372 Subsection 6.0.0 gives six predicate versions of *nat* induction. Prove that they are equivalent.

After trying the question, scroll down to the solution.

§ The six predicate versions of *nat* induction are:

(a) $P\,0 \land \forall n{:}\, nat\cdot P\,n \Rightarrow P\,(n{+}1) \;\;\Longrightarrow\;\; \forall n{:}\, nat\cdot P\,n$

(b) $P\,0 \lor \exists n{:}\, nat\cdot \lnot\, P\,n \land P\,(n{+}1) \;\;\Longleftarrow\;\; \exists n{:}\, nat\cdot P\,n$

(c) $\forall n{:}\, nat\cdot P\,n \Rightarrow P\,(n{+}1) \;\;\Longrightarrow\;\; \forall n{:}\, nat\cdot P\,0 \Rightarrow P\,n$

(d) $\exists n{:}\, nat\cdot \lnot\, P\,n \land P\,(n{+}1) \;\;\Longleftarrow\;\; \exists n{:}\, nat\cdot \lnot\, P\,0 \land P\,n$

(e) $\forall n{:}\, nat\cdot (\forall m{:}\, nat\cdot m{<}n \Rightarrow P\,m) \Rightarrow P\,n \;\;\Longrightarrow\;\; \forall n{:}\, nat\cdot P\,n$

(f) $\exists n{:}\, nat\cdot (\forall m{:}\, nat\cdot m{<}n \Rightarrow \lnot\, P\,m) \land P\,n \;\;\Longleftarrow\;\; \exists n{:}\, nat\cdot P\,n$

Proof that (a) = (b), starting with (a).

$(P\,0 \land \forall n{:}\, nat\cdot P\,n \Rightarrow P\,(n{+}1) \;\Longrightarrow\; \forall n{:}\, nat\cdot P\,n)$ contrapositive

$= \lnot(P\,0 \land \forall n{:}\, nat\cdot P\,n \Rightarrow P\,(n{+}1)) \;\Leftarrow\; \lnot\forall n{:}\, nat\cdot P\,n$ duality

$= \lnot\, P\,0 \lor \lnot(\forall n{:}\, nat\cdot P\,n \Rightarrow P\,(n{+}1)) \;\Leftarrow\; \lnot\forall n{:}\, nat\cdot P\,n$ duality twice

$= \lnot\, P\,0 \lor (\exists n{:}\, nat\cdot \lnot(P\,n \Rightarrow P\,(n{+}1))) \;\Leftarrow\; \exists n{:}\, nat\cdot \lnot\, P\,n$ material implication

$= \lnot\, P\,0 \lor (\exists n{:}\, nat\cdot \lnot(\lnot\, P\,n \lor P\,(n{+}1))) \;\Leftarrow\; \exists n{:}\, nat\cdot \lnot\, P\,n$ duality

$= \lnot\, P\,0 \lor (\exists n{:}\, nat\cdot \lnot\lnot\, P\,n \land \lnot\, P(n{+}1)) \;\Leftarrow\; \exists n{:}\, nat\cdot \lnot\, P\,n$ double negation

P is implicitly universally quantified, so rename it with its negation

$= (P\,0 \lor \exists n{:}\, nat\cdot \lnot\, P\,n \land P\,(n{+}1) \;\Longleftarrow\; \exists n{:}\, nat\cdot P\,n)$

Proof that (a) = (c), starting with (a).

$(P\,0 \land \forall n{:}\, nat\cdot P\,n \Rightarrow P\,(n{+}1) \;\Longrightarrow\; \forall n{:}\, nat\cdot P\,n)$ portation

$= \forall n{:}\, nat\cdot P\,n \Rightarrow P\,(n{+}1) \;\Rightarrow\; (P\,0 \Rightarrow \forall n{:}\, nat\cdot P\,n)$ distributive

$= (\forall n{:}\, nat\cdot P\,n \Rightarrow P\,(n{+}1) \;\Longrightarrow\; \forall n{:}\, nat\cdot P\,0 \Rightarrow P\,n)$

Proof that (b) = (d) starting with (b).

$(P\,0 \lor \exists n{:}\, nat\cdot \lnot\, P\,n \land P\,(n{+}1) \;\Longleftarrow\; \exists n{:}\, nat\cdot P\,n)$ double negation

$= (\lnot\lnot\, P\,0 \lor \exists n{:}\, nat\cdot \lnot\, P\,n \land P\,(n{+}1) \;\Longleftarrow\; \exists n{:}\, nat\cdot P\,n)$ portation

$= (\exists n{:}\, nat\cdot \lnot\, P\,n \land P\,(n{+}1) \;\Longleftarrow\; \lnot\, P\,0 \land \exists n{:}\, nat\cdot P\,n)$ distribution

$= (\exists n{:}\, nat\cdot \lnot\, P\,n \land P\,(n{+}1) \;\Longleftarrow\; \exists n{:}\, nat\cdot \lnot\, P\,0 \land P\,n)$

Proof that (e) = (f) starting with (e).

$(\forall n{:}\, nat\cdot (\forall m{:}\, nat\cdot m{<}n \Rightarrow P\,m) \Rightarrow P\,n \;\Longrightarrow\; \forall n{:}\, nat\cdot P\,n)$ contrapositive

$= (\lnot\forall n{:}\, nat\cdot (\forall m{:}\, nat\cdot m{<}n \Rightarrow P\,m) \Rightarrow P\,n) \;\Leftarrow\; \lnot\forall n{:}\, nat\cdot P\,n$ duality twice

$= (\exists n{:}\, nat\cdot \lnot((\forall m{:}\, nat\cdot m{<}n \Rightarrow P\,m) \Rightarrow P\,n)) \;\Leftarrow\; \exists n{:}\, nat\cdot \lnot\, P\,n$ duality

$= (\exists n{:}\, nat\cdot (\forall m{:}\, nat\cdot m{<}n \Rightarrow P\,m) \land \lnot\, P\,n) \;\Leftarrow\; \exists n{:}\, nat\cdot \lnot\, P\,n$ rename P to $\lnot P$

$= (\exists n{:}\, nat\cdot (\forall m{:}\, nat\cdot m{<}n \Rightarrow \lnot\, P\,m) \land P\,n \;\Longleftarrow\; \exists n{:}\, nat\cdot P\,n)$

It remains to prove that one of (a) or (b) or (c) or (d) is equivalent to one of (e) or (f).