Monotonicity and Antimonotonicity

Monotonicity and Antimonotonicity

covariance and contravariance

Monotonicity and Antimonotonicity

covariance	**and**	**contravariance**
varies directly as	**and**	**varies inversely as**

Monotonicity and Antimonotonicity

covariance	**and**	**contravariance**
varies directly as	**and**	**varies inversely as**
nondecreasing	**and**	**nonincreasing**

Monotonicity and Antimonotonicity

covariance	**and**	**contravariance**
varies directly as	**and**	**varies inversely as**
nondecreasing	**and**	**nonincreasing**
sorted	**and**	**sorted backwards**

Monotonicity and Antimonotonicity

covariance	**and**	**contravariance**
varies directly as	**and**	**varies inversely as**
nondecreasing	**and**	**nonincreasing**
sorted	**and**	**sorted backwards**

$x \le y \;\Rightarrow\; f\,x \le f\,y$

$x \le y \;\Rightarrow\; f\,x \ge f\,y$

Monotonicity and Antimonotonicity

covariance	**and**	**contravariance**
varies directly as	**and**	**varies inversely as**
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$x \le y \;\Rightarrow\; f\,x \le f\,y$

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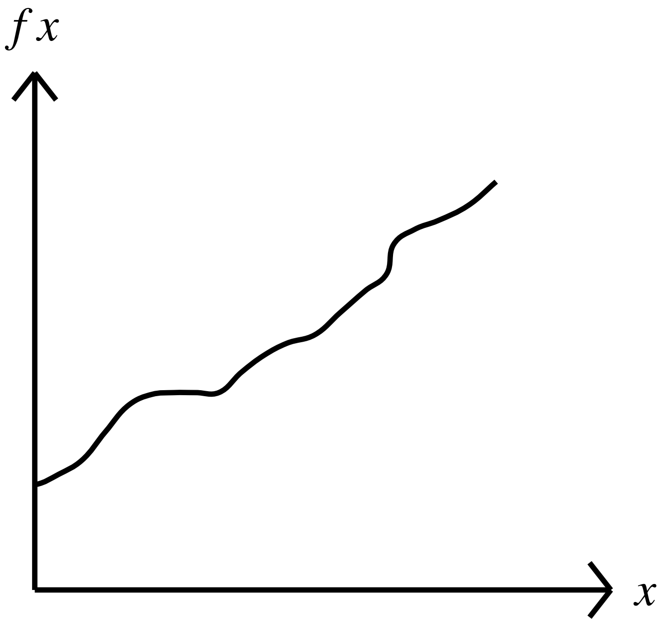

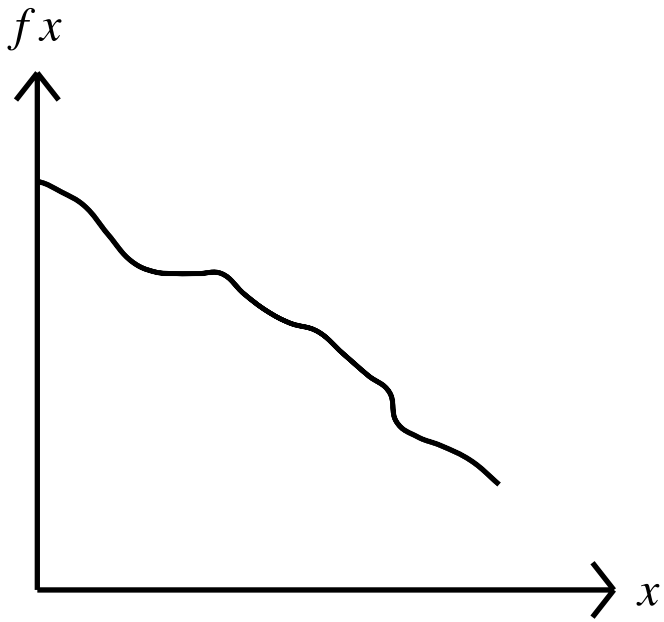

Monotonicity and Antimonotonicity

number:

binary:

Monotonicity and Antimonotonicity

number: $x \le y$

binary: $x \Rightarrow y$

Monotonicity and Antimonotonicity

number: $x \le y$ — x is less than or equal to y

binary: $x \Rightarrow y$ — x implies y

Monotonicity and Antimonotonicity

number: $x \le y$ — x is less than or equal to y

binary: $x \Rightarrow y$ — x implies y — x is falser than or equal to y

Monotonicity and Antimonotonicity

number: $x \le y$ — x is less than or equal to y

binary: $x \Rightarrow y$ — x implies y — x is stronger than or equal to y

Monotonicity and Antimonotonicity

number:	$x \le y$	x is less than or equal to y	
	$-\infty \le +\infty$ $\quad 0 \le 1$	smaller $\le$ larger	
binary:	$x \Rightarrow y$	x implies y	x is stronger than or equal to y
	$\bot \Rightarrow \top$	stronger $\Rightarrow$ weaker	

Monotonicity and Antimonotonicity

number:	$x \le y$	x is less than or equal to y	
	$-\infty \le +\infty \quad 0 \le 1$	smaller $\le$ larger	
	$x \le y \Rightarrow f\,x \le f\,y$	f is monotonic	
binary:	$x \Rightarrow y$	x implies y	x is stronger than or equal to y
	$\bot \Rightarrow \top$	stronger $\Rightarrow$ weaker	
	$x \Rightarrow y \;\Rightarrow\; f\,x \Rightarrow f\,y$	f is monotonic	

Monotonicity and Antimonotonicity

number:	$x \le y$	x is less than or equal to y
	$-\infty \le +\infty \quad 0 \le 1$	smaller $\le$ larger
	$x \le y \;\Rightarrow\; f\,x \le f\,y$	f is monotonic
		as x gets larger, $f\,x$ gets larger (or equal)

binary:	$x \Rightarrow y$	x implies y x is stronger than or equal to y
	$\bot \Rightarrow \top$	stronger $\Rightarrow$ weaker
	$x \Rightarrow y \;\Longrightarrow\; f\,x \Rightarrow f\,y$	f is monotonic
		as x gets weaker, $f\,x$ gets weaker (or equal)

Monotonicity and Antimonotonicity

number:	$x \le y$	x is less than or equal to y
	$-\infty \le +\infty \quad 0 \le 1$	smaller $\le$ larger
	$x \le y \Rightarrow f\,x \le f\,y$	f is monotonic
		as x gets larger, $f\,x$ gets larger (or equal)
	$x \le y \Rightarrow f\,x \ge f\,y$	f is antimonotonic
		as x gets larger, $f\,x$ gets smaller (or equal)
binary:	$x \Rightarrow y$	x implies y $\quad$ x is stronger than or equal to y
	$\bot \Rightarrow \top$	stronger $\Rightarrow$ weaker
	$x \Rightarrow y \implies f\,x \Rightarrow f\,y$	f is monotonic
		as x gets weaker, $f\,x$ gets weaker (or equal)
	$x \Rightarrow y \implies f\,x \Leftarrow f\,y$	f is antimonotonic
		as x gets weaker, $f\,x$ gets stronger (or equal)

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
$a \vee b$	monotonic in a	monotonic in b

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
$a \vee b$	monotonic in a	monotonic in b
$a \Rightarrow b$	antimonotonic in a	monotonic in b

Monotonicity and Antimonotonicity

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$a \wedge b$	monotonic in a	monotonic in b
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Monotonicity and Antimonotonicity

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$a \wedge b$	monotonic in a	monotonic in b
$a \vee b$	monotonic in a	monotonic in b
$a \Rightarrow b$	antimonotonic in a	monotonic in b
$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a **then** b **else** c **fi**	monotonic in b	monotonic in c

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
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$a \Rightarrow b$	antimonotonic in a	monotonic in b
$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a **then** b **else** c **fi**	monotonic in b	monotonic in c

$\neg(a \wedge \neg(a \vee b))$

Monotonicity and Antimonotonicity

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$a \Rightarrow b$	antimonotonic in a	monotonic in b
$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a **then** b **else** c **fi**	monotonic in b	monotonic in c

$$\neg(a \wedge \neg(\underline{a \vee b}))$$

$$\Uparrow$$

$$\neg(a \wedge \neg \ \underline{a} \ )$$

use the Law of Generalization $a \Rightarrow a \vee b$

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
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$a \Rightarrow b$	antimonotonic in a	monotonic in b
$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a **then** b **else** c **fi**	monotonic in b	monotonic in c

$$\neg(a \wedge \underline{\neg(a \vee b)})$$

$$\Downarrow$$

$$\neg(a \wedge \overline{\neg \quad a \quad})$$

use the Law of Generalization $a \Rightarrow a \vee b$

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
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if a **then** b **else** c **fi**	monotonic in b	monotonic in c

$$\neg(\underline{a \wedge \neg(a \vee b)})$$
$$\Downarrow$$
$$\neg(\overline{a \wedge \neg \quad a \quad})$$

use the Law of Generalization $a \Rightarrow a \vee b$

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
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$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a **then** b **else** c **fi**	monotonic in b	monotonic in c

$$\frac{\neg(a \wedge \neg(a \vee b))}{\neg(a \wedge \neg\ a\ )} \Uparrow$$

use the Law of Generalization $a \Rightarrow a \vee b$

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
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$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a **then** b **else** c **fi**	monotonic in b	monotonic in c

$\neg(a \wedge \neg(a \vee b))$ use the Law of Generalization $a \Rightarrow a \vee b$

$\Leftarrow \quad \neg(a \wedge \neg \quad a \quad)$

Monotonicity and Antimonotonicity

$\neg a$	antimonotonic in a	
$a \wedge b$	monotonic in a	monotonic in b
$a \vee b$	monotonic in a	monotonic in b
$a \Rightarrow b$	antimonotonic in a	monotonic in b
$a \Leftarrow b$	monotonic in a	antimonotonic in b
if a **then** b **else** c **fi**	monotonic in b	monotonic in c

$\neg(a \wedge \neg(a \vee b))$ — use the Law of Generalization $a \Rightarrow a \vee b$

$\Leftarrow \quad \neg(a \wedge \neg \ \ a \ \ )$ — now use the Law of Noncontradiction

$= \quad \top$

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

$$\neg(\underline{a \wedge \neg(a \vee b)})$$

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

$$\neg(a \wedge \neg(a \vee b)) \qquad \text{context } a \text{ in } \neg(a \vee b)$$

$$= \neg(a \wedge \neg(\top \vee b))$$

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

	$\neg(a \wedge \neg(a \vee b))$	context a in $\neg(a \vee b)$
=	$\neg(a \wedge \neg(\top \vee b))$	Symmetry Law and Base Law for $\vee$
=	$\neg(a \wedge \neg\top)$	

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

	$\neg(a \wedge \neg(a \vee b))$	context a in $\neg(a \vee b)$
=	$\neg(a \wedge \neg(\top \vee b))$	Symmetry Law and Base Law for $\vee$
=	$\neg(a \wedge \neg\top)$	Theorem Table for $\neg$
=	$\neg(a \wedge \bot)$	

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

	$\neg(a \wedge \neg(a \vee b))$	context a in $\neg(a \vee b)$
=	$\neg(a \wedge \neg(\top \vee b))$	Symmetry Law and Base Law for $\vee$
=	$\neg(a \wedge \neg\top)$	Theorem Table for $\neg$
=	$\neg(a \wedge \bot)$	Base Law for $\wedge$
=	$\neg\bot$	

Context

In *exp0* $\land$ *exp1* , *exp0* is a local axiom within *exp1* .

	$\neg(a \land \neg(a \lor b))$	context a in $\neg(a \lor b)$
=	$\neg(a \land \neg(\top \lor b))$	Symmetry Law and Base Law for $\lor$
=	$\neg(a \land \neg\top)$	Theorem Table for $\neg$
=	$\neg(a \land \bot)$	Base Law for $\land$
=	$\neg\bot$	Binary Axiom, or Theorem Table for $\neg$
=	$\top$	

Context

In $exp0 \wedge exp1$, $exp0$ is a local axiom within $exp1$.

	$\neg(a \wedge \neg(a \vee b))$	context a in $\neg(a \vee b)$
$=$	$\neg(a \wedge \neg(\top \vee b))$	Symmetry Law and Base Law for $\vee$
$=$	$\neg(a \wedge \neg\top)$	Theorem Table for $\neg$
$=$	$\neg(a \wedge \bot)$	Base Law for $\wedge$
$=$	$\neg\bot$	Binary Axiom, or Theorem Table for $\neg$
$=$	$\top$	

If a is a theorem, then we can replace it with $\top$.

Context

In *exp0* $\land$ *exp1* , *exp0* is a local axiom within *exp1* .

	$\neg(a \land \neg(a \lor b))$	context a in $\neg(a \lor b)$
$=$	$\neg(a \land \neg(\top \lor b))$	Symmetry Law and Base Law for $\lor$
$=$	$\neg(a \land \neg\top)$	Theorem Table for $\neg$
$=$	$\neg(a \land \bot)$	Base Law for $\land$
$=$	$\neg\bot$	Binary Axiom, or Theorem Table for $\neg$
$=$	$\top$	

If a is a theorem, then we can replace it with $\top$.

If a is an antitheorem, then $a \land$ *anything* is an antitheorem.

Context

In $exp0 \land exp1$, $exp0$ is a local axiom within $exp1$.

	$\neg(a \land \neg(a \lor b))$	context a in $\neg(a \lor b)$
$=$	$\neg(a \land \neg(\top \lor b))$	Symmetry Law and Base Law for $\lor$
$=$	$\neg(a \land \neg\top)$	Theorem Table for $\neg$
$=$	$\neg(a \land \bot)$	Base Law for $\land$
$=$	$\neg\bot$	Binary Axiom, or Theorem Table for $\neg$
$=$	$\top$	

	$x{=}y \land x{=}z$	context $x{=}y$ in $x{=}z$
$=$	$x{=}y \land y{=}z$	

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

In *exp0* ∧ *exp1* , *exp1* is a local axiom within *exp0* .

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

In *exp0* ∧ *exp1* , *exp1* is a local axiom within *exp0* .

	$a \wedge a$	use right conjunct as context in left conjunct
	$\downarrow$	
=	$\top \wedge a$	

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

In *exp0* ∧ *exp1* , *exp1* is a local axiom within *exp0* .

	$a \wedge a$	use right conjunct as context in left conjunct
$=$	$\top \wedge a$	

	$a \wedge a$	use left conjunct as context in right conjunct
$=$	$a \wedge \top$	

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

In *exp0* ∧ *exp1* , *exp1* is a local axiom within *exp0* .

$a \wedge a$ use right conjunct as context in left conjunct

$= \quad \top \wedge a$

$a \wedge a$ use left conjunct as context in right conjunct

$= \quad a \wedge \top$

$a \wedge a$

$= \quad \top \wedge \top$ ✗

Context

In *exp0* ∧ *exp1* , *exp0* is a local axiom within *exp1* .

In *exp0* ∧ *exp1* , *exp1* is a local axiom within *exp0* .

In *exp0* ∨ *exp1* , *exp0* is a local antiaxiom within *exp1* .

In *exp0* ∨ *exp1* , *exp1* is a local antiaxiom within *exp0* .

In *exp0* ⇒ *exp1* , *exp0* is a local axiom within *exp1* .

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In *exp0* ⇐ *exp1* , *exp1* is a local axiom within *exp0* .

In **if** *exp0* **then** *exp1* **else** *exp2* **fi** , *exp0* is a local axiom within *exp1* .

In **if** *a* **then** *b* **else** *c* **fi** , *exp0* is a local antiaxiom within *exp2* .

In **if** *a* **then** *b* **else** *c* **fi** , *exp1=exp2* is a local antiaxiom within *exp0* .

Number Theory

number expressions represent quantity

Number Theory

number expressions represent quantity

number expressions

$0 \quad 1 \quad 2 \quad 597 \quad 1.2 \quad 1e10 \quad \infty$

$-x \quad x+y \quad x-y \quad x \times y \quad x/y \quad \frac{x}{y} \quad x^y \quad x \uparrow y \quad x \downarrow y$

if a **then** x **else** y **fi**

Number Theory

number expressions represent quantity

number expressions

0   1   2   597   1.2   1e10 ∞

$-x$ $x+y$ $x-y$ $x\times y$ x/y $\frac{x}{y}$ x^y $x\uparrow y$ $x\downarrow y$

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Number Theory

number expressions represent quantity

number expressions

0 1 2 597 1.2 1e10 ∞

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↑

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$\uparrow$

if *a* **then** *x* **else** *y* **fi**

Number Theory

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number expressions

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$\uparrow$

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Number Theory

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$0 \quad 1 \quad 2 \quad 597 \quad 1.2 \quad 1e10 \quad \infty$

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if a **then** x **else** y **fi**

↑

Number Theory

number expressions represent quantity

number expressions

$0 \quad 1 \quad 2 \quad 597 \quad 1.2 \quad 1e10 \quad \infty$

$-x \quad x+y \quad x-y \quad x \times y \quad x/y \quad \frac{x}{y} \quad x^y \quad x \uparrow y \quad x \downarrow y$

if a **then** x **else** y **fi**

binary expressions

$x=y \quad x \neq y \quad x<y \quad x>y \quad x \le y \quad x \ge y$

Character Theory

“A” “a” “ ” ““““”” “”””””

succ *pred* **if then else fi**

$= \quad \neq \quad < \quad > \quad \leq \quad \geq$

Character Theory

“A” “a” “ ” “““” “”””

succ *pred* **if then else fi**

= ≠ < > ≤ ≥

Character Theory

“A”   “a”   “ ”   ‘‘‘‘‘‘’’   ‘‘’’’’’’

succ   *pred*   **if then else fi**

= ≠ < > ≤ ≥

Character Theory

“A”　　“a”　　“ ”　　“““”　　“”””

succ　　*pred*　　**if then else fi**

= 　≠ 　< 　> 　≤ 　≥

Character Theory

“A”   “a”   “ ”   “““”   “”””

↑ ↑

succ   *pred*   **if then else fi**

= ≠ < > ≤ ≥

Character Theory

“A”   “a”   “ ”   “““”   “”””

succ   *pred*   **if then else fi**

↑   ↑

$= \quad \neq \quad < \quad > \quad \leq \quad \geq$

Character Theory

“A” “a” “ ” “““” “”””

succ *pred* **if then else fi**

$=$ $\neq$ $<$ $>$ $\leq$ $\geq$

Character Theory

“A”　　“a”　　“ ”　　“““”　　“”””

succ　　*pred*　　**if then else fi**

$=$　$\neq$　$<$　$>$　$\leq$　$\geq$　$\leftarrow$