292 Let $k$ be a natural constant, and let $x$ and $n$ be natural variables. Suppose one unit of space is allocated before each recursive call (for the return address), and freed after the call. Find and prove a maximum space bound for the refinement

$$P \Leftarrow \textbf{if } n{=}0 \textbf{ then } x{:=}\,0 \textbf{ else } n{:=}\,n{-}1.\ P.\ x{:=}\,x{+}k \textbf{ fi}$$

After trying the question, scroll down to the solution.

§ Adding space variable $s$ and maximum space variable $m$ ,

$P \Leftarrow$ **if** $n{=}0$ **then** $x{:=}\ 0$
**else** $n{:=}\ n{-}1.\ \ s{:=}\ s{+}1.\ \ m{:=}\ m{\uparrow}s.\ \ P.\ \ s{:=}\ s{-}1.\ \ x{:=}\ x{+}k$ **fi**

and define $P\ =\ s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n$ . Proof by cases. First case:

$(s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n\ \Leftarrow\ n{=}0 \wedge (x{:=}\ 0))$ portation

$=\quad n{=}0 \wedge (x{:=}\ 0) \wedge s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n$ assignment

$=\quad n{=}0 \wedge x'{=}0 \wedge n'{=}n \wedge s'{=}s \wedge m'{=}m \wedge s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n$
context $n{=}0$ in $s \le m \le s{+}n$

$=\quad n{=}0 \wedge x'{=}0 \wedge n'{=}n \wedge s'{=}s \wedge m'{=}m \wedge s{=}m\ \Rightarrow\ m' = s{+}n$
context $n{=}0 \wedge\ m'{=}m \wedge s{=}m$ in $m' = s{+}n$

$=\quad n{=}0 \wedge x'{=}0 \wedge n'{=}n \wedge s'{=}s \wedge m'{=}m \wedge s{=}m\ \Rightarrow\ m{=}m$ reflexivity

$=\quad n{=}0 \wedge x'{=}0 \wedge n'{=}n \wedge s'{=}s \wedge m'{=}m \wedge s{=}m\ \Rightarrow\ \top$ base

$=\quad \top$

Last case:

$(\quad s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n$
$\Leftarrow\ n{\neq}0 \wedge (n{:=}\ n{-}1.\ \ s{:=}\ s{+}1.\ \ m{:=}\ m{\uparrow}s.\ \ s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n.$
$s{:=}\ s{-}1.\ \ x{:=}\ x{+}k))$ substitution 3 times

$=\quad (\quad s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n$
$\Leftarrow\ n{\neq}0 \wedge (s{+}1 \le m{\uparrow}(s{+}1) \le s{+}1{+}n{-}1\ \Rightarrow\ m' = s{+}1{+}n{-}1.$
$s{:=}\ s{-}1.\ \ x{:=}\ x{+}k))$ arithmetic

$=\quad (\quad s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n$
$\Leftarrow\ n{\neq}0 \wedge (s{+}1 \le m{\uparrow}(s{+}1) \le s{+}n\ \Rightarrow\ m' = s{+}n.$
$s{:=}\ s{-}1.\ \ x{:=}\ x{+}k))$ sequential composition twice

$=\quad (\quad s \le m \le s{+}n\ \Rightarrow\ m' = s{+}n$
$\Leftarrow\ n{\neq}0 \wedge (s{+}1 \le m{\uparrow}(s{+}1) \le s{+}n\ \Rightarrow\ m' = s{+}n))$ portation

$=\quad s \le m \le s{+}n \wedge n{\neq}0 \wedge (s{+}1 \le m{\uparrow}(s{+}1) \le s{+}n\ \Rightarrow\ m' = s{+}n)\ \Rightarrow\ m' = s{+}n$
$s{+}1 \le m{\uparrow}(s{+}1)$ and in context $n{\neq}0$ we have $s{+}1 \le s{+}n$

$=\quad s \le m \le s{+}n \wedge n{\neq}0 \wedge (m \le s{+}n\ \Rightarrow\ m' = s{+}n)\ \Rightarrow\ m' = s{+}n$ context and identity

$=\quad s \le m \le s{+}n \wedge n{\neq}0 \wedge m' = s{+}n\ \Rightarrow\ m' = s{+}n$ specialize

$=\quad \top$