

291 (factorial space) We can compute $x := n!$ (factorial) as follows.

$x := n! \Leftarrow \mathbf{if } n=0 \mathbf{ then } x:= 1 \mathbf{ else } n:= n-1. x:= n!. n:= n+1. x:= x \times n \mathbf{ fi}$

Each call $x := n!$ pushes a return address onto a stack, and each return pops an address from the stack. Add a space variable s and a maximum space variable m , with appropriate assignments to them in the program. Find and prove an upper bound on the maximum space used.

After trying the question, scroll down to the solution.

§

$$s \leq m \leq s+n \Rightarrow m' = s+n \Leftarrow$$

if $n=0$ **then** $x:=1$
else $n:=n-1.$
 $s:=s+1. m:=m \uparrow s. s \leq m \leq s+n \Rightarrow m' = s+n. s:=s-1.$
 $n:=n+1. x:=x \times n$ **fi**

Proof: by cases. First case:

$$(n=0 \wedge (x:=1) \Rightarrow (s \leq m \leq s+n \Rightarrow m' = s+n)) \quad \text{portation and expand}$$

$$= n=0 \wedge x'=1 \wedge n'=n \wedge s'=s \wedge m'=m \wedge s \leq m \leq s+n \Rightarrow m' = s+n \quad \text{context}$$

$$= n=0 \wedge x'=1 \wedge n'=n \wedge s'=s \wedge m'=m \wedge s \leq m \leq s+n \Rightarrow \top \quad \text{base}$$

$$= \top$$

Second case:

$$(s \leq m \leq s+n \Rightarrow m' = s+n$$

$$\Leftarrow (n \neq 0 \wedge (n:=n-1. s:=s+1.$$

$$m:=m \uparrow s. s \leq m \leq s+n \Rightarrow m' = s+n. s:=s-1.$$

$$n:=n+1. x:=x \times n))) \quad \text{portation}$$

$$= n \neq 0 \wedge s \leq m \leq s+n$$

$$\wedge (n:=n-1. s:=s+1. m:=m \uparrow s. s \leq m \leq s+n \Rightarrow m' = s+n.$$

$$s:=s-1. n:=n+1. x:=x \times n)$$

$$\Rightarrow m' = s+n$$

three substitutions; expand final assignment and two more substitutions

$$= n \neq 0 \wedge s \leq m \leq s+n$$

$$\wedge (s+1 \leq m \uparrow (s+1) \leq s+1+n-1 \Rightarrow m' = s+1+n-1.$$

$$x' = x \times (n+1) \wedge n' = n+1 \wedge s' = s-1 \wedge m' = m)$$

$$\Rightarrow m' = s+n \quad \text{simplify } +1-1 \text{ twice}$$

$$= n \neq 0 \wedge s \leq m \leq s+n$$

$$\wedge (s+1 \leq m \uparrow (s+1) \leq s+n \Rightarrow m' = s+n.$$

$$x' = x \times (n+1) \wedge n' = n+1 \wedge s' = s-1 \wedge m' = m)$$

$$\Rightarrow m' = s+n \quad \text{eliminate sequential composition}$$

$$= n \neq 0 \wedge s \leq m \leq s+n$$

$$\wedge (\exists n'', m'', s'', x''. (s+1 \leq m \uparrow (s+1) \leq s+n \Rightarrow m'' = s+n)$$

$$\wedge x' = x \times (n+1) \wedge n' = n+1 \wedge s' = s-1 \wedge m' = m)$$

$$\Rightarrow m' = s+n \quad \text{some arithmetic}$$

$$= n \neq 0 \wedge s \leq m \leq s+n$$

$$\wedge (\exists n'', m'', s'', x''. (s+1 \leq m \uparrow (s+1) \leq s+n \Rightarrow m'' = s+n)$$

$$\wedge x'' = x' / (n'' + 1) \wedge n'' = n' - 1 \wedge s'' = s' + 1 \wedge m'' = m')$$

$$\Rightarrow m' = s+n \quad \text{one-point 4 times}$$

$$= n \neq 0 \wedge s \leq m \leq s+n$$

$$\wedge (s+1 \leq m \uparrow (s+1) \leq s+n \Rightarrow m' = s+n)$$

$$\Rightarrow m' = s+n \quad \begin{array}{l} s+1 \leq m \uparrow (s+1) \text{ simplifies to } \top \\ \text{and in the context } n \neq 0 \wedge s \leq m \leq s+n \text{ we simplify } m \uparrow (s+1) \leq s+n \text{ to } \top \end{array}$$

$$= n \neq 0 \wedge s \leq m \leq s+n$$

$$\wedge (\top \Rightarrow m' = s+n)$$

$$\Rightarrow m' = s+n \quad \text{identity and specialize}$$

$$= \top$$