260 (machine multiplication) Given two natural numbers, write a program to find their product using only addition, subtraction, doubling, halving, test for even, and zero test.

After trying the question, scroll down to the solution.

§ For a solution with linear time, use the identity

$x \times y = (x-1) \times y + y$

For a solution with logarithmic time, use the identities

$x \times y = x/2 \times y \times 2$ (for even x)

$x \times y = (x-1)/2 \times y \times 2 + y$ (for odd x)

Let all variables be natural.

$x := x \times y \Leftarrow$ **if** $x=0$ **then** *ok*
else if *even* x **then** $x := x/2.\ x := x \times y.\ x := x \times 2$
else $x := (x-1)/2.\ x := x \times y.\ x := x \times 2.\ x := x+y$ **fi fi**

Note that in the solution, the occurrences of $x := x \times y$ are recursive calls. Note also that in the usual binary representation of natural numbers, $x := x \times 2$ is just shift left, and both $x := x/2$ (for even x) and $x := (x-1)/2$ (for odd x) are just shift right. The execution time is **if** $x=0$ **then** 0 **else** $1 +$ *floor* (*log* x) **fi**.

Here is another solution in which the recursive calls can be implemented as branches. Let *nat* variables a and b have the given numbers as their initial values, and let *nat* variable c have their product as its final value.

$c' = a \times b \Leftarrow c := 0.\ c' = c + a \times b$

$c' = c + a \times b \Leftarrow$ **if** $a=0$ **then** *ok*
else if *even* a **then** $a := a/2.\ b := b \times 2.\ c' = c + a \times b$
else $c := c+b.\ a := a-1.\ c' = c + a \times b$ **fi fi**

with execution time **if** $a=0$ **then** 0 **else** $1 +$ *floor* (*log* a) **fi**

Both of these solutions can be improved by testing for evenness before testing for zeroness. If a is not even, then it's not zero, and we save a test each iteration. Here's the second program with this improvement.

$c' = a \times b \Leftarrow c := 0.\ c' = c + a \times b$

$c' = c + a \times b \Leftarrow$ **if** *even* a
then if $a=0$ **then** *ok*
else $a := a/2.\ b := b \times 2.\ c' = c + a \times b$ **fi**
else $c := c+b.\ a := a-1.\ c' = c + a \times b$ **fi**