Function Fine Points

partial:	sometimes no result
total:	always at least one result

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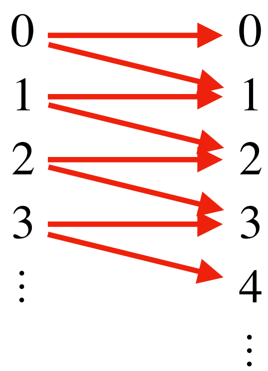
$\langle n{:}\ nat\cdot\ n, n+1\rangle$	total and nondeterministic



Function Fine Points

partial:	sometimes no result
total:	always at least one result
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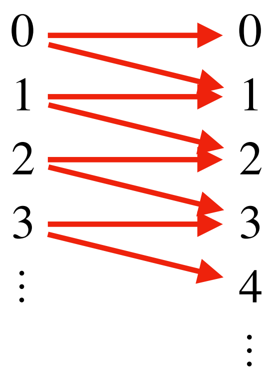
$\langle n{:}\ nat\cdot\ n, n{+}1\rangle$	total and nondeterministic



$$\langle n{:}\ nat\cdot\ n, n{+}1\rangle\ 3 \;=\; 3, 4$$

distribution

$$(f, g)\, x \;=\; f\,x, g\,x$$

distribution

$$
\begin{aligned}
(f, g)\, x &= f\, x, g\, x \\
f\,(x, y) &= f\, x, f\, y
\end{aligned}
$$

distribution

$$(f, g)\, x \;=\; f\,x, g\,x$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n\colon nat\cdot\ n{+}n\rangle$$

distribution

$$(f, g)\ x \;=\; f\,x,\ g\,x$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n\!: nat\cdot\ n{+}n\rangle$$

$$double\ 2 \;=\; 4$$

distribution

$$(f, g)\, x \;=\; f\,x, g\,x$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n\!: nat\cdot\, n{+}n\rangle$$

$$double\ 2 \;=\; 4$$

$$double\ (2, 3) \;=\; double\ 2, double\ 3 \;=\; 4, 6$$

distribution

$$(f, g)\ x \;=\; f\,x, g\,x$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n{:}\ nat\cdot\ n{+}n\rangle$$

$$double\ 2 \;=\; 4$$

$$double\ (2, 3) \;=\; double\ 2, double\ 3 \;=\; 4, 6$$

$$double\ (2, 3) \;\neq\; (2, 3) + (2, 3) \;=\; 4, 5, 6$$

distribution

$$(f, g)\ x \;=\; f\,x, g\ x$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n: nat\cdot\ n{+}n \rangle$$

$$double\ 2 \;=\; 4$$

$$double\ (2, 3) \;=\; double\ 2, double\ 3 \;=\; 4, 6$$

$$double\ (2, 3) \;\neq\; (2, 3) + (2, 3) \;=\; 4, 5, 6$$

$$tiny \;=\; \langle S: {\text{↯}}nat\cdot\ \$S < 3 \rangle$$

distribution

$(f, g)\, x \;=\; f\, x, g\, x$

$f\, (x, y) \;=\; f\, x, f\, y$

$double \;=\; \langle n{:}\ nat\cdot\ n{+}n \rangle$

$double\ 2 \;=\; 4$

$double\ (2, 3) \;=\; double\ 2, double\ 3 \;=\; 4, 6$

$double\ (2, 3) \;\ne\; (2, 3) + (2, 3) \;=\; 4, 5, 6$

$tiny \;=\; \langle S{:}\ ϟnat\cdot\ \$S < 3 \rangle$

$tiny\ \{null\} \;=\; \top$

distribution

$$(f, g)\,x \;=\; f\,x, g\,x$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n\!: nat\cdot\; n{+}n\rangle$$

$$double\;2 \;=\; 4$$

$$double\;(2, 3) \;=\; double\;2, double\;3 \;=\; 4, 6$$

$$double\;(2, 3) \;\neq\; (2, 3) + (2, 3) \;=\; 4, 5, 6$$

$$tiny \;=\; \langle S\!: \lightning nat\cdot\; \$S < 3\rangle$$

$$tiny\;\{null\} \;=\; \top$$

$$tiny\;\{0, 1, 2, 3\} \;=\; \bot$$

distribution

$$(f, g)\, x \;=\; f\,x, g\,x$$

$$f\,(x, y) \;=\; f\,x, f\,y$$

$$double \;=\; \langle n\colon nat\cdot\; n{+}n \rangle$$

$$double\; 2 \;=\; 4$$

$$double\; (2, 3) \;=\; double\; 2, double\; 3 \;=\; 4, 6$$

$$double\; (2, 3) \;\ne\; (2, 3) + (2, 3) \;=\; 4, 5, 6$$

$$tiny \;=\; \langle S\colon \text{\textlightning} nat\cdot\; \$S < 3 \rangle$$

$$tiny\; \{null\} \;=\; \top$$

$$tiny\; \{0, 1, 2, 3\} \;=\; \bot$$

$$tiny\; null \;=\; null$$

function inclusion

$$f\colon g \;=\; \square f\colon\colon \square g \;\wedge\; \forall x\colon \square g\cdot f\,x\colon g\,x$$

function inclusion

$$f\colon g \;=\; \square f\colon\colon \square g \;\wedge\; \forall x\colon \square g\cdot f\,x\colon g\,x$$

$$f\colon A\to B$$

function inclusion

$f\colon g \;=\; \square f\colon\colon \square g \;\wedge\; \forall x\colon \square g \cdot f\, x\colon g\, x$

$A \rightarrow B \;=\; \langle a\colon A \cdot B \rangle$

$f\colon A \rightarrow B$

function inclusion

$$f\colon g \;=\; \square f\colon\colon \square g \;\wedge\; \forall x\colon \square g\cdot f\,x\colon g\,x$$

$$A\rightarrow B \;=\; \langle a\colon A\cdot B\rangle$$

$A\rightarrow B$ is a function whose domain is A and whose result is B .

$$f\colon A\rightarrow B$$

function inclusion

$$f: g \;=\; \Box f:: \Box g \;\wedge\; \forall x: \Box g \cdot f\,x: g\,x$$

$$A \rightarrow B \;=\; \langle a: A \cdot B \rangle$$

$A \rightarrow B$ is a function whose domain is A and whose result is B .

$$f: A \rightarrow B \;=\; \Box f:: A \;\wedge\; \forall a: A \cdot f\,a: B$$

function inclusion

$$f: g \quad = \quad \Box f:: \Box g \;\wedge\; \forall x: \Box g \cdot f\, x: g\, x$$

$$A \rightarrow B \quad = \quad \langle a: A \cdot B \rangle$$

$A \rightarrow B$ is a function whose domain is A and whose result is B .

$$f: A \rightarrow B \quad = \quad \Box f:: A \;\wedge\; \forall a: A \cdot f\, a: B$$

$A \rightarrow B$ is all those functions whose domain is at least A and whose result is at most B .

function inclusion

$$f: g \ = \ \Box f:: \Box g \ \wedge \ \forall x: \Box g \cdot f\ x: g\ x$$

$$A \rightarrow B \ = \ \langle a: A \cdot B \rangle$$

$A \rightarrow B$ is a function whose domain is A and whose result is B .

$$f: A \rightarrow B \ = \ \Box f:: A \ \wedge \ \forall a: A \cdot f\ a: B$$

$A \rightarrow B$ is all those functions whose domain is at least A and whose result is at most B .

$A \rightarrow B$ is antimonotonic in A and monotonic in B .

function inclusion

$$f: g \ = \ \square f:: \square g \ \wedge \ \forall x: \square g \cdot f\,x: g\,x$$

$$A \rightarrow B \ = \ \langle a: A \cdot B \rangle$$

$A \rightarrow B$ is a function whose domain is A and whose result is B .

$$f: A \rightarrow B \ = \ \square f:: A \ \wedge \ \forall a: A \cdot f\,a: B$$

$A \rightarrow B$ is all those functions whose domain is at least A and whose result is at most B .

$A \rightarrow B$ is antimonotonic in A and monotonic in B .

$suc: nat \rightarrow nat$

function inclusion

$$f: g \;=\; \square f:: \square g \;\wedge\; \forall x: \square g \cdot f\,x: g\,x$$

$$A{\to}B \;=\; \langle a: A\cdot B\rangle$$

$A{\to}B$ is a function whose domain is A and whose result is B .

$$f: A{\to}B \;=\; \square f:: A \;\wedge\; \forall a: A\cdot f\,a: B$$

$A{\to}B$ is all those functions whose domain is at least A and whose result is at most B .

$A{\to}B$ is antimonotonic in A and monotonic in B .

$suc: nat{\to}nat$ function inclusion

$=$ $\square suc:: nat \;\wedge\; \forall n: nat\cdot suc\,n: nat$

function inclusion

$$f: g \;=\; \square f:: \square g \;\wedge\; \forall x: \square g\cdot f\,x: g\,x$$

$$A\to B \;=\; \langle a: A\cdot B\rangle$$

$A\to B$ is a function whose domain is A and whose result is B .

$$f: A\to B \;=\; \square f:: A \;\wedge\; \forall a: A\cdot f\,a: B$$

$A\to B$ is all those functions whose domain is at least A and whose result is at most B .

$A\to B$ is antimonotonic in A and monotonic in B .

	$suc: nat\to nat$	function inclusion
$=$	$\square suc:: nat \;\wedge\; \forall n: nat\cdot suc\,n: nat$	definition of suc
$=$	$nat:: nat \;\wedge\; \forall n: nat\cdot n+1: nat$	

function inclusion

$$f\colon g \;=\; \Box f\colon\colon \Box g \;\wedge\; \forall x\colon \Box g\cdot f\,x\colon g\,x$$

$$A\rightarrow B \;=\; \langle a\colon A\cdot B\rangle$$

$A\rightarrow B$ is a function whose domain is A and whose result is B .

$$f\colon A\rightarrow B \;=\; \Box f\colon\colon A \;\wedge\; \forall a\colon A\cdot f\,a\colon B$$

$A\rightarrow B$ is all those functions whose domain is at least A and whose result is at most B .

$A\rightarrow B$ is antimonotonic in A and monotonic in B .

	$suc\colon nat\rightarrow nat$	function inclusion
$=$	$\Box suc\colon\colon nat \;\wedge\; \forall n\colon nat\cdot suc\,n\colon nat$	definition of suc
$=$	$nat\colon\colon nat \;\wedge\; \forall n\colon nat\cdot n{+}1\colon nat$	reflexivity and definition of nat
$=$	$\top$	

function inclusion

$$f\colon g \;=\; \Box f\colon\colon \Box g \;\wedge\; \forall x\colon \Box g \cdot f\,x\colon g\,x$$

function inclusion

$$f: g \;=\; \Box f:: \Box g \;\land\; \forall x: \Box g \cdot f\,x: g\,x$$

suc: *nat*→*nat*

even: *int*→*bin*

avg: *rat*→*rat*→*rat*

function inclusion

$$f\colon g \;=\; \square f\colon\colon \square g \;\land\; \forall x\colon \square g\cdot f\,x\colon g\,x$$

suc: *nat*→*nat*

even: *int*→*bin*

avg: *rat*→*rat*→*rat*

$$A\colon B \;\land\; f\colon B\to C \;\land\; C\colon D \;\;\Rightarrow\;\; f\colon A\to D$$

function inclusion

$$f: g \quad = \quad \Box f :: \Box g \;\land\; \forall x: \Box g \cdot f\, x: g\, x$$

$suc: nat \to nat$

$even: int \to bin$

$avg: rat \to rat \to rat$

$$A: B \;\land\; f: B \to C \;\land\; C: D \quad \Rightarrow \quad f: A \to D$$

$$(0,..10): nat \;\land\; suc: nat \to nat \;\land\; nat: int \quad \Rightarrow \quad suc: (0,..10) \to int$$

function inclusion

$f: g \;=\; \Box f:: \Box g \;\wedge\; \forall x: \Box g \cdot f\,x: g\,x$

$suc: nat \to nat$

$even: int \to bin$

$avg: rat \to rat \to rat$

$A: B \;\wedge\; f: B \to C \;\wedge\; C: D \quad\Rightarrow\quad f: A \to D$

$(0,..10): nat \;\wedge\; suc: nat \to nat \;\wedge\; nat: int \quad\Rightarrow\quad suc: (0,..10) \to int$

$\langle f: (0,..10) \to int \cdot \forall n: 0,..10 \cdot even\ (f\,n)\rangle$

function inclusion

$$f: g \;=\; \Box f:: \Box g \;\wedge\; \forall x: \Box g \cdot f\, x: g\, x$$

$suc: nat \rightarrow nat$

$even: int \rightarrow bin$

$avg: rat \rightarrow rat \rightarrow rat$

$$A: B \;\wedge\; f: B \rightarrow C \;\wedge\; C: D \;\Rightarrow\; f: A \rightarrow D$$

$$(0,..10): nat \;\wedge\; suc: nat \rightarrow nat \;\wedge\; nat: int \;\Rightarrow\; suc: (0,..10) \rightarrow int$$

$$\langle \underline{f}: (0,..10) \rightarrow int \cdot \forall n: 0,..10 \cdot even\,(f\, n) \rangle$$

function inclusion

$f: g \quad = \quad \Box f:: \Box g \ \wedge \ \forall x: \Box g \cdot f\, x: g\, x$

$suc: nat \to nat$

$even: int \to bin$

$avg: rat \to rat \to rat$

$A: B \ \wedge \ f: B \to C \ \wedge \ C: D \quad \Rightarrow \quad f: A \to D$

$(0,..10): nat \ \wedge \ suc: nat \to nat \ \wedge \ nat: int \quad \Rightarrow \quad suc: (0,..10) \to int$

$\langle f: \underline{(0,..10) \to int} \cdot \forall n: 0,..10 \cdot even\ (f\, n)\rangle$

function inclusion

$f: g \;=\; \Box f :: \Box g \;\wedge\; \forall x: \Box g \cdot f\,x: g\,x$

$suc: nat \rightarrow nat$

$even: int \rightarrow bin$

$avg: rat \rightarrow rat \rightarrow rat$

$A: B \;\wedge\; f: B \rightarrow C \;\wedge\; C: D \quad\Rightarrow\quad f: A \rightarrow D$

$(0,..10): nat \;\wedge\; suc: nat \rightarrow nat \;\wedge\; nat: int \quad\Rightarrow\quad suc: (0,..10) \rightarrow int$

$\langle f: (0,..10) \rightarrow int \cdot \underline{\forall n: 0,..10} \cdot even\ (\underline{f\,n}) \rangle$

function inclusion

$f: g \;=\; \Box f:: \Box g \;\wedge\; \forall x: \Box g \cdot f\,x: g\,x$

$suc: nat \to nat$

$even: int \to bin$

$avg: rat \to rat \to rat$

$A: B \;\wedge\; f: B \to C \;\wedge\; C: D \;\Rightarrow\; f: A \to D$

$(0,..10): nat \;\wedge\; suc: nat \to nat \;\wedge\; nat: int \;\Rightarrow\; suc: (0,..10) \to int$

$\langle f: (0,..10) \to int \cdot \forall n: 0,..10 \cdot \underline{even\ (f\ n)} \rangle$

function inclusion

$f \colon g \;=\; \Box f \colon\colon \Box g \;\wedge\; \forall x \colon \Box g \cdot f\,x \colon g\,x$

$suc \colon nat \to nat$

$even \colon int \to bin$

$avg \colon rat \to rat \to rat$

$A \colon B \;\wedge\; f \colon B \to C \;\wedge\; C \colon D \quad\Rightarrow\quad f \colon A \to D$

$(0,..10) \colon nat \;\wedge\; suc \colon nat \to nat \;\wedge\; nat \colon int \quad\Rightarrow\quad suc \colon (0,..10) \to int$

$\langle f \colon (0,..10) \to int \cdot \forall n \colon 0,..10 \cdot even\,(f\,n) \rangle\; suc$

function inclusion

$f\colon g \;=\; \Box f\colon\colon \Box g \;\wedge\; \forall x\colon \Box g\cdot f\,x\colon g\,x$

$suc\colon nat\to nat$

$even\colon int\to bin$

$avg\colon rat\to rat\to rat$

$A\colon B \;\wedge\; f\colon B\to C \;\wedge\; C\colon D \quad\Rightarrow\quad f\colon A\to D$

$(0,..10)\colon nat \;\wedge\; suc\colon nat\to nat \;\wedge\; nat\colon int \quad\Rightarrow\quad suc\colon (0,..10)\to int$

$\langle f\colon (0,..10)\to int\cdot \forall n\colon 0,..10\cdot even\,(f\,n)\rangle\; suc$

$=\quad \forall n\colon 0,..10\cdot even\,(suc\,n)$

function inclusion

$$f: g \;=\; \Box f:: \Box g \;\wedge\; \forall x: \Box g\cdot f\,x: g\,x$$

$suc: nat\to nat$

$even: int\to bin$

$avg: rat\to rat\to rat$

$$A: B \;\wedge\; f: B\to C \;\wedge\; C: D \;\Rightarrow\; f: A\to D$$

$$(0,..10): nat \;\wedge\; suc: nat\to nat \;\wedge\; nat: int \;\Rightarrow\; suc: (0,..10)\to int$$

$$\langle f: (0,..10)\to int\cdot \forall n: 0,..10\cdot even\,(f\,n)\rangle\; suc$$

$$=\quad \forall n: 0,..10\cdot even\,(suc\,n)$$

$$=\quad \bot$$

function equality

$$f = g \;\; = \;\; \square f = \square g \;\wedge\; \forall x\colon \square f \cdot f\, x = g\, x$$

function composition

If ¬ *f*: □*g* then

□(*g f*) = §*x*: □*f*· *f x*: □*g*

(*g f*) *x* = *g* (*f x*)

function composition

If ¬ f: □ g then

$$□(g\,f) \;=\; §x: □f\cdot f\,x: □\,g$$

$$(g\,f)\,x \;=\; g\,(f\,x) \qquad \longleftarrow$$

function composition

If $\neg f: \square g$ then

$$\square(g\,f) \;=\; \S x: \square f \cdot f\,x: \square g$$

$$(g\,f)\,x \;=\; g\,(f\,x)$$

function composition

If ¬ f: □g then

$\square(g\ f) \ = \ \S x{:}\ \square f\cdot\ f\ x{:}\ \square g$

$(g\ f)\ x \ = \ g\ (f\ x)$

$\square(\mathit{even}\ \mathit{suc})$

$= \ \ \S x{:}\ \square \mathit{suc}\cdot\ \mathit{suc}\ x{:}\ \square \mathit{even}$

$= \ \ \S x{:}\ \mathit{nat}\cdot\ x{+}1{:}\ \mathit{int}$

$= \ \ \mathit{nat}$

function composition

If $\neg f\colon \square g$ then

$$\square(g\,f) \;=\; \S x\colon \square f\cdot f\,x\colon \square g$$

$$(g\,f)\,x \;=\; g\,(f\,x)$$

$$
\begin{aligned}
&\square(even\ suc)\\
=\quad & \S x\colon \square suc\cdot\ suc\ x\colon \square even\\
=\quad & \S x\colon nat\cdot\ x{+}1\colon int\\
=\quad & nat
\end{aligned}
$$

$(even\ suc)\ 3$

function composition

If $\neg f: \square g$ then

$$\square(g\,f) \;=\; \S x: \square f\cdot\ f\,x: \square g$$

$$(g\,f)\,x \;=\; g\,(f\,x)$$

$$\square(\mathit{even}\ \mathit{suc})$$

$$=\quad \S x: \square \mathit{suc}\cdot\ \mathit{suc}\ x: \square \mathit{even}$$

$$=\quad \S x: \mathit{nat}\cdot\ x{+}1: \mathit{int}$$

$$=\quad \mathit{nat}$$

$$(\mathit{even}\ \mathit{suc})\ 3 \;=\; \mathit{even}\ (\mathit{suc}\ 3)$$

function composition

If $\neg f\colon \square g$ then

$$\square(g\ f) \ = \ \S x\colon \square f\cdot\ f\ x\colon \square g$$

$$(g\ f)\ x \ = \ g\ (f\ x)$$

$$\square(even\ suc)$$

$$=\quad \S x\colon \square suc\cdot\ suc\ x\colon \square even$$

$$=\quad \S x\colon nat\cdot\ x{+}1\colon int$$

$$=\quad nat$$

$$(even\ suc)\ 3 \ = \ even\ (suc\ 3) \ = \ even\ 4$$

function composition

If $\neg f\colon \square g$ then

$$\square(g\,f) \;=\; \S x\colon \square f\cdot\, f\,x\colon \square g$$

$$(g\,f)\,x \;=\; g\,(f\,x)$$

$$\square(even\ suc)$$

$$=\quad \S x\colon \square suc\cdot\ suc\ x\colon \square even$$

$$=\quad \S x\colon nat\cdot\ x{+}1\colon int$$

$$=\quad nat$$

$$(even\ suc)\ 3 \;=\; even\ (suc\ 3) \;=\; even\ 4 \;=\; \top$$

function composition

If ¬ f: □g then

$□(g\ f) = §x{:}\ □f \cdot f\ x{:}\ □g$

$(g\ f)\ x = g\ (f\ x)$

$□(even\ suc)$

$= §x{:}\ □suc \cdot suc\ x{:}\ □even$

$= §x{:}\ nat \cdot x{+}1{:}\ int$

$= nat$

$(even\ suc)\ 3 = even\ (suc\ 3) = even\ 4 = \top$

$(-suc)\ 3 = -(suc\ 3) = -4$

function composition

If $\neg\, f\colon \square g$ then

$$\square(g\,f) \;=\; \S x\colon \square f\cdot\, f\,x\colon \square g$$

$$(g\,f)\,x \;=\; g\,(f\,x)$$

$$\begin{aligned}
& \square(\mathit{even}\ \mathit{suc}) \\
= \quad & \S x\colon \square \mathit{suc}\cdot\ \mathit{suc}\ x\colon \square \mathit{even} \\
= \quad & \S x\colon \mathit{nat}\cdot\ x{+}1\colon \mathit{int} \\
= \quad & \mathit{nat}
\end{aligned}$$

$$(\mathit{even}\ \mathit{suc})\ 3 \;=\; \mathit{even}\ (\mathit{suc}\ 3) \;=\; \mathit{even}\ 4 \;=\; \top$$

$$(-\mathit{suc})\ 3 \;=\; -(\mathit{suc}\ 3) \;=\; -4$$

$$(\neg\mathit{even})\ 3 \;=\; \neg(\mathit{even}\ 3) \;=\; \neg\bot \;=\; \top$$

function composition

function composition

Suppose x, y: *int*

f, g: $int \rightarrow int$

h: $int \rightarrow int \rightarrow int$

function composition

Suppose $x, y: int$

$f, g: int \to int$

$h: int \to int \to int$

Then

$$h\ f\ x\ g\ y$$

function composition

Suppose $x, y: int$

$f, g: int \to int$

$h: int \to int \to int$

Then

$$h\ f\ x\ g\ y$$

$$= \quad (((h\ f)\ x)\ g)\ y$$

function composition

Suppose x, y: *int*

f, g: *int*→*int*

h: *int*→*int*→*int*

Then

$$h\ f\ x\ g\ y$$

$$= \quad (((h\ f)\ x)\ g)\ y$$

$$= \quad ((h\ (f\ x))\ g)\ y$$

function composition

Suppose x, y: *int*

f, g: *int*→*int*

h: *int*→*int*→*int*

Then

$$h\ f\ x\ g\ y$$

$$= \quad (((h\ f)\ x)\ g)\ y$$

$$= \quad ((h\ (f\ x))\ g)\ y$$

$$= \quad (h\ (f\ x))\ (g\ y)$$

function composition

Suppose x, y: *int*

f, g: *int*→*int*

h: *int*→*int*→*int*

Then

$$
\begin{aligned}
& h\ f\ x\ g\ y \\
= \quad & (((h\ f)\ x)\ g)\ y \\
= \quad & ((h\ (f\ x))\ g)\ y \\
= \quad & (h\ (f\ x))\ (g\ y) \\
= \quad & h\ (f\ x)\ (g\ y)
\end{aligned}
$$

list as function

If $m: \Box L$ then

$$\langle n: \Box L \cdot L\, n \rangle$$

list as function

If m: $\square L$ then

$$\langle n: \square L \cdot L\, n \rangle\, m$$

list as function

If $m\colon \Box L$ then

$$\langle n\colon \Box L \cdot L\, n\rangle\, m \;=\; L\, m$$

list as function

If $m: \square L$ then

$$\underline{\langle n: \square L \cdot L\,n \rangle}\; m \;=\; \underline{L}\; m$$

function ≈ list

application ≈ indexing

list as function

If $m: \square L$ then

$$\langle n: \square L \cdot L\, n \rangle\, m \;=\; L\, m$$

function ≈ list

application ≈ indexing

function composition ≈ list composition

list as function

If $m: \square L$ then

$\langle n: \square L \cdot L\, n \rangle\, m \;=\; L\, m$

function ≈ list

application ≈ indexing

function composition ≈ list composition

– [3; 5; 2]

suc [3; 5; 2]

list as function

If m: $\square L$ then

$\langle n: \square L \cdot L\ n \rangle\ m\ =\ L\ m$

function ≈ list

application ≈ indexing

function composition ≈ list composition

$-[3;\ 5;\ 2]\ =\ [-3;\ -5;\ -2]$

$suc\ [3;\ 5;\ 2]\ =\ [4;\ 6;\ 3]$

list as function

If $m: \Box L$ then

$\langle n: \Box L \cdot L\, n \rangle\, m \;=\; L\, m$

function $\approx$ list

application $\approx$ indexing

function composition $\approx$ list composition

$-\,[3; 5; 2] \;=\; [-3; -5; -2]$

$suc\,[3; 5; 2] \;=\; [4; 6; 3]$

$1 \rightarrow 21 \mid [10; 11; 12] \quad = \quad [10; 21; 12]$

list as function

If m: □L then

$\langle n\text{: }\square L\cdot L\,n\rangle\, m \;=\; L\, m$

function ≈ list

application ≈ indexing

function composition ≈ list composition

– [3; 5; 2] = [–3; –5; –2]

suc [3; 5; 2] = [4; 6; 3]

$1{\rightarrow}21 \mid [10;\, 11;\, 12] \;=\; [10;\, 21;\, 12]$

list as function

If $m: \square L$ then

$\langle n: \square L \cdot L\, n \rangle\, m \;=\; L\, m$

function ≈ list

application ≈ indexing

function composition ≈ list composition

$-\,[3;\, 5;\, 2] \;=\; [-3;\, -5;\, -2]$

$suc\,[3;\, 5;\, 2] \;=\; [4;\, 6;\, 3]$

$1 \rightarrow 21 \mid \underline{[10;\, 11;\, 12]} \;=\; [10;\, 21;\, 12]$

list as function

If $m: \square L$ then

$\langle n: \square L \cdot L\, n \rangle\, m \;=\; L\, m$

function ≈ list

application ≈ indexing

function composition ≈ list composition

$-[3; 5; 2] \;=\; [-3; -5; -2]$

$suc\ [3; 5; 2] \;=\; [4; 6; 3]$

$1 \rightarrow 21 \mid [10; 11; 12] \quad = \quad [10; 21; 12]$

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$\Sigma L \;=\; \Sigma\langle n: \square L \cdot L\, n\rangle \;=\; \Sigma n: \square L \cdot L\, n$

limit

f: $nat \rightarrow rat$

limit

f: $nat \to rat$

$f\,0; f\,1; f\,2; \ldots$ is a sequence of rationals

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$f\,0;\ f\,1;\ f\,2;\ ...$ is a sequence of rationals

$\Updownarrow f$

limit

f: $nat \to rat$

$f\,0;\ f\,1;\ f\,2;$ ... is a sequence of rationals

$$(\Uparrow m\cdot\ \Downarrow n\cdot\ f(m+n)) \ \ \le\ \ \Updownarrow f \ \ \le\ \ (\Downarrow m\cdot\ \Uparrow n\cdot\ f(m+n))$$

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$$\Updownarrow n\cdot (-1)^n$$

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$$\Downarrow f \ \le\ \Updownarrow f \ \le\ \Uparrow f$$

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$\Downarrow f\ \le\ \Updownarrow f\ \le\ \Uparrow f$

x: $xreal\ \ =\ \ \exists f{:}\ nat \to rat\cdot\ \ x = \Updownarrow f$

limit

$p: nat \rightarrow bin$

limit

p: $nat \to bin$

$p\ 0$; $p\ 1$; $p\ 2$; ... is a sequence of binary values

limit

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$p\ 0;\ p\ 1;\ p\ 2;\ ...$ is a sequence of binary values

$$\exists m\cdot\ \forall n\cdot\ p(m+n)\ \ \Rightarrow\ \ \Updownarrow p\ \ \Rightarrow\ \ \forall m\cdot\ \exists n\cdot\ p(m+n)$$

limit

p: $nat \rightarrow bin$

$p\ 0;\ p\ 1;\ p\ 2;\ ...$ is a sequence of binary values

$$\exists m \cdot\ \forall n \cdot\ p(m+n) \ \Longrightarrow\ \Updownarrow p \ \Longrightarrow\ \forall m \cdot\ \exists n \cdot\ p(m+n)$$

$$\exists m \cdot\ \forall i \cdot\ i \ge m \Rightarrow p\ i \ \Longrightarrow\ \Updownarrow p$$

limit

p: $nat \rightarrow bin$

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$$\exists m\cdot\ \forall i\cdot\ i \geq m \Rightarrow \neg\ p\ i \ \ \Longrightarrow\ \ \neg\ \Updownarrow p$$

limit

p: $nat \to bin$

$p\ 0;\ p\ 1;\ p\ 2;\ ...$ is a sequence of binary values

$$\exists m\cdot\ \forall n\cdot\ p(m+n) \quad\Longrightarrow\quad \Updownarrow p \quad\Longrightarrow\quad \forall m\cdot\ \exists n\cdot\ p(m+n)$$

$$\exists m\cdot\ \forall i\cdot\ i\ge m \Rightarrow p\ i \quad\Longrightarrow\quad \Updownarrow p$$

$$\exists m\cdot\ \forall i\cdot\ i\ge m \Rightarrow \neg\ p\ i \quad\Longrightarrow\quad \neg\ \Updownarrow p$$

$$\Updownarrow n\cdot\ 1/(n+1) = 0 \quad = \quad \bot$$