

219 (natural division) The natural quotient of natural  $n$  and positive integer  $p$  is the natural number  $q$  satisfying

$$q \leq n/p < q+1$$

Write a program to find the natural quotient of  $n$  and  $p$  in  $\log n$  time without using functions *div*, *mod*, *floor*, or *ceil*.

After trying the question, scroll down to the solution.

§ Natural  $n$  and positive integer  $p$  are given constants. Let  $q$  be a natural variable. The specification is  $S$ , defined as

$$S = q' \leq n/p < q'+1$$

Since  $0 \leq q' \leq n$  we can do a binary search for  $q'$  and thus achieve  $\log n$  time. Modeling the solution on Subsection 4.2.5, let  $r$  and  $s$  be natural variables. Define specification  $R$  as

$$R = q \leq n/p < s+1 \Rightarrow q \leq q' \leq n/p < q'+1 \leq s+1$$

Now refine.

$$S \Leftarrow q:=0. s:=n. R$$

$$R \Leftarrow \text{if } q=s \text{ then } ok \text{ else } r:=\text{div}(q+s) \ 2. \text{ if } r \times p \leq n \text{ then } q:=r. R \text{ else } s:=r-1. R \text{ fi fi}$$

I have used  $\text{div}$ , contrary to instructions. But it's easy to get rid of it.

$$\text{div}(q+s) \ 2 = \text{if even}(q+s) \text{ then } (q+s)/2 \text{ else } (q+s-1)/2 \text{ fi}$$

We got rid of  $\text{div}(q+s) \ 2$  with 2 cases. If we had used  $\text{div } n \ p$ , we wouldn't be able to get rid of it because we can't make  $p$  cases for arbitrary  $p$ .

The solution is missing the proofs.