Concurrent Composition

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and the state variables of the composition are those of both P and Q

Ignoring time and space variables

$$P \| Q \;=\; P \land Q$$

Concurrent Composition

example in integer variables x , y , and z

$$x:= x+1 \parallel y:= y+2$$

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$x := x+1 \| y := y+2$ partition the variables:

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partition the variables:

put x in left part, put y and z in right part

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$x:= x+1 \,\|\, y:= y+2$ partition the variables:

put x in left part, put y and z in right part

$=$ $x' = x+1 \;\|\; y' = y+2 \;\land\; z'=z$

Concurrent Composition

example in integer variables x , y , and z

$x{:=}\ x{+}1 \parallel y{:=}\ y{+}2$ partition the variables:

put x in left part, put y and z in right part

$= \quad x' = x{+}1 \parallel y' = y{+}2 \land z'{=}z$

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Concurrent Composition

example in integer variables x , y , and z

$x{:=}\ x{+}1 \parallel y{:=}\ y{+}2$ partition the variables:

put x in left part, put y and z in right part

$= \quad x' = x{+}1 \parallel y' = y{+}2 \land z'{=}z$

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reasonable partition rule

If either x' or $x{:=}$ appears in a process specification, then x belongs to that process

(then neither x' nor $x{:=}$ can appear in the other process specification).

If neither x' nor $x{:=}$ appears at all, then x can be placed on either side of the partition.

Concurrent Composition

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$$x := y \parallel y := x$$

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Concurrent Composition

example in variables x , y , and z

$x := y \parallel y := x$ partition: put x in left, y in right, z in either

$= \quad x'=y \wedge y'=x \wedge z'=z$

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example in variables x , y , and z

$x:= y \parallel y:= x$ partition: put x in left, y in right, z in either

$= \quad x'=y \land y'=x \land z'=z$

implementation of a process makes a private copy of the initial value of a variable belonging to the other process if the other process contains an assignment to that variable

Concurrent Composition

example in binary variable b and integer variable x

$$b := x = x \;\|\; x := x+1$$

Concurrent Composition

example in binary variable b and integer variable x

$b{:=}\ x{=}x \ \| \ x{:=}\ x{+}1$ replace $x{=}x$ by $\top$

$=$ $b{:=}\ \top \ \| \ x{:=}\ x{+}1$

Concurrent Composition

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example in integer variables x and y

$(x{:=}\ x{+}1.\ \ x{:=}\ x{-}1) \ \| \ y{:=}\ x$

Concurrent Composition

example in binary variable b and integer variable x

$$b{:=}\ x{=}x \ \| \ x{:=}\ x{+}1 \qquad \text{replace } x{=}x \text{ by } \top$$

$$= \quad b{:=}\ \top \ \| \ x{:=}\ x{+}1$$

example in integer variables x and y

$$(x{:=}\ x{+}1.\ \ x{:=}\ x{-}1) \ \| \ y{:=}\ x$$

$$= \quad ok \ \| \ y{:=}\ x$$

Concurrent Composition

example in binary variable b and integer variable x

$b{:=}\ x{=}x \ \| \ x{:=}\ x{+}1$ replace $x{=}x$ by $\top$

$=$ $b{:=}\ \top \ \| \ x{:=}\ x{+}1$

example in integer variables x and y

$(x{:=}\ x{+}1.\ \ x{:=}\ x{-}1)\ \| \ y{:=}\ x$

$=$ $ok \,\|\, y{:=}\ x$

$=$ $y{:=}\ x$

Concurrent Composition

$(x:= x+y.\ \ x:= x\times y)\ \ \|\ \ (y:= x-y.\ \ y:= x/y)$

Concurrent Composition

$(x := x+y.\ \ x := x \times y) \ \| \ (y := x-y.\ \ y := x/y)$

$\qquad\qquad\qquad\quad\uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$

Concurrent Composition

$(x{:=}\ x+y.\ \ x{:=}\ x\times y)\ \ \|\ \ (y{:=}\ x-y.\ \ y{:=}\ x/y)$

Concurrent Composition

$$(x{:=}\ x{+}y.\ \ x{:=}\ x{\times}y)\ \ \|\ \ (y{:=}\ x{-}y.\ \ y{:=}\ x/y)$$

You should have written

$$(x{:=}\ x{+}y\ \ \|\ \ y{:=}\ x{-}y).\ \ (x{:=}\ x{\times}y\ \ \|\ \ y{:=}\ x/y)$$

Concurrent Composition

Concurrent Composition

$$
\begin{aligned}
P \| Q \;=\; \exists tP, tQ \cdot \quad & (\text{substitute } tP \text{ for } t' \text{ in } P\,) \\
\wedge\; & (\text{substitute } tQ \text{ for } t' \text{ in } Q\,) \\
\wedge\; & t' = tP \uparrow tQ
\end{aligned}
$$

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laws

$(x{:=}\,e \,\|\, y{:=}\,f).\ P \;=\; (\text{for } x \text{ substitute } e \text{ and concurrently for } y \text{ substitute } f \text{ in } P)$

$P \,\|\, Q \;=\; Q \,\|\, P$ — symmetry

$P \,\|\, (Q \,\|\, R) \;=\; (P \,\|\, Q) \,\|\, R$ — associativity

$P \,\|\, Q \lor R \;=\; (P \,\|\, Q) \lor (P \,\|\, R)$ — distributivity

$P \,\|\, \textbf{if } b \textbf{ then } Q \textbf{ else } R \textbf{ fi} \;=\; \textbf{if } b \textbf{ then } P \,\|\, Q \textbf{ else } P \,\|\, R \textbf{ fi}$ — distributivity

$\textbf{if } b \textbf{ then } P\|Q \textbf{ else } R\|S \textbf{ fi} \;=\; \textbf{if } b \textbf{ then } P \textbf{ else } R \textbf{ fi} \,\|\, \textbf{if } b \textbf{ then } Q \textbf{ else } S \textbf{ fi}$ — distributivity

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$P \,\|\,$ **if** b **then** Q **else** R **fi** $=$ **if** b **then** $P \,\|\, Q$ **else** $P \,\|\, R$ **fi** distributivity

if b **then** $P\|Q$ **else** $R\|S$ **fi** $=$ **if** b **then** P **else** R **fi** $\|$ **if** b **then** Q **else** S **fi** distributivity

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List Concurrency

List Concurrency

$$L\ i{:=}\ e \quad = \quad L'i{=}e\ \land\ (\forall j{:}\ 0,..\#L\cdot\ j{\neq}i \Rightarrow L'j = L\ j)\ \land\ x'{=}x\ \land\ y'{=}y\ \land\ ...$$

List Concurrency

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example find the maximum item in a nonempty list

List Concurrency

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example find the maximum item in a nonempty list

$$\mathit{findmax} \ = \ \langle i, j\cdot\ i{<}j \Rightarrow L'\ i = \Uparrow(L\ [i;..j])\rangle$$

List Concurrency

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example find the maximum item in a nonempty list

findmax 0 (#*L*) where

$\mathit{findmax} \quad = \quad \langle i, j\cdot\ \ i{<}j \Rightarrow L'\ i = \Uparrow(L\ [i;..j])\rangle$

List Concurrency

$$L\ i{:=}\ e \quad = \quad L'i{=}e\ \wedge\ (\forall j{:}\ 0,..\#L\cdot\ j{\neq}i \Rightarrow L'j = L\,j)\ \wedge\ x'{=}x\ \wedge\ y'{=}y\ \wedge\ ...$$

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example find the maximum item in a nonempty list

$\mathit{findmax}\ 0\ (\#L)$ where

$$\mathit{findmax}\ =\ \langle i, j\cdot\ i{<}j \Rightarrow L'\ i = \Uparrow(L\ [i;..j])\rangle$$

$$\mathit{findmax}\ i\ j \ \Leftarrow\ \textbf{if}\ j{-}i = 1\ \textbf{then}\ ok$$
$$\textbf{else}\ \ (\mathit{findmax}\ i\ (\mathit{div}\ (i{+}j)\ 2)\ \|\ \mathit{findmax}\ (\mathit{div}\ (i{+}j)\ 2)\ j).$$
$$L\ i := (L\ i)\uparrow(L\ (\mathit{div}\ (i{+}j)\ 2))\ \ \textbf{fi}$$

List Concurrency

$L\ i{:=}\ e \quad = \quad L'i{=}e \ \land\ (\forall j{:}\ 0,..\#L\cdot\ j{\neq}i \Rightarrow L'j = L\ j)\ \land\ x'{=}x\ \land\ y'{=}y\ \land\ ...$

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else $(\mathit{findmax}\ i\ (\mathit{div}\ (i{+}j)\ 2)\ \|\ \mathit{findmax}\ (\mathit{div}\ (i{+}j)\ 2)\ j).$

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List Concurrency

$L\,i{:=}\,e \quad = \quad L'i{=}e \;\land\; (\forall j{:}\; 0,..\#L\cdot\; j{\ne}i \Rightarrow L'j = L\,j) \;\land\; x'{=}x \;\land\; y'{=}y \;\land\; ...$

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example find the maximum item in a nonempty list

$\mathit{findmax}\ 0\ (\#L)$ where

$\mathit{findmax} \;=\; \langle i, j\cdot\;\; i{<}j \Rightarrow L'\ i = {\Uparrow}(L\ [i;..j])\rangle$

$\mathit{findmax}\ i\ j \;\Leftarrow\;$ **if** $j{-}i = 1$ **then** ok

$\longrightarrow$ **else** $(\mathit{findmax}\ i\ (\mathit{div}\ (i{+}j)\ 2) \;\|\; \mathit{findmax}\ (\mathit{div}\ (i{+}j)\ 2)\ j).$

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$$findmax\ i\ j \ \Leftarrow \ \textbf{if}\ j{-}i = 1\ \textbf{then}\ ok$$

$$\textbf{else}\quad (findmax\ i\ (div\ (i{+}j)\ 2)\ \|\ findmax\ (div\ (i{+}j)\ 2)\ j).$$

$$L\ i := (L\ i){\uparrow}(L\ (div\ (i{+}j)\ 2))\ \ \textbf{fi}$$

recursive time $= ceil\ (log\ (j{-}i))$