16 (degenerate) There are 256 operators with 3 binary operands and a binary result. How many of them are degenerate? An operator is degenerate if its result can be expressed without using all the operands.

After trying the question, scroll down to the solution.

§ Let $D\ n$ be the number of degenerate binary operators with n operands. Let $G\ n$ be the number of nondegenerate (good) binary operators with n operands.

$$G\ n + D\ n \ = \ 2^{2^n}$$

To begin with, there are 2 operators with 0 operands, namely $\top$ and $\bot$, and both of them are good. So

$$G\ 0 = 2 \ \text{ and } \ D\ 0 = 0$$

Next, there are 4 operators with 1 operand, and 2 of them are good (identity and negation), and two of them are degenerate (the two constant functions). So

$$G\ 1 = 2 \ \text{ and } \ D\ 1 = 2$$

Next, there are 16 operators with 2 operands. There are 2 degenerate operators that use 0 of the operands (the two good operators with 0 operands), and 2 degenerate operators that use only the left operand (the two good operators with 1 operand), and 2 degenerate operators that use only the right operand (the two good operators with 1 operand again). So that's 6 degenerate operators that don't use both the operands.

$$G\ 2 = 10 \ \text{ and } \ D\ 2 = 6$$

More generally, the number of degenerate operators with n operands is the number of good operators for each combination of fewer than n operands. Let $C\ n\ m$ be the number of ways of choosing m things from among n things. Let $n!$ be the factorial function.

$$n! \ = \ \prod i{:}\ 0,..n\cdot\ i{+}1 \ = \ 1{\times}2{\times}3{\times}...{\times}n$$

$$C\ n\ m \ = \ n!\ /\ (m! \times (n{-}m)!)$$

$$\begin{aligned} D\ n \ &= \ \Sigma i{:}\ 0,..n\cdot\ C\ n\ i \times G\ i \\ &= \ C\ n\ 0 \times G\ 0 \ + \ C\ n\ 1 \times G\ 1 \ + \ ... \ + \ C\ n\ (n{-}1) \times G\ (n{-}1) \end{aligned}$$

We want $D\ 3$, which is

$$\begin{aligned} D\ 3 \ &= \ C\ 3\ 0 \times G\ 0 \ + \ C\ 3\ 1 \times G\ 1 \ + \ C\ 3\ 2 \times G\ 2 \\ &= \ 1{\times}2 + 3{\times}2 + 3{\times}10 \\ &= \ 38 \end{aligned}$$

Therefore 38 of the 256 operators with 3 binary operands and a binary result are degenerate.