# Function Theory

$\langle v\!: D\cdot\, b\rangle$ “map $v$ in $D$ to $b$”

body (may use $v$ )

domain, type (a bunch)

variable, parameter (a fresh name)

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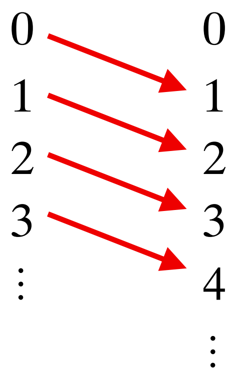
body (may use $v$ )

domain, type (a bunch)

variable, parameter (a fresh name)

$v\text{: } D$ is a local axiom within $b$

$\langle n\text{: } nat\cdot\ n{+}1\rangle$



# Renaming

$$\langle n\colon nat\cdot\ n{+}1\rangle \quad = \quad \langle m\colon nat\cdot\ m{+}1\rangle$$

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# Domain and Size

| | |
|---|---|
| $\Box f$ | "domain of $f$" |
| $\# f$ | "size of $f$" |

# Renaming

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# Domain and Size

$\Box f$ "domain of $f$"

$\#f$ "size of $f$"

$$\Box\langle n\colon nat\cdot\ n{+}1\rangle \quad = \quad nat$$

$$\#\langle n\colon nat\cdot\ n{+}1\rangle \quad = \quad \infty$$

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$$\langle n\text{: } nat\cdot\ n{+}1\rangle \quad = \quad \langle m\text{: } nat\cdot\ m{+}1\rangle$$

## Domain and Size

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## Application

$f\,x$ "$f$ applied to $x$"

# Renaming

$\langle n: nat \cdot n+1 \rangle \quad = \quad \langle m: nat \cdot m+1 \rangle$

# Domain and Size

$\square f$ "domain of $f$"

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$\square \langle n: nat \cdot n+1 \rangle \quad = \quad nat$

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# Application

$f\, x$ "$f$ applied to $x$" or "$f$ of $x$"

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$\langle n\text{: } nat\cdot\ n{+}1\rangle \ = \ \langle m\text{: } nat\cdot\ m{+}1\rangle$

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$fx$ ✗

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$fx$ ✗ $f(x)$

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$fx$ ✗ $f\,(x)$ $(f)x$

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$\langle n{:}\ nat\cdot\ n{+}1\rangle \quad = \quad \langle m{:}\ nat\cdot\ m{+}1\rangle$

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$fx$ ✗ $\quad f\,(x) \quad (f)x \quad f\,(x{+}1) \quad -x$

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$\langle n: nat\cdot n+1\rangle\ 3$

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$\langle n\text{: } nat \cdot n{+}1\rangle\ 3 \quad = \quad 3{+}1$

## Renaming

$\langle n: nat\cdot\ n+1\rangle \quad = \quad \langle m: nat\cdot\ m+1\rangle$

## Domain and Size

$\Box f$ "domain of $f$"

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$\langle n: nat\cdot\ n+1\rangle\ 3 \quad = \quad 3+1 \quad = \quad 4$

## two variables

$$avg \;=\; \langle x\text{: } xrat\cdot \langle y\text{: } xrat\cdot (x+y)/2\rangle\rangle$$

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# two variables

$avg = \langle x\text{: } xrat\cdot \langle y\text{: } xrat\cdot (x{+}y)/2\rangle\rangle$

$avg\ 3 = \langle y\text{: } xrat\cdot (3{+}y)/2\rangle$

# two variables

$avg\ =\ \langle x\colon xrat\cdot\ \langle y\colon xrat\cdot\ (x+y)/2\rangle\rangle$

$avg\ 3\ =\ \langle y\colon xrat\cdot\ (3+y)/2\rangle$

$avg\ 3\ 5\ =\ (3+5)/2$

# two variables

$avg = \langle x: xrat \cdot \langle y: xrat \cdot (x+y)/2 \rangle\rangle$

$avg\ 3 = \langle y: xrat \cdot (3+y)/2 \rangle$

$avg\ 3\ 5 = (3+5)/2 = 4$

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$$avg\ 3\ 5 \;=\; (3+5)/2 \;=\; 4$$

$$avg(3, 5)$$

## two variables

$$avg = \langle x{:}\ xrat\cdot \langle y{:}\ xrat\cdot (x+y)/2\rangle\rangle$$

$$avg\ 3 = \langle y{:}\ xrat\cdot (3+y)/2\rangle$$

$$avg\ 3\ 5 = (3+5)/2 = 4$$

$$avg(3, 5) = avg\ 3, avg\ 5$$

## two variables

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$$avg\ 3\ 5 \ = \ (3+5)/2 \ = \ 4$$

$$avg(3, 5) \ = \ avg\ 3, avg\ 5$$

## predicate

function with binary result

$$even \ = \ \langle i\text{: } int\cdot \ i/2\text{: } int\rangle$$

## two variables

$avg\ =\ \langle x: xrat\cdot \langle y: xrat\cdot (x+y)/2\rangle\rangle$

$avg\ 3\ =\ \langle y: xrat\cdot (3+y)/2\rangle$

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## predicate

function with binary result

$even\ =\ \langle i: int\cdot i/2: int\rangle$

## relation

function with predicate result

$divides\ =\ \langle n: nat+1\cdot \langle i: int\cdot\ i/n: int\rangle\rangle$

## two variables

$avg\ =\ \langle x\colon xrat\cdot\ \langle y\colon xrat\cdot\ (x{+}y)/2\rangle\rangle$

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## predicate   function with binary result

$even\ =\ \langle i\colon int\cdot\ i/2\colon int\rangle$

## relation   function with predicate result

$divides\ =\ \langle n\colon nat{+}1\cdot\ \langle i\colon int\cdot\ \ i/n\colon int\rangle\rangle$

$even\ =\ divides\ 2$

## selective union

$f \mid g$ "$f$ otherwise $g$"

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## abbreviated function notations

## selective union

$f \mid g$ "$f$ otherwise $g$"

$\square(f \mid g) \;=\; \square f, \square g$

$(f \mid g)\, x \;=\;$ **if** $x$: $\square f$ **then** $f\, x$ **else** $g\, x$ **fi**

## abbreviated function notations

$\langle x\!: xrat \cdot \langle y\!: xrat \cdot (x+y)/2\rangle\rangle$

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$\langle x\text{: } xrat\cdot \langle y\text{: } xrat\cdot (x+y)/2\rangle\rangle = \langle x, y\text{: } xrat\cdot (x+y)/2\rangle$

$\langle n\text{: } nat\cdot n+1\rangle$

## selective union

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$\langle n\text{: } nat\cdot n+1\rangle \;=\; \langle n\cdot n+1\rangle$

## selective union

$f \mid g$ "$f$ otherwise $g$"

$\square(f \mid g) \;=\; \square f, \square g$

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## abbreviated function notations

$\langle x{:}\ xrat\cdot\ \langle y{:}\ xrat\cdot\ (x+y)/2\rangle\rangle \;=\; \langle x, y{:}\ xrat\cdot\ (x+y)/2\rangle$

$\langle n{:}\ nat\cdot\ n+1\rangle \;=\; \langle n\cdot\ n+1\rangle$

$\langle n{:}\ 2\cdot\ 3\rangle$

## selective union

$f \mid g$ "$f$ otherwise $g$"

$$\square(f \mid g) \ = \ \square f, \square g$$

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$$\langle x{:}\ xrat \cdot \langle y{:}\ xrat \cdot (x+y)/2\rangle\rangle \ = \ \langle x, y{:}\ xrat \cdot (x+y)/2\rangle$$

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$\langle n{:}\ 2 \cdot 3\rangle \ = \ 2 \rightarrow 3$ scope brackets go with variable, dot becomes arrow

## selective union

$f \mid g$ "$f$ otherwise $g$"

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## abbreviated function notations

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$\langle x{:}\ int\cdot\ \langle y{:}\ int\cdot\ x{+}3\rangle\rangle$

## selective union

$f \mid g$ "$f$ otherwise $g$"

$\square(f \mid g) \;=\; \square f, \square g$

$(f \mid g)\; x \;=\;$ **if** $x\!: \square f$ **then** $f\; x$ **else** $g\; x$ **fi**

## abbreviated function notations

$\langle x\!: xrat\cdot \langle y\!: xrat\cdot (x+y)/2\rangle\rangle \;=\; \langle x, y\!: xrat\cdot (x+y)/2\rangle$

$\langle n\!: nat\cdot n+1\rangle \;=\; \langle n\cdot n+1\rangle$

$\langle n\!: 2\cdot 3\rangle \;=\; 2 \rightarrow 3$ scope brackets go with variable, dot becomes arrow

$\langle x\!: int\cdot \langle y\!: int\cdot x+3\rangle\rangle \;=\; x+3$ ? but we can't apply it

# Scope and Substitution

**local**

**nonlocal**

# Scope and Substitution

**local**

**nonlocal** global

# Scope and Substitution

**local** bound

**nonlocal** global, free

# Scope and Substitution

**local** bound, hidden

**nonlocal** global, free, visible

# Scope and Substitution

**local** bound, hidden, private

**nonlocal** global, free, visible, public

# Scope and Substitution

**local** bound, hidden, private

introduction is inside the expression (formal)

**nonlocal** global, free, visible, public

introduction is outside the expression (formal or informal)

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$$\langle x\cdot \quad x \quad y \quad \rangle ( \quad x \quad y \quad )$$

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$$\langle x \cdot \quad x \quad\quad y \quad \rangle ( \quad x \quad y \quad )$$

↑ ↑

local

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nonlocal

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$$\langle x\cdot \quad x \quad \langle x\cdot \quad x \quad \rangle \quad x \quad \rangle\ 3$$

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$$\langle x \cdot \quad x \quad \langle x \cdot \quad x \quad \rangle \quad x \quad \rangle\, 3$$

$$= \quad ( \quad 3 \quad \langle x \cdot \quad x \quad \rangle \quad 3 \quad )$$

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$\langle y \cdot \quad x \quad y \quad \langle x \cdot \quad x \quad y \quad \rangle \quad x \quad y \quad \rangle\, x$

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$$\langle y\cdot \quad x \quad y \quad \langle x\cdot \quad x \quad y \quad \rangle \quad x \quad y \quad \rangle\, x \qquad \text{rename inner } x \text{ to } z$$

$$= \quad \langle y\cdot \quad x \quad y \quad \langle z\cdot \quad z \quad y \quad \rangle \quad x \quad y \quad \rangle\, x$$

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$$\langle y\cdot \quad x \quad y \quad \langle x\cdot \quad x \quad y \quad \rangle \quad x \quad y \quad \rangle\, x$$ rename inner $x$ to $z$

$$= \quad \langle y\cdot \quad x \quad y \quad \langle z\cdot \quad z \quad y \quad \rangle \quad x \quad y \quad \rangle\, x$$ now apply

$$= \quad ( \quad x \quad x \quad \langle z\cdot \quad z \quad x \quad \rangle \quad x \quad x \quad )$$