135 Let x and n be natural variables. Find a specification P such that both the following refinements can be proven:

$$x = x' \times 2^{n'} \Leftarrow n := 0.\ P$$

$$P \Leftarrow \textbf{if } even\ x \textbf{ then } x := x/2.\ n := n+1.\ P \textbf{ else } ok \textbf{ fi}$$

After trying the question, scroll down to the solution.

§ The main specification $x = x' \times 2^{n'}$ looks a bit funny. Usually we write x'=something . So we can rewrite it as $x' = x/2^{n'}$. This looks reasonable because we see that x is repeatedly divided by 2 . But it's still a bit funny because the number of times x is divided by 2 is n' , which is the final value of n . What we want for P is like the main specification, but it's what remains to be done. Variable n is counting how many times x is divided by 2 . The remaining computation is to divide x by 2 another $n'-n$ times.

$$P \;=\; x' = x/2^{n'-n}$$

Or, to write it like the main specification,

$$P \;=\; x = x' \times 2^{n'-n}$$

Proof:

	$n := 0.\ \ x = x' \times 2^{n'-n}$	Substitution Law
$=$	$x = x' \times 2^{n'}$	

The second one is proven by cases. First,

	$even\ x \ \wedge\ (x := x/2.\ \ n := n+1.\ \ x = x' \times 2^{n'-n})$	use Substitution Law twice
$=$	$even\ x \ \wedge\ x/2 = x' \times 2^{n'-(n+1)}$	number theory
$=$	$even\ x \ \wedge\ x = x' \times 2^{n'-n}$	specialization
$\Rightarrow$	P	

Now the other case:

	$(x = x' \times 2^{n'-n} \ \Leftarrow\ \neg\ even\ x \ \wedge\ ok)$	expand ok
$=$	$(x = x' \times 2^{n'-n} \ \Leftarrow\ \neg\ even\ x \ \wedge\ x'=x \ \wedge\ n'=n)$	use antecedent as context in consequent
$=$	$(x = x \times 2^{n-n} \ \Leftarrow\ \neg\ even\ x \ \wedge\ x'=x \ \wedge\ n'=n)$	number theory
$=$	$(\top \ \Leftarrow\ \neg\ even\ x \ \wedge\ x'=x \ \wedge\ n'=n)$	base law for $\Leftarrow$
$=$	$\top$	