19 (pigs) Here is a sentence: “If this sentence is true, then pigs can fly.”. What are the consequences of the fact that this sentence has been written?

After trying the question, scroll down to the solution.

§ Let s be the quoted sentence.
Within sentence s , the phrase “this sentence is true” is formalized as $s{=}\top$.
(The occurrence of $=\top$ is unnecessary; it is there just to match the English.)
Let p be the sentence “Pigs can fly.”.
So sentence s is $(s{=}\top)\Rightarrow p$.
Therefore $s{=}((s{=}\top)\Rightarrow p)$. Word by word:
The quoted sentence (s) is ($=$):
if this sentence (s) is ($=$) true ($\top$), then ($\Rightarrow$) pigs can fly (p).

As far as I know right now, the quoted sentence might be true, or it might be false, or it might be either true or false (underdetermined, incomplete), or it might be neither true nor false (overdetermined, inconsistent; see Exercise 3). I am not supposing that the quoted sentence (s , or $(s{=}\top)\Rightarrow p$) is true. But, by the definitions of s and p , and the fact that the quoted sentence appears above, $s{=}((s{=}\top)\Rightarrow p)$ is definitely true. Let's start with that, and see where it goes.

$$\top \qquad \text{use the definitions of } s \text{ and } p$$
$$= \quad s{=}((\underline{s{=}\top})\Rightarrow p) \qquad \text{identity law}$$
$$= \quad s{=}(s\Rightarrow p) \qquad \text{antisymmetry (double implication)}$$
$$= \quad (\underline{s\Rightarrow(s\Rightarrow p)}) \wedge ((s\Rightarrow p)\Rightarrow s) \qquad \text{portation}$$
$$= \quad (\underline{s\wedge s} \Rightarrow p) \wedge ((s\Rightarrow p)\Rightarrow s) \qquad \text{idempotence}$$
$$= \quad (\underline{s\Rightarrow p}) \wedge ((\underline{s\Rightarrow p})\Rightarrow s) \qquad \text{discharge, or context and identity}$$
$$= \quad (\underline{s}\Rightarrow p) \wedge \underline{s} \qquad \text{symmetry and discharge, or context and identity}$$
$$= \quad s\wedge p$$

We conclude that the quoted sentence is true, and pigs can fly.

With only two variables, we can easily verify the calculation.

Let $s{=}\top$ and $p{=}\top$.
Then $s{=}((s{=}\top)\Rightarrow p) \;=\; \top{=}((\underline{\top{=}\top})\Rightarrow\top) \;=\; \top{=}(\underline{\top\Rightarrow\top}) \;=\; \underline{\top{=}\top} \;=\; \top$.
And $s\wedge p \;=\; \top\wedge\top \;=\; \top$.

Let $s{=}\top$ and $p{=}\bot$.
Then $s{=}((s{=}\top)\Rightarrow p) \;=\; \top{=}((\underline{\top{=}\top})\Rightarrow\bot) \;=\; \top{=}(\underline{\top\Rightarrow\bot}) \;=\; \underline{\top{=}\bot} \;=\; \bot$.
And $s\wedge p \;=\; \top\wedge\bot \;=\; \bot$.

Let $s{=}\bot$ and $p{=}\top$.
Then $s{=}((s{=}\top)\Rightarrow p) \;=\; \bot{=}((\underline{\bot{=}\top})\Rightarrow\top) \;=\; \bot{=}(\underline{\bot\Rightarrow\top}) \;=\; \underline{\bot{=}\top} \;=\; \bot$.
And $s\wedge p \;=\; \bot\wedge\top \;=\; \bot$.

Let $s{=}\bot$ and $p{=}\bot$.
Then $s{=}((s{=}\top)\Rightarrow p) \;=\; \bot{=}((\underline{\bot{=}\top})\Rightarrow\bot) \;=\; \bot{=}(\underline{\bot\Rightarrow\bot}) \;=\; \underline{\bot{=}\top} \;=\; \bot$.
And $s\wedge p \;=\; \bot\wedge\bot \;=\; \bot$.

This exercise includes a quoted sentence. And now we are obliged to conclude that the quoted sentence is true, and pigs can fly. And that bothers me a lot because I'm pretty sure pigs cannot fly. You could put anything in place of “pigs can fly” and prove it the same way. For example, if we write “If this sentence is true, then Santa Claus exists.”, then we have to conclude that Santa Claus exists.

I am bothered, distressed, dismayed, and upset. I don't see anything wrong with the argument except that I don't like the conclusion. The calculation is correct and verified. The definition of p looks quite harmless. That leaves only the definition of s as suspicious. The definition of s is recursive, but that, by itself, is not damning. Counting (0, 1, 2, 3, ...) and all arithmetic operations are defined recursively. Every loop in every program is a recursive definition. According to the earlier calculation, the definition $s=((s=\top)\Rightarrow p)$ is just an elaborate way of defining $s=\top$, and there's no logical harm in that. The trouble is that this elaborate definition of s has the side-effect of saying $p=\top$. In logician-speak, the definition of s is not conservative.

We introduced the names s and p ; there is no harm in introducing names. Then we defined s as $s=((s=\top)\Rightarrow p)$, and that was the trouble. A definition is an axiom. Whenever we introduce an axiom, we must always be careful that it says only what we intend it to say. This axiom says too much. Whenever an axiom says too much, we should withdraw it, or at least weaken it so that it no longer says what we don't want it to say. Problem solved, right? Unfortunately not.

We design a formal world (a theory) by choosing names and choosing axioms. But we don't design the real physical world we live in that way. We don't have to believe the quoted sentence, but we do have to believe that the sentence appears above. Just scroll back and it's staring you in the face. That's a cold, hard, physical fact. The axiom $s=((s=\top)\Rightarrow p)$ formalizes that fact. The quoted sentence (s) is (=): if this sentence (s) is (=) true ($\top$), then ($\Rightarrow$) pigs can fly (p). If we want to reason about this physical world we live in, then we must accept this axiom. And we thereby accept that pigs can fly. I am not happy.

And it gets worse. Here is another sentence: “If this sentence is true, then pigs cannot fly.”. Let r be this new quoted sentence. Let p continue to be the sentence “Pigs can fly.”. So sentence r is $(r=\top)\Rightarrow\neg p$. I am not supposing that this new quoted sentence is true, but it's a cold, hard, physical fact that it appears here. The axiom $r=((r=\top)\Rightarrow\neg p)$ formalizes that fact. So, if we want to reason about the real physical world, we must accept that axiom. It is equivalent to $r\wedge\neg p$, so we must accept that this sentence is true too, and that pigs cannot fly. We now have an inconsistency: $p\wedge\neg p$; we have proven $\bot$.

If we want to use mathematics to reason about the real physical world, we have an inconsistency. That makes mathematics useless. Will somebody please show me what's wrong?

Here's a simpler sentence, without the pigs: “This sentence is false.”. This is the well-known Liar's Paradox.

Let L be the sentence “This sentence is false.”. Formally, L is the sentence $L=\bot$. So

$$L=(L=\bot)$$

This sentence (L) is (=): this sentence (L) is (=) false ($\bot$).

It is a cold hard physical fact that the sentence “This sentence is false.” appears here, and I have called it L . So we must accept $L=(L=\bot)$ as an axiom if we want to represent reality. We can simplify it.

	$\top$	axiom
=	$L=(L=\bot)$	binary associative
=	$(L=L)=\bot$	reflexive
=	$\top=\bot$	binary

$= \quad \bot$

We have proved $\bot$. Representing reality is inconsistent. How is that possible?