218 (local minimum) You are given a list L of at least 3 numbers such that $L\,0 \ge L\,1$ and $L\,(\#L{-}2) \le L\,(\#L{-}1)$. A local minimum is an interior index i: 1,..#L–1 such that

$$L\,(i{-}1) \ge L\,i \le L\,(i{+}1)$$

Write a program to find a local minimum of L .

After trying the question, scroll down to the solution.

§ Specification P is defined as

$$P \;=\; i'\!: 1,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1)$$

Here is a linear search solution. Let i be a natural variable.

$$P \;\Leftarrow\; i{:=}\,1.\;\; Q$$

$$Q \;\Leftarrow\; \textbf{if}\; L\,i \le L(i+1)\; \textbf{then}\; ok\; \textbf{else}\; i{:=}\,i+1.\;\; Q\; \textbf{fi}$$

Now we need to define specification Q . Here is the first attempt: make it just like P except change the 1 to i .

$$Q \;=\; i'\!: i,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1)$$

Proof of P refinement:

$$\begin{array}{lll}
 & i{:=}\,1.\;\; Q & \text{expand } Q\\
= & i{:=}\,1.\;\; i'\!: i,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1) & \text{substitution law}\\
= & i'\!: 1,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1) & \\
= & P &
\end{array}$$

Proof of Q refinement, first case:

$$\begin{array}{lll}
 & L\,i \le L(i+1) \;\wedge\; ok & \text{expand } ok\\
= & L\,i \le L(i+1) \;\wedge\; i'=i & \text{portation}
\end{array}$$

This is not quite enough to imply Q . We also need $i < \#L-1$ and $L(i-1) \ge L\,i$. So weaken Q .

$$Q \;=\; i < \#L-1 \;\wedge\; L(i-1) \ge L\,i \;\Rightarrow\; i'\!: i,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1)$$

Now I have to redo the previous proof.

Proof of P refinement:

$$\begin{array}{lll}
 & i{:=}\,1.\;\; Q & \text{expand } Q\\
= & i{:=}\,1.\;\; i < \#L-1 \;\wedge\; L(i-1) \ge L\,i \;\Rightarrow\; i'\!: i,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1) & \\
 & & \text{substitution law}\\
= & 1 < \#L-1 \;\wedge\; L\,0 \ge L\,1 \;\Rightarrow\; i'\!: 1,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1) & \\
 & & \text{given } \#L \ge 3 \text{ and given } L\,0 \ge L\,1\\
= & \top \;\Rightarrow\; i'\!: 1,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1) & \text{identity}\\
= & i'\!: 1,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1) & \\
= & P &
\end{array}$$

Proof of Q refinement, first case:

$$\begin{array}{lll}
 & Q \;\Leftarrow\; L\,i \le L(i+1) \;\wedge\; ok & \text{expand } Q \text{ and } ok\\
= & \quad (i < \#L-1 \;\wedge\; L(i-1) \ge L\,i \;\Rightarrow\; i'\!: i,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1)) & \\
 & \Leftarrow\; L\,i \le L(i+1) \;\wedge\; i'=i & \text{portation}\\
= & \quad L\,i \le L(i+1) \;\wedge\; i'=i \;\wedge\; i < \#L-1 \;\wedge\; L(i-1) \ge L\,i & \\
 & \Rightarrow\; i'\!: i,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1) & \text{context}\\
= & \top &
\end{array}$$

Proof of Q refinement, last case:

$$\begin{array}{lll}
 & Q \;\Leftarrow\; L\,i > L(i+1) \;\wedge\; (i{:=}\,i+1.\;\; Q) & \text{expand first } Q \text{ and portation}\\
= & \quad i < \#L-1 \;\wedge\; L(i-1) \ge L\,i \;\wedge\; L\,i > L(i+1) \;\wedge\; (i{:=}\,i+1.\;\; Q) & \\
 & \Rightarrow\; i'\!: i,..\#L-1 \;\wedge\; L(i'-1) \ge L\,i' \le L(i'+1) & \text{expand remaining } Q
\end{array}$$

$$= \quad i<\#L-1 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1)$$
$$\land\ (i{:=}\,i+1.\ i<\#L-1 \ \land\ L(i-1)\ge L\,i \ \Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1))$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1) \qquad \text{substitution law}$$
$$= \quad i<\#L-1 \ \land\ L(i-1)\ge L i \ \land\ L\,i>L(i+1)$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1))$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1)$$

Here's my informal thinking. I see that the consequent of the inner implication

$$i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1)$$

implies the main consequent

$$i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1)$$

So I need to get rid of the antecedent of the inner implication. I can discharge it if I can show

$$i<\#L-1 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1)$$

which is the same as

$$(i<\#L-2 \ \lor\ i=\#L-2) \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1)$$

which is the same as

$$(i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1))$$
$$\land\ (i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1))$$

The top line is $\top$. So let's work on the bottom line.

$$i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1) \qquad \text{context}$$
$$= \quad i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L(\#L-2)>L(\#L-1) \ \Rightarrow\ i<\#L-2 \ \land\ L\,i\ge L(i+1)$$

But we are given $L(\#L-2)\le L(\#L-1)$. So the antecedent is $\bot$. So the bottom line is $\top$.
So that's the plan. Now write it formally. Resuming from where I left off,

$$= \quad \underline{i<\#L-1} \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1) \qquad \text{rewrite underlined bit}$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1))$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1)$$
$$= \quad (i<\#L-2 \ \lor\ i=\#L-2) \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1) \qquad \text{distribution this line}$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1))$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1)$$
$$= \quad (\quad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1)$$
$$\lor\ \ i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1)\ ) \qquad \text{context this line}$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1))$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1)$$
$$= \quad (\quad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1)$$
$$\lor\ \ i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L(\#L-2)>L(\#L-1)\ )$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1))$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1) \qquad \text{We are given } L(\#L-2)\le L(\#L-1)$$
$$= \quad (\quad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1)$$
$$\lor\ \ i=\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ \bot\ )$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1))$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1) \qquad \text{base, identity}$$
$$= \quad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1)$$
$$\land\ (i<\#L-2 \ \land\ L\,i\ge L(i+1) \ \Rightarrow\ i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1))$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1) \qquad \text{discharge}$$
$$= \quad i<\#L-2 \ \land\ L(i-1)\ge L\,i \ \land\ L\,i>L(i+1)$$
$$\land\ i'\colon i+1,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1)$$
$$\Rightarrow\ i'\colon i,..\#L-1 \ \land\ L(i'-1)\ge L\,i'\le L(i'+1) \qquad \text{specialization}$$
$$= \quad \top$$

If the list is decreasing right up to the last item, and then increases, linear search will look through the whole list, whereas binary search would take log time. But if the list starts with one decrease, and then increases all the way to the end, linear search will take time 1 and binary search will still take log time. On average, I don't know which is better.