Leftarrow \quad x \neq 1 \land (div\ x\ 2 < 1 \ \Rightarrow\ t'=\infty)$ ✔

Termination

$x'=2 \quad \Leftarrow$

Termination

$x'=2 \quad \Leftarrow \quad x:= 2$

Termination

$x' = 2 \quad \Longleftarrow$

Termination

$x'{=}2 \quad \Leftarrow \quad x'{=}2$

Termination

$x'=2 \quad \Longleftarrow \quad x'=2$

complain only if $x' \neq 2$

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ \ x'=2$

complain only if $x' \neq 2$

Termination

$x'=2 \;\Leftarrow\; t:=t+1.\; x'=2$

complain only if $x' \neq 2$

$x'=2 \;\land\; t'<\infty$

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ x'=2$

complain only if $x' \neq 2$

$x'=2 \ \land \ t'<\infty$

unimplementable

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ x'=2$

complain only if $x' \neq 2$

$x'=2 \ \land \ t' < \infty$

unimplementable

(infinite loop). $x'=2 \ \land \ t' < \infty$

Termination

$x'=2 \;\; \Leftarrow \;\; t:= t+1.\; x'=2$

complain only if $x' \ne 2$

$x'=2 \;\wedge\; t'<\infty$

unimplementable

(infinite loop). $x'=2 \;\wedge\; t'<\infty$

↑ this part starts at time ∞ (it never starts)

so it can't stop at a finite time

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ x'=2$

complain only if $x' \neq 2$

$x'=2 \ \land \ t'<\infty$

unimplementable

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ x'=2$

complain only if $x' \neq 2$

$x'=2 \ \land \ t'<\infty$

unimplementable

$x'=2 \land (t<\infty \Rightarrow t'<\infty)$

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ x'=2$

complain only if $x' \neq 2$

$x'=2 \ \land \ t'<\infty$

unimplementable

$x'=2 \land (t<\infty \Rightarrow t'<\infty) \quad \Leftarrow \quad t:= t+1.\ x'=2 \land (t<\infty \Rightarrow t'<\infty)$

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ \ x'=2$

complain only if $x' \ne 2$

$x'=2 \ \land \ t'<\infty$

unimplementable

$x'=2 \land (t<\infty \Rightarrow t'<\infty) \quad \Leftarrow \quad t:= t+1.\ \ x'=2 \land (t<\infty \Rightarrow t'<\infty)$

complain only if $x' \ne 2 \ \lor \ t<\infty \land t'=\infty$

Termination

$x'=2 \Leftarrow t:= t+1.\ x'=2$

complain only if $x' \ne 2$

$x'=2 \wedge t'<\infty$

unimplementable

$x'=2 \wedge (t<\infty \Rightarrow t'<\infty) \Leftarrow t:= t+1.\ x'=2 \wedge (t<\infty \Rightarrow t'<\infty)$

complain only if $x' \ne 2 \vee t<\infty \wedge t'=\infty$

$x'=2 \wedge t' \le t+1$

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ x'=2$

complain only if $x' \neq 2$

$x'=2 \ \land \ t'<\infty$

unimplementable

$x'=2 \land (t<\infty \Rightarrow t'<\infty) \quad \Leftarrow \quad t:= t+1.\ x'=2 \land (t<\infty \Rightarrow t'<\infty)$

complain only if $x' \neq 2 \ \lor \ t<\infty \land t'=\infty$

$x'=2 \land t' \le t+1 \quad \Leftarrow \quad t:= t+1.\ x'=2 \land t' \le t+1$ ✗

Termination

$x'=2 \quad \Leftarrow \quad t:= t+1.\ x'=2$

complain only if $x' \neq 2$

$x'=2 \ \land \ t'<\infty$

unimplementable

$x'=2 \land (t<\infty \Rightarrow t'<\infty) \quad \Leftarrow \quad t:= t+1.\ x'=2 \land (t<\infty \Rightarrow t'<\infty)$

complain only if $x' \neq 2 \ \lor \ t<\infty \land t'=\infty$

$x'=2 \land t' \le t+1 \quad \Leftarrow \quad x:= 2$