294 Let n be a natural variable. Find the average space occupied by the execution of

$$P \Leftarrow \textbf{if } n \ge 2 \textbf{ then } n{:=}\ n-2.\ P.\ n{:=}\ n+1.\ P.\ n{:=}\ n+1 \textbf{ else } ok \textbf{ fi}$$

if each call increases one space, and each return decreases one space.

After trying the question, scroll down to the solution.

§ We need the time, the space, and the space-time product. We found the time in Exercise 258.

$t' = t + f\,n$

where

$f\,0 \;=\; 0$

$f\,1 \;=\; 0$

$f(n+2) \;=\; 1 + f\,n + f(n+1)$

For the space, introduce variable s . For the space-time product, introduce variable p . As in Exercise 258, we just need a time increase before the first call, so that is the only increase in p .

$P \;\Leftarrow\;$ **if** $n \ge 2$ **then** $n{:=}\, n-2.$
$\quad s{:=}\, s+1.\;\; p{:=}\, p + s\times 1.\;\; P.\;\; s{:=}\, s-1.$
$\quad n{:=}\, n+1.$
$\quad s{:=}\, s+1.\;\; P.\;\; s{:=}\, s-1.$
$\quad n{:=}\, n+1$
else ok **fi**

Now we need to define P and it should have the form $p{:=}\, p+$(something) because it has to leave n and s unchanged. The “something” has to be the initial space-time product due to the prior computation, plus the space-time product during our computation but due to the initial space, plus the space-time product during our computation due to the space required by our computation, which is the quantity we need for the answer.

The initial space-time product due to the prior computation is p .
The space-time product during our computation but due to the initial space is $s\times$(time) .
For the space-time product during our computation due to the space required by our computation, I'll say $g\,n$ where g is some function I hope to find from the proof.

$P \;=\; p{:=}\, p \;+\; s\times f\,n \;+\; g\,n$

Proof by cases. First case:

$n\ge 2 \;\wedge\; (n{:=}\, n-2.\;\; s{:=}\, s+1.\;\; p{:=}\, p + s\times 1.\;\; P.$
$\quad s{:=}\, s-1.\;\; n{:=}\, n+1.\;\;\; s{:=}\, s+1.\;\; P.$
$\quad s{:=}\, s-1.\;\; n{:=}\, n+1)$ — replace P twice and replace final two assignments

$=\;\; n\ge 2 \;\wedge\; (n{:=}\, n-2.\;\; s{:=}\, s+1.\;\; p{:=}\, p+s.\;\; p{:=}\, p \;+\; s\times f\,n \;+\; g\,n.$
$\quad s{:=}\, s-1.\;\; n{:=}\, n+1.\;\;\; s{:=}\, s+1.\;\; p{:=}\, p \;+\; s\times f\,n \;+\; g\,n.$
$\quad s' = s-1 \;\wedge\; n' = n+1 \;\wedge\; p'=p)$ — substitution law 9 times

$=\;\; n\ge 2 \;\wedge\; (\;\; s'=s \;\wedge\; n'=n$
$\quad \wedge\; p' = p \;+ s + 1 +\; (s+1)\times f(n-2) \;+\; g(n-2) \;+\; (s+1)\times f(n-1) \;+\; g(n+1))$ — rearrange

$=\;\; n\ge 2 \;\wedge\; (\;\; s'=s \;\wedge\; n'=n$
$\quad \wedge\; p' = p \;+ (s+1)\times(1 + f(n-2) + f(n-1)) \;+\; g(n-2) \;+\; g(n+1))$ — use $f(n+2) \;=\; 1 + f\,n + f(n+1)$

$=\;\; n\ge 2 \;\wedge\; (\;\; s'=s \;\wedge\; n'=n$
$\quad \wedge\; p' = p \;+ (s+1)\times f\,n \;+\; g(n-2) \;+\; g(n+1))$ — now we need

$\Rightarrow\;\; p{:=}\, p \;+\; s\times f\,n \;+\; g\,n$

$=\;\; P$

So we see that for $n\ge 2$, $g\,n \;=\; f\,n +\; g(n-2) \;+\; g(n+1)$. Second case:

$n<2 \;\wedge\; ok$ — replace ok

$=\;\; n<2 \;\wedge\; s'=s \;\wedge\; n'=n \;\wedge\; p'=p$ — $f\,0 = f\,1 = 0$

$=\;\; n<2 \;\wedge\; s'=s \;\wedge\; n'=n \;\wedge\; p' = p \;+\; s\times f\,n \;+\; 0$ — now we need

$\Rightarrow\;\; p{:=}\, p \;+\; s\times f\,n \;+\; g\,n$

So we see that for $n<2$, $g\,n = 0$.

The average space occupied by the computation is $g\ n\ /\ f\ n$ where

$$f\ 0 = 0$$
$$f\ 1 = 0$$
$$f(n+2) = 1 + f\ n + f(n+1)$$

$$g\ 0 = 0$$
$$g\ 1 = 0$$
$$g(n+2) = f(n+2) + g\ n + g(n+1)$$