11 (dual) One operator is the dual of another operator if it negates the result when applied to the negated operands. The zero-operand operators $\top$ and $\bot$ are each other's duals. If $op_0\ \neg a \ = \ \neg\ op_1\ a$ then op_0 and op_1 are duals. If $(\neg a)\ op_0\ (\neg b) \ = \ \neg(a\ op_1\ b)$ then op_0 and op_1 are duals. And so on for more operands.

(a) Of the 4 one-operand binary operators, there is 1 pair of duals, and 2 operators that are their own duals. Find them.

(b) Of the 16 two-operand binary operators, there are 6 pairs of duals, and 4 operators that are their own duals. Find them.

(c) What is the dual of the three-operand operator **if then else fi** ? Express it using only the operator **if then else fi** .

(d) The dual of a binary expression without variables is formed as follows: replace each operator with its dual, adding parentheses if necessary to maintain the precedence. Explain why the dual of a theorem is an antitheorem, and vice versa.

(e) Let P be a binary expression without variables. From part (d) we know that every binary expression without variables of the form

$$\text{(dual of } P\text{)} \ = \ \neg P$$

is a theorem. Therefore, to find the dual of a binary expression with variables, we must replace each operator by its dual and negate each variable. For example, if a and b are binary variables, then the dual of $a \wedge b$ is $\neg a \vee \neg b$. And since

$$\text{(dual of } a \wedge b\text{)} \ = \ \neg(a \wedge b)$$

we have one of the Duality Laws:

$$\neg a \vee \neg b \ = \ \neg(a \wedge b)$$

The other of the Duality Laws is obtained by equating the dual and negation of $a \vee b$. Obtain five laws that do not appear in this book by equating a dual with a negation.

(f) Dual operators have value tables that are each other's vertical mirror reflections. For example, the value table for $\wedge$ (below left) is the vertical mirror reflection of the value table for $\vee$ (below right).

$\wedge$:

$\top\top$	$\top$
$\top\bot$	$\bot$
$\bot\top$	$\bot$
$\bot\bot$	$\bot$

$\vee$:

$\top\top$	$\top$
$\top\bot$	$\top$
$\bot\top$	$\top$
$\bot\bot$	$\bot$

Design symbols (you may redesign existing symbols where necessary) for the 4 one-operand and 16 two-operand binary operators according to the following criteria.

(i) Dual operators should have symbols that are vertical mirror reflections (like $\wedge$ and $\vee$). This implies that self-dual operators have vertically symmetric symbols, and all others have vertically asymmetric symbols.

(ii) If $a\ op_0\ b \ = \ b\ op_1\ a$ then op_0 and op_1 should have symbols that are horizontal mirror reflections (like $\Rightarrow$ and $\Leftarrow$). This implies that symmetric operators have horizontally symmetric symbols, and all others have horizontally asymmetric symbols.

After trying the question, scroll down to the solution.

(a) Of the 4 one-operand binary operators, there is 1 pair of duals, and 2 operators that are their own duals. Find them.

§ To answer this question, I'll use the symbols I introduce in part (f). The pair of duals is: $\bar{\top}$ (always $\top$) and $\underline{\bot}$ (always $\bot$). The two self-duals are: $\mathtt{I}$ (identity) and $\ddagger$ (negation).

(b) Of the 16 two-operand binary operators, there are 6 pairs of duals, and 4 operators that are their own duals. Find them.

§ To answer this question, I'll use the symbols I will introduce in part (f). The six dual pairs are: $\bar{\triangle}\,\underline{\triangledown}$, $\vee\,\wedge$, $\geqslant\,\gtrdot$, $\leqslant\,\lessdot$, $\triangleq\,\bar{\triangledown}$, $\triangle\,\triangledown$. The four self-duals are: $<$, $>$, $\triangleright$, $\triangleleft$.

(c) What is the dual of the three-operand operator **if then else fi** ? Express it using only the operator **if then else fi** .

§ Its value table is

$\top\top\top$	$\top\top\bot$	$\top\bot\top$	$\top\bot\bot$	$\bot\top\top$	$\bot\top\bot$	$\bot\bot\top$	$\bot\bot\bot$
$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\bot$	$\bot$

The dual of **if** a **then** b **else** c **fi** is equivalent to **if** a **then** c **else** b **fi** .

(d) The dual of a binary expression without variables is formed as follows: replace each operator with its dual, adding parentheses if necessary to maintain the precedence. Explain why the dual of a theorem is an antitheorem, and vice versa.

§ I will show that for every expression P without variables, (dual of P) $= \neg P$. I do so by induction on the structure of expression P . The two binary values give us two base cases.

(dual of $\top$) — use the dual-forming rules
$=$ $\bot$
$=$ $\neg\top$

(dual of $\bot$) — use the dual-forming rules
$=$ $\top$
$=$ $\neg\bot$

There is an induction step for each of the binary operators. Suppose (this is an induction hypothesis) that (dual of P) $= \neg P$. Then

(dual of $\neg P$) — use the dual-forming rules
$=$ $\neg$(dual of P) — use the induction hypothesis
$=$ $\neg\neg P$

Suppose that (dual of P) $= \neg P$ and (dual of Q) $= \neg Q$. Then

(dual of $P \wedge Q$) — use the dual-forming rules
$=$ (dual of P) $\vee$ (dual of Q) — use the induction hypotheses
$=$ $\neg P \vee \neg Q$ — use duality law
$=$ $\neg(P \wedge Q)$

And similarly for all other operators.

§(e) From $a=b$ we get $\neg a \ddagger \neg b \;=\; \neg(a=b)$
From **if** a **then** b **else** c **fi** we get **if** $\neg a$ **then** $\neg c$ **else** $\neg b$ **fi** $=$ $\neg$**if** a **then** b **else** c **fi**
From $a=b \wedge c$ we get $\neg a \ddagger \neg b \vee \neg c \;=\; \neg(a=b \wedge c)$
From $a=b \vee c$ we get $\neg a \ddagger \neg b \wedge \neg c \;=\; \neg(a=b \vee c)$
From $a = (b \wedge c)$ we get $\neg a \ddagger (\neg b \vee \neg c) \;=\; \neg(a = (b \wedge c))$

§(f)

old			¬	
new	⫪	**I**	ǂ	⫫
⊤	⊤	⊤	⊥	⊥
⊥	⊤	⊥	⊤	⊥

old		∨	⇐		⇒		=	∧		ǂ						
new	$\bar{\triangle}$	∨	⩾	<	⩽	>	≜	∧	△	$\bar{\nabla}$	▷	⩾	◁	⩽	▽	$\underline{\nabla}$
⊤ ⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊥	⊥	⊥	⊥
⊤ ⊥	⊤	⊤	⊤	⊤	⊥	⊥	⊥	⊥	⊤	⊤	⊤	⊤	⊥	⊥	⊥	⊥
⊥ ⊤	⊤	⊤	⊥	⊥	⊤	⊤	⊥	⊥	⊤	⊤	⊥	⊥	⊤	⊤	⊥	⊥
⊥ ⊥	⊤	⊥	⊤	⊥	⊤	⊥	⊤	⊥	⊤	⊥	⊤	⊥	⊤	⊥	⊤	⊥